African Diaspora Journal of Mathematics
Volume 21, Number 1, pp. 57–72 (2018)
ISSN 1539-854X
www.math-res-pub.org/adjm

A Locally Asymptotically Optimal Test With Application to Financial Data

Tewfik Lounis*
Insitut Elie Cartan de Lorraine, Université de Lorraine,
Vandoeuvre-lès-Nancy, FRANCE

Joseph Ngatchou-Wandji†
Insitut Elie Cartan de Lorraine, Université de Lorraine,
Vandoeuvre-lès-Nancy, FRANCE

Abstract

A locally asymptotically optimal test is constructed for log-return processes. The behavior of the test statistic is studied under the null and under a sequence of local alternatives. A local asymptotic normality (LAN) result is previously established. Applying the test to log-return data, one rejects the hypothesis that they are independent and identically distributed (iid).

AMS Subject Classification: 62M10; 62N02; 62N03; 62P05.

Keywords: ARCH models, Black Scholes model, Brownian Motion, Contiguity, LAN, Log-returns, Samuelson Model.

1 Introduction

Several phenomena in biology, physics and finance are modelled by time series models. In order to discriminate between the possible competing models, various statistical tests are proposed. For a review, see, among others, [29], [12], [16], [13] and references therein.

Many of the proposed tests are likelihood-based and their power is generally not studied. For local alternatives contiguous to the null hypothesis, the study of the theoretical power can be possible if the LAN property is established. For details on the notions of contiguity and LAN, see for instance [8] or [19].

Since Le Cam [18], several versions of LAN have been developed, amongst which those of [30], [11], [14] and [15]. However, except perhaps the works in [2], [20], [3], [4] and [6], LAN has not been much considered in the context of non-linearity.

In [14] is studied an efficient test of linearity against contiguous alternatives. The authors study its local power after establishing a modified version of the Le Cam's LAN.

*E-mail address: tewfik.lounis@gmail.com
†E-mail address: joseph.ngatchou-wandji@univ-lorraine.fr

However, they assume the parameters of the model to be known. Their work is extended by Chebana and Laïb [7] to a larger class of non-linear models with the assumption that the parameter vector is unknown. The proposed test statistic is then based on an estimator of the parameter. Unfortunately, this complicates the study of its local asymptotic optimality and the authors do not investigate this problem. In an effort to solve it, Lounis [24] proposes a new estimator for which the asymptotic optimality of the test is obtained. His idea consists in making inference on a sub-sample of the observed sample which is assumed to be large enough. In practice, such a technique can work well in financial markets where large numbers of the data are often avalaible.

In the present work, we study a log-likelihood ratio test for checking the iid hypothesis of log-return processes. In discrete time, these processes play an important role in the statistical analysis of financial prices (see, e.g., [9]). Precisely, we aim at testing a particular and simple form of the linearity of log-return processes. In this purpose, following the idea of [24], we construct a likelihood ratio test. We study its local asymptotic power using the LAN property given in [14], also considered in [7].

The remaining of this paper is structured as follows. In Section 2, we list our assumptions. In Section 3, testing the iid hypothesis of log-return processes is studied. In Section 4, our results are applied to testing the iid hypothesis of the daily, weekly and monthly log-returns processes. The proof of the results are relegated to the Section 5.

2 Notation and general assumptions

Let $(X_i : i \in \mathbb{Z})$ be a sequence of second-order random variables solutions of the stochastic equation

$$X_i = T(\mathbf{X}_{i-1}) + V(\mathbf{X}_{i-1})\varepsilon_i,\ i \in \mathbb{Z}, \tag{2.1}$$

where $(\varepsilon_i : i \in \mathbb{Z})$ is a sequence of standard iid random variables with density $f > 0$ and for all $i \geq 1$, ε_i is independent of the σ-field $\mathcal{F}_i = \sigma(\mathbf{X}_j, j < i)$; $T(\cdot)$ and $V(\cdot)$ are real-valued functions assumed to be unkown; $\mathbf{X}_i = (X_{i-1}, \dots, X_{i-d})^\top$ with the superscript "$^\top$" denoting the transpose.

Consider the parametric class of functions

$$\mathcal{M} = \left\{ \left(m_\theta(\cdot), \sigma_\rho(\cdot) \right), (\theta, \rho)^\top \in \Theta_1 \times \Theta_2 \right\},$$

where both functions $m_\theta(\cdot)$ and $\sigma_\rho(\cdot)$ have known forms and $(\rho^\top, \theta^\top)^\top \in \Theta_1 \times \Theta_2$, with Θ_1 and Θ_2 standing for open sets of $\mathbb{R}^q$ and $\mathbb{R}^\ell$ respectively, equipped with the Euclidian norm "$\|\cdot\|$". It is further assumed that interior sets $\mathrm{int}(\Theta_1)$ and $\mathrm{int}(\Theta_2)$ of Θ_1 and Θ_2 are not empty.

A problem of interest is that of testing $H_0\,[(T(\cdot), V(\cdot)) \in \mathcal{M}]$ versus $H_1\,[(T(\cdot), V(\cdot)) \notin \mathcal{M}]$. As already observed in [25], this is equivalent to testing

$$H_0\,[T(\cdot), V(\cdot)] \quad = \quad \left(m_{\theta_0}(\cdot), \sigma_{\rho_0}(\cdot) \right),$$

against

$$H_1\,[T(\cdot), V(\cdot)] \quad \neq \quad \left[m_{\theta_0}(\cdot), \sigma_{\rho_0}(\cdot) \right],$$

for some true, but unkown $(\theta_0,\rho_0)^\top \in \Theta_1 \times \Theta_2$.

