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006

CLOSED MINIMAL WILLMORE HYPERSURFACES OF $\mathbb{S}^5(1)$ WITH CONSTANT SCALAR CURVATURE*

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Abstract. We consider minimal closed hypersurfaces $M^4 \subset \mathbb{S}^5(1)$ with constant scalar curvature. We prove that, if M^4 is additionally a Willmore hypersurface, then it is isoparametric. This gives a positive answer to the question made by Chern about the pinching of the scalar curvature for closed minimal Willmore hypersurfaces in dimension 4.

Key words. Chern's conjecture, Willmore hypersurfaces, constant scalar curvature, minimal hypersurfaces in spheres.

AMS subject classifications. 53B25, 53C40

1. Introduction. S. S. Chern proposed the following question (see [7] and [8]): Let $M^n \subset \mathbb{S}^{n+1}(1)$ be an n-dimensional closed minimally immersed hypersurface of $\mathbb{S}^{n+1}(1)$ ($n \geq 2$) with constant scalar curvature. Let A be the set of possible values for the (constant) scalar curvature of M^n. Question: Is A a discrete set of real numbers?

First non-trivial case is $n = 3$. This case has been completely solved combining results from [2] and [6] in the more general context of local constant mean curvature. The answer is: for fixed H (constant mean curvature), A is finite.

For $n \geq 4$ the problem remains open. In this note we study the subclass of closed minimal Willmore hypersurfaces of $\mathbb{S}^5(1)$ with constant scalar curvature. Precisely, we prove the following:

THEOREM 1. *Let $M^4 \subset \mathbb{S}^5(1)$ be a closed minimal Willmore hypersurface of $\mathbb{S}^5(1)$ with constant scalar curvature, then M^4 is isoparametric.*

An immediate consequence of Theorem 1 is the following corollary which gives the possible values for squared length of the second fundamental form of closed minimal Willmore hypersurface with constant scalar curvature in $\mathbb{S}^5(1)$.

COROLLARY 1. *Let $M^4 \subset \mathbb{S}^5(1)$ be a closed minimal Willmore hypersurface of $\mathbb{S}^5(1)$ with constant scalar curvature. If S denotes the squared norm of the second fundamental form, then $S = 0, 4$ or 12.*

REMARK 1. *In dimension $n = 2$, the minimality implies the Willmore condition, in other words, minimal surfaces are examples of Willmore surfaces in $\mathbb{S}^3(1)$. In dimension $n = 3$, it was proved in [3] that every closed minimally immersed hypersurface of $\mathbb{S}^4(1)$ with identically zero Gauß-Kronecker curvature and nowhere zero second fundamental form is the boundary of a tube of a minimally immersed 2-dimensional surface in $\mathbb{S}^4(1)$, whose geodesic radius is $\frac{\pi}{2}$ and whose second fundamental form in*

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each normal direction is never zero. This means, by taking a non-isoparametric surface (close to the veronese surface), one can build a non-isoparametric minimal Willmore hypersurface of $\mathbb{S}^4(1)$. *This shows that the condition* $S \equiv const.$ *is essential to proving that in dimension* $n = 4$, *minimal Willmore hypersurfaces are isoparametric in* $\mathbb{S}^5(1)$.

2. Preliminaries. Let M^4 be a 4-dimensional hypersurface in a unit sphere $\mathbb{S}^5(1)$. We choose a local orthonormal frame field $\{e_1, ..., e_5\}$ in $\mathbb{S}^5(1)$, so that restricted to M^4, $e_1, ..., e_4$ are tangent to M^4. Let $\omega_1, ..., \omega_5$ denote the dual co-frame field in $\mathbb{S}^5(1)$. We use the following convention for the indices: A, B, C, D range from 1 to 5 and i, j, k range from 1 to 4. The structure equations of $\mathbb{S}^5(1)$ as a hypersurface of the Euclidean space $\mathbb{R}^6$, are given by

$$d\,\omega_A = -\sum_B \omega_{AB} \wedge \omega_B, \;\; \omega_{AB} + \omega_{BA} = 0,$$
$$d\,\omega_{AB} = -\sum_C \omega_{AC} \wedge \omega_{CB} + \frac{1}{2} \sum_{C,D} \bar{R}_{ABCD} \omega_C \wedge \omega_D,$$

where $\bar{R}$ is the Riemannian curvature tensor

$$\bar{R}_{ABCD} = \delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}.$$

The contractions $\bar{R}_{AC} = \sum_B \bar{R}_{ABCB}$ and $\bar{R} = \sum_{A,B} \bar{R}_{ABAB}$ are the Ricci curvature tensor and the scalar curvature of $\mathbb{S}^5(1)$, respectively. Next, we restrict all the tensors to M^4. First of all, $\omega_5 = 0$ on M^4, then $\sum_i \omega_{5i} \wedge \omega_i = d\omega_5 = 0$. By Cartan's lemma, we can write

$$\omega_{5i} = \sum_j h_{ij}\omega_i, \;\; h_{ij} = h_{ji}. \tag{2.1}$$

Here $h = \sum_{i,j} h_{ij}\omega_i\omega_j$ denotes the second fundamental form of M^4 and the principal curvatures λ_i are the eigenvalues of the matrix (h_{ij}). Furthermore, the mean curvature is given by $H = \frac{1}{4}\sum_i h_{ii} = \frac{1}{4}\sum_i \lambda_i$ and $K = \det(h_{ij}) = \prod_i \lambda_i$ is the Gauß-Kronecker curvature. On M^4 we have

$$d\omega_i = -\sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0,$$
$$d\omega_{ij} = -\sum_k \omega_{ik} \wedge \omega_{kj} + \frac{1}{2} \sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l,$$

where R is the Riemannian curvature tensor on M^4 with components satisfying

$$0 = R_{ijkl} + R_{ijlk}.$$

These structure equations imply the following integrability condition (Gauß equation):

$$R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + (h_{ik}h_{jl} - h_{il}h_{jk}).$$

For the scalar curvature we have

$$\kappa = 12 + 16H^2 - S,$$

where $S = \sum_{i,j} h_{ij}^2$ is the square norm of h.