Here we are interested to the particular forms of H_1 given by

$$H_1^{(n)}[(T(\cdot),V(\cdot)] \quad = \quad \left(m_{\theta_0}(\cdot)+h n^{-\frac{1}{2}}G(\cdot),\sigma(\rho_0,\cdot)+h' n^{-\frac{1}{2}}S(\cdot)\right),$$

where $G(\cdot)$ and $S(\cdot)$ are given real-valued functions, h and h' are non-zero real numbers.

As can be seen, the $H_1^{(n)}$'s are close to H_0 for larger values of n. They belong to a large class of *local alternatives* to H_0. The advantage of considering them lies in the fact that the statistical asymptotic theory of Le Cam (see, e.g., [18, 19]) can allow for the study of the asymptotic power.

We make the following assumptions, also made in [7] :

(C.1) For all $(a,b) \in \mathbb{R}\times\mathbb{R}^*$, there exists a measurable and positive function M, and two positives real numbers γ and δ, with $\max(|a|,|b-1|) < \delta$ and $\mathbf{E}[M^{1+\gamma}(\varepsilon_d)] < \infty$, such that for all $x \in \mathbb{R}$,

$$\left|\frac{1}{f(x)}\frac{\partial^2}{\partial a^j \partial b^k}\left[\frac{1}{b}f\left(\frac{x-a}{b}\right)\right]\right| \le M(x), \;\; i,j \in \mathbb{N}, \;\; j+k=2.$$

(C.2) There exists a positive real number γ' such that

$$\max\left(\mathbf{E}\left|\frac{G(\mathbf{X}_d)}{\sigma_{\rho_0}(\mathbf{X}_d)}\right|^{2+\gamma'} \;\; ; \;\; \mathbf{E}\left|\frac{S(\mathbf{X}_d)}{\sigma_{\rho_0}(\mathbf{X}_d)}\right|^{2+\gamma'}\right) < \infty.$$

(C.3) There exists a positive real number γ'' such that for $k = 0,1$,

$$\mathbf{E}\left|M_f(\varepsilon_d)\varepsilon_d{}^k\right|^{2+\gamma''} < \infty.$$

where $M_f(x) = \dfrac{f'(x)}{f(x)}$, $x \in \mathbb{R}$.

(C.4)

- $\displaystyle\int M_f(x)f(x)dx = \int\left[\dot{M}_f(x)+M_f^2(x)f(x)\right]dx = \int\left[\dot{M}_f(x)+M_f^2(x)\right]f(x)dx = 0$
- $\displaystyle\int xM_f(x)f(x)dx = -1, \qquad \int x^2\left[\dot{M}_f(x)+M_f^2(x)\right]f(x)dx = 2$

Note that assumption **(C.1)** is also done in [14], and that assumptions **(C.3)** and **(C.4)** are satisfied by a large class of density functions, including the standard normal distribution.

We recall that if **(C.1)-(C.4)** hold, then a LAN property is etablished under H_0 (see Theorem 2.1 of [7]) and the logarithm of the likelihood ratio, Λ_n, decomposes into

$$\Lambda_n \quad = \quad \Lambda_n(\theta_0,\rho_0) = \mathcal{V}_n(\theta_0,\rho_0) - \frac{\tau^2(\theta_0,\rho_0)}{2} + o_P(1),$$

where the central sequence $(\mathcal{V}_n(\theta_0,\rho_0) : n \ge 1)$ converges in distribution to a zero-mean Gaussian random variable with variance $\tau^2(\theta_0,\rho_0)$.

In the present setting we have,

$$\begin{aligned}
\mathcal{V}_n(\theta_0,\rho_0) &= -\frac{1}{\sqrt{n}}\sum_{t=1}^{n} h\frac{G(\mathbf{X}_0)M_f(\varepsilon_t)}{\sigma_{\rho_0}(\mathbf{X}_t)} - \frac{1}{\sqrt{n}}\sum_{t=1}^{n} h'\frac{S(\mathbf{X}_0)\left(1+\varepsilon_t M_f(\varepsilon_t)\right)}{\sigma_{\rho_0}(\mathbf{X}_t)} && (2.2)\\
\tau^2(\theta_0,\rho_0) &= h^2 I_0 \mathbf{E}\left[\frac{G(\mathbf{X}_0)}{\sigma_{\rho_0}(\mathbf{X}_0)}\right]^2 + h'^2(I_2-1)\mathbf{E}\left[\frac{S(\mathbf{X}_0)}{\sigma_{\rho_0}(X_d)}\right]^2\\
&+ hh' I_1 \mathbf{E}\left[\frac{G(\mathbf{X}_0)S(\mathbf{X}_0)}{\sigma^2_{\rho_0}(\mathbf{X}_0)}\right] && (2.3)\\
I_j &= \mathbf{E}\left[\varepsilon_0^j M_f^2(\varepsilon_0)\right],\ j\in\{0,1,2\}.
\end{aligned}$$

For $\alpha \in (0,1)$, the Neyman-Pearson test we study is defined by

$$T_n = T_n(\theta_0,\rho_0) = I\left[\frac{\mathcal{V}_n(\theta_0,\rho_0)}{\tau(\theta_0,\rho_0)} \geq c_\alpha\right], \tag{2.4}$$

where c_α is the $(1-\alpha)$-quantile of the standard normal distribution.

Recall that the test consists in rejecting H_0 whenever $T_n = 1$. It is known from [14] that whenever the parameter (ρ_0,θ_0) is assumed to be known, the test T_n is locally asymptotically optimal with local asymptotic power function $1-\Phi\left[c_\alpha - \tau(\theta_0,\rho_0)\right]$, where Φ and c_α are respectively the cumulative distribution function and the $(1-\alpha)$-quantile of a standard Gaussian distribution. Also, from [7], if (ρ_0,θ_0) is no more known the optimality is not insured, as the power is affected by the substitution of the parameter for an estimator. This problem is solved in [22] and [24] by constructing a new estimator based on a discrete estimator (see [18] and [17]). It is shown there that the difference between the central sequence with the true parameter and its estimated version obtained by plugging in the new estimator, is asymptotically negligible.

3 Testing log-returns

In order to test the iid assumption of the log-returns, we apply the results of [7], [22] and [24] to the particular case where

$$m_{\theta_0}(\mathbf{X}_{i-1}) = m_{(\mu_0,\sigma_0)}(\mathbf{X}_{i-1}) = \mu_0 - (\sigma_0{}^2)/2 \quad \text{and} \quad \sigma_{\rho_0}(\mathbf{X}_{i-1}) = \sigma_0, \tag{3.1}$$

with $\theta_0 = (\mu_0,\sigma_0)$, $\rho_0 = \sigma_0 > 0$, $q = 2$ and $\ell = 1$.
We first introduce some preliminary concepts and results used to modelling prices in financial markets. Next, we present our testing problem and prove our main thoeretical results.

3.1 The log-returns process

The price of a stock (exchanged) is modelized in the thesis of Bachelier [1] by the following continuous-time stochastic model

$$\tilde{S}_t = \tilde{S}_0 + \mu_0 t + \sigma_0 B_t,$$

where μ_0 and σ_0 are two constants and $(B_t : t \geq 0)$ is a Brownian motion (BM). A more important model proposed in [27] is

$$\tilde{S}_t = \tilde{S}_0 \exp\left[\left(\mu_0 - \frac{\sigma_0{}^2}{2}\right)t + \sigma_0 B_t\right]. \tag{3.2}$$

Also, based on (3.2), and assuming that there exists some constant r such that $\tilde{S}_t = \exp(rt)$ for all $t = 0, \ldots, n$, using Itô calculus, Black and Scholes [5] derived an explicit formula for price $\tilde{S}_t$ in the pay-off of European call options.

The process $\tilde{S}_t$ driven by the BM, also called geometric Brownian motion, is the unique solution of the following stochastic differential equation

$$d\tilde{S}_t = \mu_0 \tilde{S}_t dt + \sigma_0 \tilde{S}_t dB_t, \tag{3.3}$$

where μ_0 and σ_0 are respectively the drift and the volatility.

Under the conditions of [5], taking the logarithms of both sides of (3.2) gives

$$\log(\tilde{S}_t) = \log(\tilde{S}_0) + \left(\mu_0 - \frac{\sigma_0{}^2}{2}\right)t + \sigma_0 B_t, \;\; t = 1, \ldots, n. \tag{3.4}$$

From (3.4), one finds

$$\log(\tilde{S}_t/\tilde{S}_{t-1}) = \mu_0 - \frac{\sigma_0{}^2}{2} + \sigma_0 W_t, \;\; t = 1 \ldots, n, \tag{3.5}$$

where the $W_t = B_t - B_{t-1}$'s are the zero-mean iid Gaussian increments of the BM.

The process $\left(X_t = \log(\tilde{S}_t/\tilde{S}_{t-1}) : t \geq 0\right)$ is called log-returns discrete time process. In the past, there has been a great interest in estimating the parameters of this model. This estimation is often based on the historical data (see [26]). A natural estimator for (μ_0, σ_0^2) is (μ_n, σ_n^2), with

$$\begin{aligned}
\mu_n &= \bar{X}_n + \sigma_n^2/2 && (3.6)\\
\sigma_n^2 &= \frac{1}{n}\sum_{t=1}^{n}[log(\tilde{S}_t/\tilde{S}_{t-1}) - \bar{X}_n]^2 && (3.7)\\
\bar{X}_n &= \frac{1}{n}\sum_{t=1}^{n} log\left(\tilde{S}_t/\tilde{S}_{t-1}\right) = \frac{1}{n}\sum_{t=1}^{n} X_t.
\end{aligned}$$

Note that both μ_n and σ_n^2 converge almost surely to μ_0 and σ_0^2 respectively.

3.2 Testing the iid property of a log-returns process

From Wiener's (or BM's) processes properties, one easily sees that under H_0, the log-returns are iid. With the above definition, rewrite (3.5) as

$$X_t = \mu_0 - \frac{\sigma_0{}^2}{2} + \sigma_0 W_t, \;\; t = 1, \ldots n. \tag{3.8}$$

Consider the alternative models defined by

$$X_t = \mu_0 - \frac{{\sigma_0}^2}{2} + \frac{hG(\mathbf{X}_t)}{\sqrt{n}} + \left[\sigma_0 + \frac{h' S(\mathbf{X}_t)}{\sqrt{n}}\right] W_t, \;\; t = 1, \ldots n. \tag{3.9}$$

Note that in the last two models, (W_t) plays the role of (ε_t) in the models of the preceding sections. Since it represents the sequence of the increments of the BM which are stationary, independent and zero-mean normally distributed with variance 1, the models (3.8) and 3.9) are particular hypotheses of those presented in Section 2, with $\varepsilon_t \sim \mathcal{N}(0,1)$. Then, testing H_0 versus H_1^n where

$$H_0 : (h, h') = (0,0) \quad \text{and} \quad H_1^{(n)} : (h, h') \neq (0,0) \tag{3.10}$$

can be understood as testing the iid property of the X_i's given by (3.8), against some possible and particular close form of non iid property given for instance by (3.9).