From now on we will consider minimal hypersurfaces, that is the mean curvature H is identically zero on M^4. In this situation, its Ricci curvature and scalar curvature are given by, respectively,

$$R_{ij} = 3\delta_{ij} - \sum_k h_{ik}h_{jk}, \tag{2.2}$$

$$\kappa = 12 - S. \tag{2.3}$$

It follows from (2.3) that κ is constant if and only if S is constant. The covariant derivative ∇h with components h_{ijk} is given by

$$\sum_k h_{ijk}\omega_k = dh_{ij} + \sum_k h_{jk}\omega_{ik} + \sum_k h_{ik}\omega_{jk}. \tag{2.4}$$

Then the exterior derivative of (2.2) together with the structure equations yields the following Codazzi equation

$$h_{ijk} = h_{ikj} = h_{jik}. \tag{2.5}$$

For any fixed point on M^4, we can choose a local orthonormal frame $\{e_1, ..., e_4\}$, such that

$$h_{ij} = \lambda_i \delta_{ij}.$$

We define the symmetric functions f_3 and f_4 on M^4 as follows:

$$f_3 := \sum_{i,j,k} h_{ij}h_{jk}h_{ki} = \sum_i \lambda_i^3, \quad f_4 := \sum_{i,j,k} h_{ij}h_{jk}h_{kl}h_{li} = \sum_i \lambda_i^4, \tag{2.6}$$

and additionally

$$A := \sum_{i,j,k} \lambda_i^2 h_{ijk}^2 \quad \text{and} \quad B := \sum_{i,j,k} \lambda_i\lambda_j h_{ijk}^2. \tag{2.7}$$

The following formulas are taken from Peng and Terng [14] (see also [15]):

$$\frac{1}{2}\Delta S = \sum_{i,j,k} h_{ijk}^2 + (4 - S)S,$$

$$\frac{1}{3}\sum_{i,j} h_{ij}(f_3)_{ij} = Sf_4 - f_3^2 - S^2 + 2B - A + \frac{1}{2}\sum_{i,j,k} h_{ik}h_{jk}S_{ij}.$$

In particular, if S and f_3 are assumed to be constant, using the equations above, we have

$$\sum_{i,j,k} h_{ijk}^2 = (S - 4)S, \tag{2.8}$$

$$A - 2B = Sf_4 - f_3^2 - S^2. \tag{2.9}$$

Because h_{ijk} is totally symmetric, we have

$$A + 2B = \frac{1}{3}\sum_{i,j,k}(\lambda_i + \lambda_j + \lambda_k)^2 h_{ijk}^2 \geq 0. \tag{2.10}$$

3. Willmore hypersurfaces of spheres. Willmore hypersurfaces in spheres are known to be the critical points of the variational problem of the following Willmore functional (see [9]):

$$\int_M (S - nH^2)^{\frac{n}{2}} \nu.$$

H. Li computed the Euler-Lagrange equation for the Willmore functional. He obtained the following characterization of Willmore hypersurfaces (see [9]).

THEOREM 2. *Let $M^n \subset \mathbb{S}^{n+1}(1)$ be an n-dimensional compact hypersurface in an $(n+1)$-dimensional unit sphere $\mathbb{S}^{n+1}(1)$. Then M^n is a Willmore hypersurface if and only if*

$$\begin{aligned} 0 = &-\rho^{n-2}\Big(2HS - nH^3 - \sum_{i,j,k} h_{ij}h_{jk}h_{ki}\Big) + (n-1)\Delta(\rho^{n-2}H) \\ &- \sum_{i,j}(\rho^{n-2})_{ij}(nH\delta_{ij} - h_{ij}), \end{aligned}$$

where $\rho^2 = S - nH^2$, Δ is the Laplacian and $(.)_{ij}$ is the covariant derivative with respect to the induced connection.

An immediate consequence of Theorem 2 is the following characterization of Willmore hypersurfaces of spheres with constant mean curvature and constant scalar curvature:

COROLLARY 2. *Let $M^n \subset \mathbb{S}^{n+1}(1)$ be an n-dimensional compact hypersurface with constant mean curvature and constant scalar curvature in an $(n+1)$-dimensional unit sphere $\mathbb{S}^{n+1}(1)$. Then M^n is a Willmore hypersurface if and only if*

$$f_3 = \sum_{i,j,k} h_{ij}h_{jk}h_{ki} = 2HS - 4H^3.$$

In particular, the Willmore condition for minimal hypersurfaces with constant scalar curvature is equivalent to the condition $f_3 \equiv 0$.

In dimension $n = 4$, we have the following examples:

EXAMPLE 1. *The totally geodesic great sphere $\mathbb{S}^4(1) \subset \mathbb{S}^5(1)$ is a minimal Willmore hypersurface with $S = 0$;*

EXAMPLE 2. *The Clifford torus $W_{2,2} = \mathbb{S}^2\big(\frac{\sqrt{2}}{2}\big) \times \mathbb{S}^2\big(\frac{\sqrt{2}}{2}\big)$ is the only closed minimal Willmore hypersurface which is isoparametric in $\mathbb{S}^5(1)$ with two distinct principal curvature;*

EXAMPLE 3. *(Cartan's minimal hypersurface of $\mathbb{S}^5(1)$).*
Let $\mathbb{S}^5(1) = \{z \in \mathbb{C}^3 = \mathbb{R}^3 \times \mathbb{R}^3 \colon \|z\| = 1\}$ and consider the real function $F\colon \mathbb{S}^5(1) \longrightarrow \mathbb{R}$ defined by

$$F(z) = (\|x\|^2 - \|y\|^2)^2 + 4 < x, y >^2, \quad \textit{for} \quad z = x + iy.$$

Then for every t, $0 < t < \frac{\pi}{4}$, the level hypersurface of F given by

$$M_t^4 = \{z \in \mathbb{S}^5(1) : \quad F(z) = \cos^2(2t)\} = F^{-1}\Big(cos^2(2t)\Big)$$

is an isoparametric hypersurface with principal curvatures

$$\lambda_1 = \frac{1+\sin(2t)}{\cos(2t)}, \quad \lambda_2 = \frac{-1+\sin(2t)}{\cos(2t)}, \quad \lambda_3 = \tan(t) \quad \textit{and} \quad \lambda_4 = -\cot(t).$$

The hypersurfaces M_t^4 constitute the Cartan family of isoparametric hypersurfaces with four distinct principal curvatures. Among these isoparametric hypersurfaces, only the minimal one, $M^4_{\frac{\pi}{8}}$ (Cartan's minimal hypersurface), is a Willmore hypersurface. Its principal curvatures are

$$1+\sqrt{2}, \quad 1-\sqrt{2}, \quad -1+\sqrt{2} \quad \textit{and} \quad -1-\sqrt{2}.$$

Note that isoparametric hypersurfaces with four distinct principal curvatures in $\mathbb{S}^5(1)$ and $\mathbb{S}^9(1)$ were constructed by E. Cartan [5], with the property that all the principal curvatures have the same multiplicity. Such hypersurfaces are homogeneous and do exist only in $\mathbb{S}^5(1)$ and $\mathbb{S}^9(1)$. Nomizu (see[12] and [13] for details) generalized Cartan's construction to higher odd dimension.

4. Proof of Theorem 1. Obviously, if $S = 0$ (trivial case), then M^4 is the totally geodesic great sphere $\mathbb{S}^4(1)$. Suppose from now on that $S > 0$. Because the hypersurface is assumed to be minimal and by the Willmore condition $f_3 = 0$, the characteristic polynomial of the matrix (h_{ij}) corresponding to the second fundamental form is given by

$$p(\lambda) = \lambda^4 - \frac{S}{2}\lambda^2 + K. \tag{4.1}$$

It is clear that this fourth order polynomial $p(\lambda)$ has real roots (principal curvatures of M^4) if and only if $S^2 \geq 16K$ everywhere and M^4 has non-negative Gauß-Kronecker curvature function, i.e, $K \geq 0$.

Remark 2. *To get the condition $S^2 \geq 16K$ under Willmore condition for minimal hypersurfaces in $\mathbb{S}^4(1)$ with constant scalar curvature, one can use Lagrange multipliers method to minimize the functional $f_4 = \frac{S^2}{2} - 4K$ under $H = 0$, $S^2 \equiv const.$ and $f_3 = 0$.*

Renumbering the vector fields e_1, e_2, e_3, e_4 if necessary, we may assume that the pincipal curvatures satisfy $\lambda_1 \leq \lambda_2 \leq 0 \leq \lambda_3 \leq \lambda_4$. More precisely we have

$$\begin{cases} \lambda_4 = \frac{1}{2}\big(S + \sqrt{S^2 - 16K}\big)^{\frac{1}{2}} = -\lambda_1 \quad \text{and} \\ \lambda_3 = \frac{1}{2}\big(S - \sqrt{S^2 - 16K}\big)^{\frac{1}{2}} = -\lambda_2 \end{cases}$$

It is clear that $\lambda_i(p) = \lambda_j(p)$ for arbitrary $1 \leq i < j \leq 4$ at some point $p \in M^4$ if and only if at that point p one has $K(p) = 0$ or $\frac{S^2}{16}$.

In order to prove Theorem 1, we have to distinguish the following cases:

(i) there exists a point $p \in M^4$ such that $K(p) = \frac{S^2}{16}$;

(ii) $0 \leq K < \frac{S^2}{16}$ everywhere on M^4.

The following result will play a crucial role in the proof of our main result.

THEOREM 3. *Let $M^4 \subset \mathbb{S}^5(1)$ be a closed minimal Willmore hypersurface with constant scalar curvature. If there exists a point p of M^4 such that $K(p) = \frac{S^2}{16} > 0$, where K denotes the Gauß-Kronecker curvature function and S the squared length of the second fundamental form, then M^4 is isoparametric with two distinct principal curvatures; in this case, M^4 is the Clifford torus $\mathbb{S}^2\big(\frac{\sqrt{2}}{2}\big) \times \mathbb{S}^2\big(\frac{\sqrt{2}}{2}\big)$.*

Proof. Suppose that at a point $p \in M^4$ we have $K(p) = \frac{S^2}{16} > 0$. At such a point p the principal curvatures are given by

$$-\lambda_1 = -\lambda_2 = \lambda_3 = \lambda_4 = \frac{\sqrt{S}}{2} > 0. \tag{4.2}$$

Using the Codazzi equations (see integrability conditions from section 2), we obtain the following at p:

$$h_{123} = h_{124} = h_{134} = h_{234} = h_{112} = h_{221} = h_{334} = h_{443} = 0. \tag{4.3}$$

Since M^4 is minimal and has constant scalar curvature, we have for $1 \le k \le 4$

$$\sum_i h_{iik} = \sum_i \lambda_i h_{iik} = 0. \tag{4.4}$$

It follows from (4.2), (4.3) and (4.4) that

$$h_{iii} = 0 \quad \text{for all} \quad i \quad \text{at} \quad p. \tag{4.5}$$

Another consequence of the Willmore condition for minimal hypersurfaces with constant scalar curvature, i.e., $f_3 = 0$, is that $f_4 = \frac{S^2}{2} - 4K$. Therefore, inserting this expression of f_4 into the equation (2.9) with $f_3 = 0$, we get

$$A - 2B = \frac{S^2}{4}(S - 4). \tag{4.6}$$

Because of (4.3), the only eventual non-zero h_{ijk} are h_{113}, h_{114}, h_{223}, h_{224}, h_{331}, h_{332}, h_{441} and h_{442}, and we use (4.2) to get

$$3(A + 2B) = \sum_{i,j,k} (\lambda_i + \lambda_j + \lambda_k)^2 h_{ijk}^2 = \frac{S}{4} \sum_{ijk} h_{ijk}^2.$$

Therefore, by (2.8) we have

$$3(A + 2B) = \frac{S^2}{4}(S - 4). \tag{4.7}$$

From the equations (4.6) and (4.7), we deduce that

$$A + 4B = 0. \tag{4.8}$$

On the other hand, we use again (4.2), (4.3) and (4.5) to compute the expressions of A and B at p explicitly. We get the following:

$$A + 4B = -\frac{S}{4} \sum_{i,j} h_{iij}^2.$$

So by (4.8), we conclude that $h_{ijk} = 0$, for all i, j, k. Thus $0 = \sum_{i,j,k} h_{ijk}^2 + S(S-4)$, i.e., $S = 4$.