We define under H_0 the process of residuals by

$$W_t(\mu_0, \sigma_0) = \frac{1}{\sigma_0}\left[X_t - \left(\mu_0 - \frac{{\sigma_0}^2}{2}\right)\right], \; t \in \mathbb{Z}. \tag{3.11}$$

From (3.6) and (3.7), a natural estimator $\widehat{W}_t$ of $W_t(\mu_0, \sigma_0)$ is

$$\widehat{W}_t = W_t(\mu_n, \sigma_n) = \frac{1}{\sigma_n}\left[X_t - \left(\mu_n - \frac{{\sigma^2}_n}{2}\right)\right], \; t \in \mathbb{Z}. \tag{3.12}$$

For the study of the testing problem (3.10), we use the test (2.4) based on the Neyman-Pearon statistic and described by the central sequence (2.2) with the estimators given by (3.6), (3.7) and (3.12).

3.3 Main results

In order to present the theoretical results derived for this particular testing problem. We firstly remind some small results of independent interest on the stationarity and the ergodicity of the model (3.8). Secondly, we give an explicit form for the central sequence $\mathcal{V}_n$. Finally, we state our main results. In a sequel, it assumed that $\sigma_0^2 < +\infty$.

Observe that the equality (3.8) can be rewritten as

$$\begin{aligned} X_t &= \mu_0 - \frac{\sigma_0^2}{2} + Z_t \\ Z_t &= \sigma_0 W_t, \;\; t = 1, \ldots n. \end{aligned} \tag{3.13}$$

From Weiner's processes properties, it is immediate that $\{Z_t : t = 1, \ldots n\}$ is an iid zero-mean Gaussian process with unit variance. It is then an easy matter to see that $\{X_t : t = 1, \ldots n\}$ is an iid Gaussian process with mean $\mu_0 - \sigma_0^2/2$ and finite variance σ_0^2. Consequently, the process $(X_t : t \in \mathbb{Z})$ solution of (3.8) is strictly stationary and ergodic.

Under H_0, assuming that **(C.1)-(C.4)** are satisfied, the expression of the central sequence $\mathcal{V}_n$ was given in (2.2). Since $(W_t : t = 1,\ldots,n)$ is a sequence of zero-mean Gaussian random variables with unit variance, the density function of the W_t's is that of a standard Gaussian distribution. Hence, for all real number x, $f'(x) = -xf(x)$, from which it is easy to check that $M_f(W_i) = -W_i$. With this, the central sequence can be rewritten as

$$\mathcal{V}_n(\mu_0,\sigma_0) = \frac{1}{\sqrt{n}}\sum_{t=1}^{n} h\frac{G(\mathbf{X}_t)W_t}{\sigma_0} - \frac{1}{\sqrt{n}}\sum_{t=1}^{n} h'\frac{S(\mathbf{X}_t)\left(1-W_t^2\right)}{\sigma_0}. \tag{3.14}$$

To state our results, we make the following additional assumptions

(C.5) $0 < {\sigma^2}_n < +\infty$.

(C.6) $\max\{\mathbf{E}(X_0), \mathbf{E}[X_0 G(X_0)], \mathbf{E}[X_0 S(X_0)]\} < \infty$.

Remark 3.1. Assumption **(C.5)** generally holds. Indeed, if $\sigma_n^2 = 0$ from (3.7) one has that for each $t = 1,\ldots,n$, $\log(\tilde{S}_t/\tilde{S}_{t-1}) = \frac{1}{n}\sum_{t=1}^{n}\log\left(\tilde{S}_t/\tilde{S}_{t-1}\right)$. Since $\sigma > 0$, this contradicts the fact that (W_t) is a Gaussian process.

Since the second-order moment are assumed to be finite, assumption **(C.6)** is readily satisfied by a large class of functions G and S including bounded functions as those considered in Section 4.

Proposition 3.2. *Assume that **(C.1)-(C.6)** are satisfied. Then, under H_0,*

$$\mathcal{V}_n(\mu_0,\sigma_0) = \mathcal{V}_n(\mu_n,\sigma_n) + D_n(\mu_0,\sigma_0,\mu_n,\sigma_n) + o_P(1) \tag{3.15}$$

$$\begin{aligned}
D_n(\mu_0,\sigma_0,\mu_n,\sigma_n) &= (\mu_0-\mu_n)\left[\frac{1}{\sqrt{n}}\sum_{t=1}^{n} h d_1(\mu_n,\sigma_n)G(\boldsymbol{X}_t) - \frac{1}{\sqrt{n}}\sum_{t=1}^{n} h' d_5(\mu_n,\sigma_n,X_t)S(\boldsymbol{X}_t)\right]\\
&+(\sigma_0-\sigma_n)\left[\frac{1}{\sqrt{n}}\sum_{t=1}^{n} h d_2(\mu_n,\sigma_n,X_t)G(\boldsymbol{X}_t) - \frac{1}{\sqrt{n}}\sum_{t=1}^{n} h' d_7(\mu_n,\sigma_n,X_t)S(\boldsymbol{X}_t)\right]\\
d_1(\mu_n,\sigma_n) &= -\frac{1}{{\sigma_n}^2}\\
d_2(\mu_n,\sigma_n,X_t) &= \frac{-2\mu_n-2X_t}{{\sigma^3}_n}\\
d_5(\mu_n,\sigma_n,X_t) &= \frac{1}{\sigma_n} - \frac{2\mu_n}{{\sigma^3}_n} + \frac{2X_t}{{\sigma^3}_n}\\
d_7(\mu_n,\sigma_n,X_t) &= -\frac{1}{4} - \frac{\mu_n}{\sigma_n^2} + \frac{X_t}{\sigma_n^2} + \frac{3{\mu_n}^2}{\sigma_n^4} - \frac{1}{\sigma_n^2} - \frac{\sigma_n\mu_n}{\sigma_n^3}.
\end{aligned} \tag{3.16}$$

Proof. The proof of this result and the next are given in Section 6.