In this case, by applying a result of Chern, do Carmo and Kobayashi (see Theorem 2, [8]), we infer that M^4 is isometric to the Clifford torus $\mathbb{S}^2\big(\frac{\sqrt{2}}{2}\big) \times \mathbb{S}^2\big(\frac{\sqrt{2}}{2}\big)$. □

Now we consider the case $K < \frac{S^2}{16}$ everywhere on M^4 and prove

THEOREM 4. *Let $M^4 \subset \mathbb{S}^5(1)$ be a closed minimal Willmore hypersurface with constant scalar curvature. If $K < \frac{S^2}{16}$ everywhere on M^4, then M^4 is isoparametric with four distinct principal curvatures; in this case, M^4 is the Cartan minimal hypersurface as described in Example 3.*

Proof. If $0 \leq S \leq 4$, our result follows immediately using a result of Chern, do Carmo and Kobayashi [8]. Assume now that $S > 4$. In this case we want to prove that $S = 12$, i.e., $\kappa = 0$. Suppose that $S \neq 12$, i.e., $|\kappa| > 0$.

Choose $p \in M^4$ such that $C_1 = K(p) = \max K$. If $K(p) = 0$ then K vanishes identically on M^4. Consequently, the characteristic polynomial (4.1) has constant coefficients, i.e., the hypersurface M^4 is isoparametric. Since $S > 0$, M^4 then is an isoparametric hypersurface of $\mathbb{S}^5(1)$ with three distinct principal curvatures. This is a contradiction as it is well known from Cartan's classification result [4] that isoparametric hypersurfaces of $\mathbb{S}^{n+1}(1)$ with three distinct principal curvatures do exist only if $n = 3, 6, 12, 24$. This proves that the open subset of M^4 defined by

$$X := K^{-1}\Big(0, \frac{S^2}{16}\Big)$$

is non-empty. We say that the pair (U, ω) is admissible if

(i) U is an open subset of X,
(ii) $\omega = (\omega_1, \omega_2, \omega_3, \omega_4)$ is a smooth orthonormal co-frame field on U,
(iii) $\omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \omega_4 = \mathrm{vol}$,
(iv) $h = \sum_i \lambda_i \omega_i \omega_i$.

From [1], we know that there is one and only one 3-form ψ on X such that if (U, ω) is admissible, then such a 3-form ψ is given on U by

$$\begin{aligned}\psi = {} & \omega_1 \wedge \omega_2 \wedge \omega_{34} + \omega_3 \wedge \omega_1 \wedge \omega_{24} + \omega_1 \wedge \omega_4 \wedge \omega_{23} + \omega_2 \wedge \omega_3 \wedge \omega_{14} \\ & + \omega_4 \wedge \omega_2 \wedge \omega_{13} + \omega_3 \wedge \omega_4 \wedge \omega_{12}.\end{aligned}$$

Define $D := \prod_{1 \leq i < j \leq 4} (\lambda_j - \lambda_i)$ and $q(w, x, y, z) := \frac{1}{4}\left((w-x)^2(w-y)(w-z)\right)^{-1}$.

LEMMA 1. *Denote by K_i the ith component of the covariant derivative dK with respect to the co-frame field $\omega = (\omega_1, \omega_2, \omega_3, \omega_4)$, i.e., $dK = \sum_{i=1}^4 K_i \omega_i$. Then on X we have:*

$$\begin{aligned}dK \wedge \psi = -4\Big(& \big(q(\lambda_4, \lambda_1, \lambda_2, \lambda_3) + q(\lambda_3, \lambda_1, \lambda_2, \lambda_4) + q(\lambda_2, \lambda_1, \lambda_3, \lambda_4)\big) K_1^2 \\ & + \big(q(\lambda_4, \lambda_2, \lambda_1, \lambda_3) + q(\lambda_3, \lambda_2, \lambda_1, \lambda_4) + q(\lambda_1, \lambda_2, \lambda_3, \lambda_4)\big) K_2^2 \\ & + \big(q(\lambda_4, \lambda_3, \lambda_1, \lambda_2) + q(\lambda_2, \lambda_3, \lambda_1, \lambda_4) + q(\lambda_1, \lambda_3, \lambda_2, \lambda_4)\big) K_3^2 \\ & + \big(q(\lambda_3, \lambda_4, \lambda_1, \lambda_2) + q(\lambda_2, \lambda_4, \lambda_1, \lambda_3) + q(\lambda_1, \lambda_4, \lambda_2, \lambda_3)\big) K_4^2 \Big) \mathit{vol}.\end{aligned} \tag{4.9}$$

Proof. Differentiating our curvature conditions

$$\begin{aligned}
&\lambda_1+\lambda_2+\lambda_3+\lambda_4=0,\\
&\lambda_1^2+\lambda_2^2+\lambda_3^2+\lambda_4^2=S=const,\\
&\lambda_1^3+\lambda_2^3+\lambda_3^3+\lambda_4^3=0
\end{aligned}$$

with respect to the direction field e_1, we obtain:

$$\begin{aligned}
0&=h_{111}+h_{221}+h_{331}+h_{441},\\
0&=\lambda_1h_{111}+\lambda_2h_{221}+\lambda_3h_{331}+\lambda_4h_{441},\\
0&=\lambda_1^2h_{111}+\lambda_2^2h_{221}+\lambda_3^2h_{331}+\lambda_4^2h_{441}.
\end{aligned}$$