The random variable $D_n(\mu_0,\sigma_0,\mu_n,\sigma_n)$ defined in Proposition (3.2) depends on the unknown (μ_0,σ_0). One would like to substitute this parameter for an appropriate known $(\bar{\mu}_n,\bar{\sigma}_n)$ such that $D_n(\mu_0,\sigma_0,\mu_n,\sigma_n) - D_n(\bar{\mu}_n,\bar{\sigma}_n,\mu_n,\sigma_n) = o_P(1)$. This is possible using the following result from [23].

Proposition 3.3. *Let μ be a positive real number. For all integer n, let $N = [1+n^{\mu+1}]$. Then*

$$\sqrt{n}(\theta_n-\theta) = \sqrt{n}(\theta_n-\theta_N) + o_P(1).$$

Proposition 3.4. *Assume that **(C.1)-(C.6)** hold. Then, under H_0,*

$$\begin{aligned}
&\mathcal{V}_n(\mu_0,\sigma_0) = \mathcal{V}_n(\mu_n,\sigma_n) + D_n(\mu_N,\sigma_N,\mu_n,\sigma_n) + o_P(1), \qquad (3.17)\\
&D_n(\mu_N,\sigma_N,\mu_n,\sigma_n) =\\
&(\mu_N-\mu_n)\left[\frac{1}{\sqrt{n}}\sum_{t=1}^{n} hd_1(\mu_n,\sigma_n)G(\boldsymbol{X}_t) - \frac{1}{\sqrt{n}}\sum_{t=1}^{n} h' d_5(\mu_n,\sigma_n,X_t)S(\boldsymbol{X}_t)\right]\\
&+(\sigma_N-\sigma_n)\left[\frac{1}{\sqrt{n}}\sum_{t=1}^{n} hd_2(\mu_n,\sigma_n,X_t)G(\boldsymbol{X}_t) - \frac{1}{\sqrt{n}}\sum_{t=1}^{n} h' d_7(\mu_n,\sigma_n,X_t)S(\boldsymbol{X}_t)\right].
\end{aligned}$$

Now we are ready to state our fundamental result.

Proposition 3.5. *Assume that **(C.1)-(C.6)** hold. Then, under H_0, there exists an estimator $\bar{\theta}_n = (\bar{\mu}_n,\bar{\sigma}_n)$ of $\theta_0 = (\mu_0,\sigma_0)$ such that*

$$\mathcal{V}_n(\mu_0,\sigma_0) \quad = \quad \mathcal{V}_n(\bar{\mu}_n;\bar{\sigma}_n) + o_P(1). \qquad (3.18)$$

The constructed estimator $(\bar{\mu}_n,\bar{\sigma}_n)$ enables obtaining the equivalence between the central sequence and its estimated version. The following theorem shows that this insures the optimality of our test.

Theorem 3.6. *Assume that **(C.1)-(C.6)** hold. Then, under H_1^n, the local asymptotic power of the test $\bar{T}_n = T_n(\bar{\mu}_n,\bar{\sigma}_n)$ is $1-\Phi[c_\alpha - \tau(\mu,\sigma)]$, Φ and c_α are respectively the cumulative distribution function and the $(1-\alpha)$-quantile of the standard Gaussian distribution. This test is locally asymptotically optimal.*

4 Application to financial data

In this section, we apply our theoretical results to testing the iid hypothesis of the daily, weekly and monthly log-returns. Our numerical results are listed in the tables below.

The data avalaible are from Forex Capital Market France (FXCM). They contain daily returns of length 500 collected from 11th may 1993 to 7th november 1995. They also contain weekly returns of lengths 500 collected from may 1993 to november 2002 and monthly returns of lengths 263 collected from 1rst may 1993 to 1rst march 2015.

We use the test statistic $\bar{T}_n$ with nominal level $\alpha = 0.05$. For the daily data, we take $\lambda = 0.07$. and $N = 140, 154, 169, 184, 214, 291, 369$ corresponding to sub-samples of lengths $n = 100, 110, 120, 130, 150, 200, 250$ from the latest consecutive indexes in the original samples. We consider $S(\mathbf{X}_i) = \sin(X_{i-1})/500$, for $i = 2,\ldots,N$ and $S(X_1) = 0$, $G(\mathbf{X}_i) = 0$ and $h = h' = 1$. It is easy to check that $\max\left[\left|\frac{G(\mathbf{X}_d)}{\sigma_{\rho_0}(\mathbf{X}_d)}\right|;\left|\frac{S(\mathbf{X}_d)}{\sigma_{\rho_0}(\mathbf{X}_d)}\right|\right] < \infty$ and then assumption **(C.2)** is satisfied. The estimators μ_n and σ_n can be derived easily from (3.6) and (3.7). In this setting and from the Proposition 3.4, we have

$$D_n(\mu_N,\sigma_N,\mu_n,\sigma_n) = (\mu_N-\mu_n)\left[-\frac{1}{\sqrt{n}}\sum_{t=2}^{n}\left\{\frac{1}{\sigma_n}-\frac{2\mu_n}{\sigma^3{}_n}+\frac{2X_t}{\sigma^3{}_n}\right\}\frac{\sin(X_{t-1})}{500}\right] \tag{4.1}$$

$$+ (\sigma_N-\sigma_n)\left[-\frac{1}{\sqrt{n}}\sum_{t=2}^{n}\left\{-\frac{1}{4}-\frac{\mu_n}{\sigma_n^2}+\frac{X_t}{\sigma_n^2}+\frac{3\mu_n{}^2}{\sigma_n^4}-\frac{1}{\sigma_n^2}-\frac{\sigma_n\mu_n}{\sigma_n^3}\right\}\frac{\sin(X_{t-1})}{500}\right]$$

$$\mathcal{V}_n(\mu_0,\sigma_0) = -\frac{1}{\sqrt{n}}\sum_{t=2}^{n}\frac{\left(1-W_t^2\right)}{\sigma_0}\frac{\sin(X_{t-1})}{500}. \tag{4.2}$$

Then, we obtain

$$\frac{\partial\mathcal{V}_n(\mu_n,\sigma_n)}{\partial\mu} = \frac{2}{\sqrt{n}}\sum_{t=2}^{n}\frac{W_t}{\sigma_n^2}\frac{\sin(X_{t-1})}{500}, \tag{4.3}$$

with $\bar{\sigma}_n=\sigma_n$, $\bar{\mu}_n=\mu_n+x(n)$, where $x(n)$ defined in (5.19) is computed from (4.1) and (4.3). Note that by the ergodicity and the stationarity of model (3.8), $\tau^2(\mu_0,\sigma_0)$ defined in (2.3) can almost surely be approximated by

$$\mathcal{U}_{n,h,h'}(\mu_0,\sigma_0) = \frac{1}{n}\sum_{t=2}^{n}\frac{\left(1-W_t^2\right)^2}{\sigma_0^2}\left(\frac{\sin(X_{t-1})}{500}\right)^2. \tag{4.4}$$

For more details about this, see for instance the proof of Theorem 1 of [14]. Then using the expression of T_n given by (2.4), the ratio $\mathcal{V}_n(\theta_0,\rho_0)/\tau(\theta_0,\rho_0)$ can be approximated by

$$B_n = -\frac{\sum_{t=2}^{n}(1-W_t^2)\,[\sin(X_{t-1})/500]}{\sqrt{\sum_{t=2}^{n}(1-W_t^2)^2\,[\sin(X_{t-1})/500]^2}}. \tag{4.5}$$

Substuting (μ_0,σ_0) for $(\bar{\mu}_n,\bar{\sigma}_n)$, one gets $T(\bar{\mu}_n,\bar{\sigma}_n)$. Then, with our computations, our test rejects the null hypothesis of the iid property of the daily log-returns in either case.

Proceeding the same way with weekly and monthly log-returns leads to the same conclusion. Note that for the monthly log-returns, since the sample size is smaller than the others, we considered N = 154, 169, 184, 214, 229, 245 and the corresponding n = 110, 120, 130, 150, 160, 170.

The above results suggest that the following model

$$X_t=\mu_n-\frac{\sigma_n^2}{2}+\left[\sigma_n+\frac{\sin(X_{t-1})}{500\sqrt{n}}\right]\varepsilon_t,\ t=1,\ldots,n, \tag{4.6}$$

where (ε_t) is a sequence of iid standard Gaussian random variables, is a possible model that can be fitted to the tree log-returns data studied.

Lounis [23] studied the cases $S(x)=G(x)=0.008$ and $S(x)=G(x)=0.05$ and concluded that the alternative models were preferable to the null model. This means that (4.6) is not the only model that can be ajusted to our log-returns data. However, looking for the more suitable of all the candidate models is a work beyond the scope of this paper.

Daily Log-returns

n	N	Data size	$\bar{T}_n$	Decision	Power
100	139	500	10.0420	H_1^n	0.8289439
110	154	500	10.4963	H_1^n	0.8289439
120	169	500	10.9636	H_1^n	0.8289439
130	184	500	11.4021	H_1^n	0.8289439
150	214	500	12.2469	H_1^n	0.8289439
200	291	500	14.2137	H_1^n	0.8289439
250	369	500	15.8113	H_1^n	0.8289439

Weekly Log-returns

n	N	Data size	$\bar{T}_n$	Decision	Power
100	139	500	10.0000	H_1^n	0.8289439
110	154	500	10.4881	H_1^n	0.8289185
120	169	500	14.5401	H_1^n	0.8289185
130	184	500	11.4018	H_1^n	0.8289185
150	214	500	12.2473	H_1^n	0.8289185
200	291	500	14.1430	H_1^n	0.8289185
250	369	500	15.8111	H_1^n	0.8289185

Monthly Log-returns

n	N	Data size	$\bar{T}_n$	Decision	Power
100	139	263	10.9548	H_1^n	0.8289185
110	154	263	10.9714	H_1^n	0.8289185
120	169	263	12.4795	H_1^n	0.8289185
130	184	263	11.0653	H_1^n	0.8289185
150	214	263	13.1369	H_1^n	0.8289185
160	229	263	14.8550	H_1^n	0.8289185
170	245	263	16.9266	H_1^n	0.8289185

5 Proof of the results

5.1 Proof of Proposition 3.2

By simple computations, one shows that

$$\frac{\partial\left(\frac{W_t}{\sigma_0}\right)}{\partial\mu} = -\frac{1}{{\sigma_0}^2} = d_1(\mu_0,\sigma_0), \tag{5.1}$$

$$\frac{\partial\left(\frac{W_t}{\sigma_0}\right)}{\partial\sigma} = \frac{-2\mu_0 - 2X_t}{\sigma^3{}_0} = d_2(\mu_0,\sigma_0,X_t), \tag{5.2}$$

$$\frac{\partial^2\left(\frac{W_t}{\sigma_0}\right)}{\partial\mu^2} = 0, \tag{5.3}$$

$$\frac{\partial^2\left(\frac{W_t}{\sigma_0}\right)}{\partial\sigma^2} = \frac{2X_t-\mu_0}{{\sigma_0}^3} = d_3(\mu_0,\sigma_0), \tag{5.4}$$