Because the four principal curvatures are distinct at every point, we can express h_{ii1}, $i=2,3,4$, in terms of h_{111}:

$$h_{ii1}=-\frac{\prod\limits_{j\neq i,1}(\lambda_j-\lambda_1)}{\prod\limits_{j\neq i,1}(\lambda_j-\lambda_i)}h_{111}.$$

This implies

$$K_1=\sum_{i=1}^{4}\frac{K}{\lambda_i}h_{ii1}=-(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)h_{111};$$

and

$$h_{ii1}=\frac{K_1}{\prod\limits_{j\neq i}(\lambda_j-\lambda_i)} \tag{4.10}$$

for $i=2,3$ or 4.
Using the equation (2.4), we deduce

$$\omega_{1j}=\frac{1}{\lambda_j-\lambda_1}\left(\sum_k h_{1jk}\,\omega_k\right). \tag{4.11}$$

To compute $d\,K\wedge\psi=\left(\sum\limits_i K_i\,\omega_i\right)\wedge\psi$, we just need to compute $\omega_1\wedge\psi$; the other terms can be determined by analogy. Using the equations (4.10) and (4.11), we get

$$\begin{aligned}
\omega_1\wedge\psi&=\omega_1\wedge\big(\omega_2\wedge\omega_3\wedge\omega_{14}+\omega_4\wedge\omega_2\wedge\omega_{13}+\omega_3\wedge\omega_4\wedge\omega_{12}\big)\\
&=\left(\sum_{i\neq1}\frac{h_{ii1}}{\lambda_i-\lambda_1}\right)\mathrm{vol}\\
&=\left(\sum_{i\neq1}\frac{K_1}{(\lambda_i-\lambda_1)\prod\limits_{j\neq i}(\lambda_j-\lambda_i)}\right)\mathrm{vol}\\
&=-4K_1\Big(q(\lambda_4,\lambda_1,\lambda_2,\lambda_3)+q(\lambda_3,\lambda_1,\lambda_2,\lambda_4)+q(\lambda_2,\lambda_1,\lambda_3,\lambda_4)\Big)\,\mathrm{vol}.
\end{aligned}$$

∎

LEMMA 2. *The exterior differential $d\psi$ of the form ψ on X is given by*

$$d\psi = \left(\frac{1}{D^2}(S^2 - 16K)|\nabla K|^2 + \frac{\kappa}{2}\right) vol. \tag{4.12}$$

Proof.

$$\begin{aligned} d\psi &= d(\omega_1 \wedge \omega_2 \wedge \omega_{34}) + ... \\ &= d\omega_1 \wedge \omega_2 \wedge \omega_{34} - \omega_1 \wedge d\omega_2 \wedge \omega_{34} + \omega_1 \wedge \omega_2 \wedge d\omega_{34} + \cdots \end{aligned}$$

From the structure equations, we have:

$$\begin{aligned} d\omega_1 &= -(\omega_{12} \wedge \omega_2 + \omega_{13} \wedge \omega_3 + \omega_{14} \wedge \omega_4) \\ &= (...) \wedge \omega_2 - \frac{1}{\lambda_3 - \lambda_1}(h_{113}\omega_1 + h_{134}\omega_4) \wedge \omega_3 \\ &\quad - \frac{1}{\lambda_4 - \lambda_1}(h_{114}\omega_1 + h_{134}\omega_3) \wedge \omega_4. \end{aligned}$$

So

$$\begin{aligned} d\omega_1 \wedge \omega_2 \wedge \omega_{34} &= -\frac{h_{113}h_{443}}{(\lambda_3 - \lambda_1)(\lambda_4 - \lambda_3)} \omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \omega_4 \\ &\quad - \frac{h_{114}h_{334}}{(\lambda_3 - \lambda_2)(\lambda_4 - \lambda_3)} \omega_1 \wedge \omega_4 \wedge \omega_2 \wedge \omega_3 \\ &= -\left(\frac{h_{113}h_{443}}{(\lambda_3 - \lambda_1)(\lambda_4 - \lambda_3)} + \frac{h_{114}h_{334}}{(\lambda_4 - \lambda_3)(\lambda_4 - \lambda_1)}\right) \text{vol}. \end{aligned}$$

In the same way (interchanging the role of ω_1 and ω_2), we have

$$\omega_1 \wedge d\omega_2 \wedge \omega_{34} = \left(\frac{h_{223}h_{443}}{(\lambda_3 - \lambda_2)(\lambda_4 - \lambda_3)} + \frac{h_{224}h_{334}}{(\lambda_4 - \lambda_3)(\lambda_4 - \lambda_2)}\right) \text{vol}.$$

We also have

$$\begin{aligned} d\omega_{34} &= -\omega_{31} \wedge \omega_{14} - \omega_{32} \wedge \omega_{24} + R_{3434}\, \omega_3 \wedge \omega_4 \\ &= -\left(\frac{h_{331}h_{441}}{(\lambda_1 - \lambda_3)(\lambda_4 - \lambda_1)} + \frac{h_{332}h_{442}}{(\lambda_2 - \lambda_3)(\lambda_4 - \lambda_2)} + \lambda_3\lambda_4 + 1\right) \omega_3 \wedge \omega_4 \\ &\quad + (\cdots) \wedge \omega_1 + (\cdots) \wedge \omega_2. \end{aligned}$$

So

$$\begin{aligned} \omega_1 \wedge \omega_2 \wedge \omega_3 \wedge d\omega_{34} = \Bigg(&\lambda_3\lambda_4 + 1 - \frac{h_{331}h_{441}}{(\lambda_1 - \lambda_3)(\lambda_4 - \lambda_1)} \\ &- \frac{h_{332}h_{442}}{(\lambda_2 - \lambda_3)(\lambda_4 - \lambda_2)}\Bigg) \text{vol}. \end{aligned}$$

Similarly one computes

$$d(\omega_3 \wedge \omega_1 \wedge \omega_{24}), \quad d(\omega_1 \wedge \omega_4 \wedge \omega_{23}) \quad \text{and} \quad d(\omega_2 \wedge \omega_3 \wedge \omega_{14})$$

to get that

$$d\psi = \left(\frac{1}{2}\kappa - \sum_{k=1}^{4} I_k\right) \text{vol},$$

where

$$I_k = \sum_{k\neq i<j\neq k} \frac{h_{iik}h_{jjk}}{(\lambda_k-\lambda_i)(\lambda_k-\lambda_i)}.$$

Recall that the principal curvatures satisfy $\lambda_1 = -\lambda_4$ and $\lambda_2 = -\lambda_3$. Thus $S^2 - 16K = 4(\lambda_4^2-\lambda_3^2)^2$ and $D = 4\lambda_3\lambda_4(\lambda_4^2-\lambda_3^2)^2$. Now using (4.10) to compute I_1, we get

$$I_1 = -\frac{1}{4\lambda_3^2\lambda_4^2(\lambda_4^2-\lambda_3^2)^2}K_1^2 = \frac{1}{D^2}(S^2-16K)K_1^2.$$

Similarly, we have

$$I_i = -\frac{1}{D^2}(S^2-16K)K_i^2, \quad \text{for} \quad i=2,3,4.$$

Therefore,

$$\sum_{k=1}^{4} I_k = \frac{1}{D^2}(S^2-16K)\sum_{i=1}^{4}K_i^2 = \frac{1}{D^2}(S^2-16K)|\nabla K|^2.$$

This establishes the formula (4.12). □

Now we are in position to continue the proof of Theorem 4. From Sard's theorem, we can obtain $\varepsilon > 0$ such that $C_1 - \varepsilon$ is a regular value of K. Take $0 < \varepsilon_1 < \varepsilon$ sufficiently small such that $D(p) \neq 0$ for all $p \in W_\varepsilon \cup W_{\varepsilon_1}$, where W_ε and W_{ε_1} are compact subsets of M^4 defined by

$$W_\varepsilon = K^{-1}[C_1-\varepsilon, C_1] \quad \text{and} \quad W_{\varepsilon_1} = K^{-1}[C_1-(\varepsilon_1+\varepsilon), C_1-\varepsilon].$$

Now we consider a smooth function $\eta_{\varepsilon,\varepsilon_1}\colon (-\infty, C_1+\varepsilon] \longrightarrow [0,1]$ with compact support such that

(i) $0 \leq \eta_{\varepsilon,\varepsilon_1}(t) \leq 1$ for all t,
(ii) $\eta_{\varepsilon,\varepsilon_1}(t) = 0$ if $t \leq C_1-(\varepsilon_1+\varepsilon)$ and $\eta_{\varepsilon,\varepsilon_1}(t) = 1$ if $C_1-\varepsilon \leq t \leq C_1+\varepsilon$,
(iii) $\eta'_{\varepsilon,\varepsilon_1}(t) \geq 0$ for all t.

In fact the function $\eta_{\varepsilon,\varepsilon_1}$ can be defined by $\eta_{\varepsilon,\varepsilon_1}(t) = \xi\Big(t - \big(C_1-(\varepsilon_1+\varepsilon)\big)\Big)$, where

$$\xi(t) = \begin{cases} 0, \quad \text{if} \quad t\leq 0 \\ \exp\left(\frac{-\varepsilon_1}{t}\exp(\frac{-\varepsilon_1}{\varepsilon_1-t})\right) \text{if} \quad 0<t<\varepsilon_1 \\ 1 \quad \text{if} \quad \varepsilon_1 \leq t \leq \varepsilon_1+2\varepsilon. \end{cases}$$

Applying Stoke's theorem to integrate

$$d\big(\eta_{\varepsilon,\varepsilon_1}(K)\psi\big) = \eta_{\varepsilon,\varepsilon_1}(K)\,d\psi + \eta'_{\varepsilon,\varepsilon_1}(K)\,dK\wedge\psi$$

on the closed hypersurface M^4, we have

$$0 = \int_{W_\varepsilon\cup W_{\varepsilon_1}} \eta_{\varepsilon,\varepsilon_1}(K)d\psi + \int_{W_{\varepsilon_1}} \eta'_{\varepsilon,\varepsilon_1}(K)dK\wedge\psi. \tag{4.13}$$

Define the numbers $C := \max \frac{1}{D^2}(S^2-16\,K)$ and $C' := \max\limits_{1\leq i\leq 4}|q_i|$ on $W_\varepsilon \cup W_{\varepsilon_1}$, where q_i is the factor of K_i^2 in the expression (4.9) of $dK\wedge\psi$ (see Lemma 1). It follows that

$|dK \wedge \psi| \leq C'|\nabla K|^2$ on $W_\varepsilon \cup W_{\varepsilon_1}$. The equation (4.13) implies

$$\begin{aligned}\left|\int_{W_\varepsilon \cup W_{\varepsilon_1}} \eta_{\varepsilon,\varepsilon_1}(K) d\psi\right| &= \left|\int_{W_{\varepsilon_1}} \eta'_{\varepsilon,\varepsilon_1}(K) dK \wedge \psi\right| \\ &\leq \int_{W_{\varepsilon_1}} \eta'_{\varepsilon,\varepsilon_1}(K)|dK \wedge \psi| \,\mathrm{vol} \\ &= C' \int_{W_{\varepsilon_1}} \eta'_{\varepsilon,\varepsilon_1}(K)|\nabla K| \,\mathrm{vol}.\end{aligned}$$