$$\frac{\partial^2\left(\frac{W_t}{\sigma_0}\right)}{\partial\sigma\partial\mu} = \frac{\partial^2\left(\frac{W_t}{\sigma_0}\right)}{\partial\mu\partial\sigma} = \frac{2}{{\sigma_0}^3} = d_4(\mu_0,\sigma_0), \tag{5.5}$$

$$\frac{\partial\left(\frac{1-W^2{}_t}{\sigma_0}\right)}{\partial\mu} = \frac{1}{\sigma_0} - \frac{2\mu_0}{\sigma^3{}_0} + \frac{2X_t}{\sigma^3{}_0} = d_5(\mu_0,\sigma_0,X_t), \tag{5.6}$$

$$\frac{\partial^2\left(\frac{1-W^2{}_t}{\sigma_0}\right)}{\partial\mu^2} = \frac{-2}{\sigma^3{}_0} = d_6(\mu_0,\sigma_0), \tag{5.7}$$

$$\frac{\partial\left(\frac{1-W^2{}_t}{\sigma_0}\right)}{\partial\sigma} = -\frac{1}{4} - \frac{\mu_0}{\sigma_0^2} + \frac{X_t}{\sigma_0^2} + \frac{3{\mu_0}^2}{\sigma_0^4} - \frac{1}{\sigma_0^2} - \frac{\sigma_0\mu_0}{\sigma_0^3} = d_7(\mu_0,\sigma_0,X_t), \tag{5.8}$$

$$\frac{\partial^2\left(\frac{1-W^2{}_t}{\sigma_0}\right)}{\partial\sigma^2} = -\frac{1}{\sigma_0^2} + \frac{6\mu_0}{\sigma_0^4} - \frac{6X_t}{\sigma_0^4} = d_8(\mu_0,\sigma_0,X_t), \tag{5.9}$$

$$\frac{\partial^2\left(\frac{1-W^2{}_t}{\sigma_0}\right)}{\partial\mu\partial\sigma} = \frac{\partial^2\left(\frac{1-W^2{}_t}{\sigma_0}\right)}{\partial\sigma\partial\mu} = -\frac{1}{\sigma_0^2} + \frac{6\mu_0}{\sigma_0^4} - \frac{6X_t}{\sigma_0^4} = d_9(\mu_0,\sigma_0,X_t). \tag{5.10}$$

By a first-order Taylor expansion of $\mathcal{V}_n$ around (μ_n,σ_n), one finds,

$$\begin{aligned}
\mathcal{V}_n(\mu_0,\sigma_0) &= \mathcal{V}_n(\mu_n,\sigma_n) + D_n(\mu_0,\sigma_0,\mu_n,\sigma_n) + R_n \\
D_n(\mu_0,\sigma_0,\mu_n,\sigma_n) &= \frac{\partial\mathcal{V}_n(\mu_n,\sigma_n)}{\partial\mu}(\mu_0-\mu_n) + \frac{\partial\mathcal{V}_n(\mu_n,\sigma_n)}{\partial\sigma}(\sigma_0-\sigma_n),
\end{aligned} \tag{5.11}$$

$$\begin{aligned}
R_n &= \frac{\partial^2\mathcal{V}_n(\tilde{\mu}_n,\tilde{\sigma}_n)}{\partial\mu^2}(\mu_0-\mu_n)^2 + \frac{\partial^2\mathcal{V}_n(\tilde{\mu}_n,\tilde{\sigma}_n)}{\partial\sigma^2}(\sigma_0-\sigma_n)^2 \\
&+ \frac{1}{2}\frac{\partial^2\mathcal{V}_n(\tilde{\mu}_n,\tilde{\sigma}_n)}{\partial\mu\partial\sigma}(\mu_0-\mu_n)\times(\sigma_0-\sigma_n) \\
&+ \frac{1}{2}\frac{\partial^2\mathcal{V}_n(\tilde{\mu}_n,\tilde{\sigma}_n)}{\partial\sigma\partial\mu}(\mu_0-\mu_n)\times(\sigma_0-\sigma_n),
\end{aligned} \tag{5.12}$$

where $\tilde{\mu} \in [\mu_0;\mu_n]$ and $\tilde{\sigma} \in [\sigma_0;\sigma_n]$.

Using the expression of central sequence given by (3.14) and the equalities (5.1), (5.2), (5.6), (5.8) and (5.11), one obtains

$$\begin{aligned}
D_n(\mu_0,\sigma_0,\mu_n,\sigma_n) &= \left[\frac{1}{\sqrt{n}}\sum_{t=1}^{n} hd_1(\mu_n,\sigma_n)G(\mathbf{X}_t) - \frac{1}{\sqrt{n}}\sum_{t=1}^{n} h'd_5(\mu_n,\sigma_n,X_t)S(\mathbf{X}_t)\right]\times(\mu_0-\mu_n) \\
&+ \left[\frac{1}{\sqrt{n}}\sum_{t=1}^{n} hd_2(\mu_n,\sigma_n,X_t)G(\mathbf{X}_t) - \frac{1}{\sqrt{n}}\sum_{t=1}^{n} h'd_7(\mu_n,\sigma_n,X_t)S(\mathbf{X}_t)\right]\times(\sigma_0-\sigma_n).
\end{aligned}$$