Because of the expression (4.13) of $d\psi$ where κ is constant, we have

$$\begin{aligned}&\left|\int_{W_\varepsilon \cup W_{\varepsilon_1}} \eta_{\varepsilon,\varepsilon_1}(K) d\psi\right| = \left|\int_{W_\varepsilon \cup W_{\varepsilon_1}} \eta_{\varepsilon,\varepsilon_1}(K) \left(\frac{1}{D^2}(S^2 - 16K)|\nabla K|^2 + \frac{\kappa}{2}\right) \mathrm{vol}\right| \\ &\geq \left|\frac{\kappa}{2} \int_{W_\varepsilon \cup W_{\varepsilon_1}} \eta_{\varepsilon,\varepsilon_1}(K) \,\mathrm{vol}\right| - \left|\int_{W_\varepsilon \cup W_{\varepsilon_1}} \frac{1}{D^2} \eta_{\varepsilon,\varepsilon_1}(K)(S^2 - 16K)|\nabla K|^2 \,\mathrm{vol}\right| \\ &\geq \frac{|\kappa|}{2} \int_{W_{\varepsilon_1}} \mathrm{vol} - C \int_{W_\varepsilon \cup W_{\varepsilon_1}} |\nabla K|^2 \,\mathrm{vol}.\end{aligned}$$

This provides the following inequality:

$$\frac{|\kappa|}{2} \int_{W_\varepsilon} \mathrm{vol} \leq C \int_{W_\varepsilon \cup W_{\varepsilon_1}} |\nabla K|^2 \mathrm{vol} + C' \int_{W_{\varepsilon_1}} \eta'_{\varepsilon,\varepsilon_1}(K)|\nabla K|^2 \mathrm{vol}. \tag{4.14}$$

The following result is well known from Analysis and Measure Theory (see for example the book [10], pp. 461):

LEMMA 3. *Let ω be a differential form on M^4 and $F \subset M^4$ a closed subset with zero measure. Then for all $\varepsilon > 0$, there exists an open subset $Z \subset M^4$ such that $F \subset Z$ and $|\int_Z \omega| < \varepsilon$.*

From Lemma 3 and Sard's theorem we can obtain $0 < \varepsilon_2 < \varepsilon_1$, such that the number $t_2 = C_1 - (\varepsilon_2 + \varepsilon)$ is a regular value of K and

$$\int_{W_{\varepsilon_1}} \eta'_{\varepsilon,\varepsilon_1}(K)|\nabla K|^2 \mathrm{vol} < \int_{Y_2} \eta'_{\varepsilon,\varepsilon_1}(K)|\nabla K|^2 \mathrm{vol} + \frac{1}{2}, \tag{4.15}$$

where $Y_2 = K^{-1}[C_1 - (\varepsilon_1 + \varepsilon), t_2]$.

Notice that $\lim_{\varepsilon_1 \to \varepsilon_2} Y_2 = K^{-1}(t_2) := X_2$, $\lim_{\varepsilon_1 \to \varepsilon_2} \eta'_{\varepsilon,\varepsilon_1}(K) = \eta'_{\varepsilon,\varepsilon_2}(K)$ and $\lim_{\varepsilon_1 \to \varepsilon_2} W_{\varepsilon_1} = W_{\varepsilon_2} := K^{-1}[t_2, C_1 - \varepsilon]$. Moreover $\int_{X_2} \eta'_{\varepsilon,\varepsilon_2}(K)|\nabla K|^2 \mathrm{vol} = 0$, hence (4.15) yields

$$\int_{W_{\varepsilon_1}} \eta'_{\varepsilon,\varepsilon_1}(K)|\nabla K|^2 \mathrm{vol} \leq \frac{1}{2}.$$

Therefore, we can define inductively a sequence (ε_i), $0 < \varepsilon_i < \varepsilon_{i-1}$, such that the number $t_i = C_1 - (\varepsilon_i + \varepsilon)$ is a regular value of K and

$$\int_{W_{\varepsilon_i}} \eta'_{\varepsilon,\varepsilon_i}(K)|\nabla K|^2 \mathrm{vol} \leq \frac{1}{i}, \tag{4.16}$$

where $W_{\varepsilon_i} = K^{-1}[t_i, C_1 - \varepsilon]$.

It follows from (4.14) and (4.16) that

$$\frac{|\kappa|}{2}\int_{W_\varepsilon} \text{vol} \le C\int_{W_\varepsilon \cup W_{\varepsilon_i}} |\nabla K|^2 \text{vol} + \frac{1}{i}.$$

And since $\lim_{i\to\infty} W_{\varepsilon_i} = K^{-1}(C_1 - \varepsilon)$, we get

$$\frac{|\kappa|}{2}\int_{W_\varepsilon} \text{vol} \le C\int_{W_\varepsilon} |\nabla K|^2 \text{vol} \le C\sup_{W_\varepsilon} |\nabla K|^2 \int_{W_\varepsilon} \text{vol}. \tag{4.17}$$

Note that $\int_{W_\varepsilon} \text{vol} > 0$ and $\lim_{\varepsilon\to 0}\sup_{W_\varepsilon} |\nabla K|^2 = 0$, thus (4.17) implies that

$$\frac{|\kappa|}{2} \le \lim_{\varepsilon\to 0}\sup_{W_\varepsilon} |\nabla K|^2 = 0,$$

which contradicts our assumption that $\kappa \neq 0$. Hence $\kappa = 0$ on M^4.

Now we want to prove that the Gauß-Kronecker curvature function K is constant on M^4 to conclude that M^4 is an isoparametric hypersurface. The proof essentially follows the pattern of de Sousa ([16], [17]). We only stress the points which may lead to some differences. We proceed as above while proving that κ is constant.

Given a small non-zero positive real number ε, we choose a smooth function $\eta_\varepsilon\colon (-\infty, C_1 + \varepsilon] \longrightarrow \mathbb{R}$ with compact support such that:

(i) $0 \le \eta_\varepsilon(t) \le 1$ for all t,
(ii) $\eta_\varepsilon(t) = 0$ if $t \in (-\infty, \frac{\varepsilon}{3}]$,
(iii) $\eta_\varepsilon(t) = 1$ if $t \in (\varepsilon, C_1 + \varepsilon]$,
(iv) $\eta'_\varepsilon(t) \ge 0$ for all $t \in (\frac{\varepsilon}{3}, \varepsilon)$.

Althought there does not exist a unique extension of the form ψ on $K^{-1}(0)$ because of $\eta_\varepsilon(K) = 0$ on $K^{-1}(0)$, we may consider the 3-form $\varphi = \eta_\varepsilon(K)\psi$ which is globally defined on M^4. Since $\kappa = 0$, by Stoke's theorem and (4.12), we have

$$\begin{aligned} 0 &= \int_{M^4} d\varphi = \int_{K^{-1}[\frac{\varepsilon}{3}, C_1+\varepsilon]} \eta_\varepsilon(K)\, d\,\psi + \int_{K^{-1}[\frac{\varepsilon}{3},\varepsilon]} \eta'_\varepsilon(K) d\,K \wedge \psi \\ &= \int_{K^{-1}[\frac{\varepsilon}{3}, C_1+\varepsilon]} \eta_\varepsilon(K) \frac{(S^2 - 16K)}{D^2} |\nabla K|^2 \text{vol} + \int_{K^{-1}[\frac{\varepsilon}{3},\varepsilon]} \eta'_\varepsilon(K) d\,K \wedge \psi \end{aligned} \tag{4.18}$$

Let α_1 be a real number such that

$$\max\{|Q_1|, |Q_2|, |Q_3|, |Q_4|\} \le \alpha_1, \tag{4.19}$$

where Q_i is the factor of $-4K_i^2$ $(1 \le i \le 4)$ in the equation (4.9).

It follows from (4.9), (4.18) and (4.19) for sufficiently small $\varepsilon > 0$ that

$$\begin{aligned} &\int_{K^{-1}[\frac{\varepsilon}{3}, C_1+\varepsilon]} \eta_\varepsilon(K) \frac{(S^2 - 16K)}{D^2} |\nabla K|^2 \text{vol} \\ &\le \alpha_1 \int_{K^{-1}[\frac{\varepsilon}{3},\varepsilon]} \eta'_\varepsilon(K) |\nabla K|^2 \text{vol}. \end{aligned} \tag{4.20}$$

Let $\xi\colon (-\infty, C_1+\varepsilon] \longrightarrow \mathbb{R}$ be the smooth function given by $\xi(t) := \eta_\varepsilon(t) - 1$. Notice that $\xi'(t) = \eta'_\varepsilon(t)$. By applying Stoke's theorem to

$$\operatorname{div}\Big((\xi \circ K)\nabla K\Big) = \eta'_\varepsilon(K)|\nabla K|^2 + \xi(K)\Delta K,$$

we get

$$0 = \int_{M^4} \operatorname{div}\Big((\xi \circ K)\nabla K\Big)\mathrm{vol} = \int_{K^{-1}[\frac{\varepsilon}{3},\varepsilon]} \eta'_\varepsilon(K)|\nabla K|^2\mathrm{vol} + \int_{K^{-1}[0,\varepsilon]} \xi(K)\Delta K\mathrm{vol},$$

which implies the following integral inequality:

$$\int_{K^{-1}[\frac{\varepsilon}{3},\varepsilon]} \eta'_\varepsilon(K)|\nabla K|^2\mathrm{vol} \le \int_{K^{-1}[0,\varepsilon]} |\Delta K|\mathrm{vol}. \tag{4.21}$$

Combining the inequalities (4.20) and (4.21), we get

$$0 \le \int_{K^{-1}[\frac{\varepsilon}{3},C_1+\varepsilon]} \eta_\varepsilon(K)\frac{(S^2-16K)}{D^2}|\nabla K|^2\mathrm{vol} \le \alpha_1 \int_{K^{-1}[0,\varepsilon]} |\Delta K|\mathrm{vol}. \tag{4.22}$$

The following lemma was proved in [2] for $n = 3$ and still holds for $n > 3$.

Lemma 4. *Let $u\colon M^4 \longrightarrow \mathbb{R}$ be a smooth function and $m = min_{M^4} u$. If $D_\varepsilon = u^{-1}\big([m, m+\varepsilon]\big)$, then*

$$\lim_{\varepsilon\to 0} \int_{D_\varepsilon} |\Delta u| vol = 0.$$

In particular,

$$\lim_{\varepsilon\to 0} \int_{K^{-1}[0,\varepsilon]} |\Delta u| vol = 0.$$

Due to Lemma 4 and the integral inequality (4.22), we infer that

$$\lim_{\varepsilon\to 0} \int_{K^{-1}[\frac{\varepsilon}{3},C_1+\varepsilon]} \eta_\varepsilon(K)\frac{(S^2-16K)}{D^2}|\nabla K|^2\mathrm{vol} = 0. \tag{4.23}$$

For $0 < \varepsilon < \varepsilon'$, we have

$$\begin{aligned} 0 &\le \int_{K^{-1}[\varepsilon',C_1+\varepsilon]} \frac{(S^2-16K)}{D^2}|\nabla K|^2\mathrm{vol} \le \int_{K^{-1}[\varepsilon,C_1+\varepsilon]} \frac{(S^2-16K)}{D^2}|\nabla K|^2\mathrm{vol} \\ &\le \int_{K^{-1}[\frac{\varepsilon}{3},C_1+\varepsilon]} \eta_\varepsilon(K)\frac{(S^2-16K)}{D^2}|\nabla K|^2\mathrm{vol}. \end{aligned}$$

So (4.23) yields $|\nabla K| \equiv 0$ identically on $M^4 \setminus K^{-1}(0)$. Since $\nabla K = 0$ on $K^{-1}(0)$, we conclude that K is a constant function on M^4. Therefore, M^4 is an isoparametric hypersurface. This completes the proof.

Our main result (Theorem 1) is proved combining Theorem 3 and Theorem 4. ∎

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