Also, using the equalities (5.3), (5.4), (5.5), (5.7), (5.9), (5.10) and (5.12), it follows that

$$\begin{aligned}
R_n &= \frac{1}{2}\left[-\frac{1}{\sqrt{n}}\sum_{t=1}^{n} h' d_6(\tilde{\mu}_n,\tilde{\sigma}_n) S(\mathbf{X}_t)\right]\times(\mu_0-\mu_n)^2 \\
&+ \frac{1}{2}\left[\frac{1}{\sqrt{n}}\sum_{t=1}^{n} h d_3(\tilde{\mu}_n,\tilde{\sigma}_n) G(\mathbf{X}_t) - \frac{1}{\sqrt{n}}\sum_{t=1}^{n} h' d_8(\tilde{\mu}_n,\tilde{\sigma}_n,X_t) S(\mathbf{X}_t)\right]\times(\sigma_0-\sigma_n)^2 \\
&+ \left[\frac{1}{\sqrt{n}}\sum_{t=1}^{n} h d_9(\tilde{\mu}_n,\tilde{\sigma}_n,X_t) G(\mathbf{X}_t) - \frac{1}{\sqrt{n}}\sum_{t=1}^{n} h' d_9(\tilde{\mu}_n,\tilde{\sigma}_n,X_t) S(\mathbf{X}_t)\right]\times(\mu_0-\mu_n)(\sigma_0-\sigma_n) \\
&=: R_{n,1}+R_{n,2}+R_{n,3}. \qquad (5.13)
\end{aligned}$$

Now, we have to show that $R_n = o_P(1)$. In this purpose, we study the asymptotic behavior of $R_{n,1}$, $R_{n,2}$ and $R_{n,3}$. Starting with that of $R_{n,1}$, one can write

$$\begin{aligned}
R_{n,1} &= \frac{1}{2\tilde{\sigma}_n^3}\left[-\frac{1}{\sqrt{n}}\sum_{t=1}^{n} h' S(\mathbf{X}_t)\right]\times(\mu_0-\mu_n)^2, \\
&= \frac{1}{2\tilde{\sigma}_n^3}\left[-\frac{1}{n}\sum_{t=1}^{n} h' S(\mathbf{X}_t)\right]\times\left(\sqrt{n}(\mu_0-\mu_n)\right)^2\times\frac{1}{\sqrt{n}}.
\end{aligned}$$

Since $\tilde{\sigma}_n \in [\sigma_0;\sigma_n]$, by assumption **(C.5)** one has $0<\tilde{\sigma}_n<\infty$. Thus, $1/\tilde{\sigma}_n^3$ is finite. By **(C6)** and making use of the ergodic theorem one has that $\frac{1}{n}\sum_{t=1}^{n} h' S(\mathbf{X}_t)$ converges almost surely to some finite constant K. The fact that $\sqrt{n}(\mu_0-\mu_n) = O_P(1)$ implies that $\left[\sqrt{n}(\mu_0-\mu_n)\right]^2 = O_P(1)$. It follows that

$$R_{n,1} = o_P(1). \qquad (5.14)$$

By a similar reasoning to that of the proof of (5.15), it easy to check that under the same assumptions, $R_{n,2} = o_P(1)$ and $R_{n,3} = o_P(1)$. In summary

$$R_n = o_P(1).$$

5.2 Proof of Proposition 3.4

From equality (3.15) one has

$$\begin{aligned}
D_n(\mu_0,\sigma_0,\mu_n,\sigma_n) &= \left[\frac{1}{\sqrt{n}}\sum_{t=1}^{n} h d_1(\mu_n,\sigma_n) G(\mathbf{X}_t) - \frac{1}{\sqrt{n}}\sum_{t=1}^{n} h' d_5(\mu_n,\sigma_n,X_t) S(\mathbf{X}_t)\right]\times(\mu_0-\mu_n) \\
&+ \left[\frac{1}{\sqrt{n}}\sum_{t=1}^{n} h d_2(\mu_n,\sigma_n,X_t) G(\mathbf{X}_t) - \frac{1}{\sqrt{n}}\sum_{t=1}^{n} h' d_7(\mu_n,\sigma_n,X_t) S(\mathbf{X}_t)\right]\times(\sigma_0-\sigma_n).
\end{aligned}$$

Proceeding as in the proof of (5.15) below, we show the differences

$$\frac{1}{\sqrt{n}}\sum_{t=1}^{n} h d_1(\mu_n,\sigma_n) G(\mathbf{X}_t) - \frac{1}{\sqrt{n}}\sum_{t=1}^{n} h' d_5(\mu_n,\sigma_n,X_t) S(\mathbf{X}_t)$$

$$\frac{1}{\sqrt{n}}\sum_{t=1}^{n} hd_2(\mu_n,\sigma_n,X_t)G(\mathbf{X}_t) - \frac{1}{\sqrt{n}}\sum_{t=1}^{n} h' d_7(\mu_n,\sigma_n,X_t)S(\mathbf{X}_t)$$

are $O_p(1)$'s. Then, applying Proposition 3.3, one substitutes $\sqrt{n}(\mu_0-\mu_n)$ and $\sqrt{n}(\sigma_0-\sigma_n)$ respectively for $\sqrt{n}(\mu_N-\mu_N N)+o_P(1)$ and $\sqrt{n}(\sigma_0-\sigma_n)+o_P(1)$. This establishes the Proposition.

5.3 Proof of Proposition 3.5

Consider again

$$\mathcal{V}_n(\mu_0,\sigma_0) \quad = \quad \mathcal{V}_n(\mu_n,\sigma_n)+D_n(\mu_n,\sigma_n,\mu_n,\sigma_n)+o_P(1), \tag{5.15}$$

where $D_n(\mu_n,\sigma_n,\mu_n,\sigma_n)$ is defined in Proposition 3.4.
The tangent space Γ of the map $\mathcal{V}_n$ at (μ_n,σ_n) is described as

$$\begin{aligned}\Gamma \quad &: \quad \Big\{(X,Y)\in\mathbf{R}\times\mathbf{R}, \text{ such that,}\\ \mathcal{V}_n(X,Y)-\mathcal{V}_n(\mu_n,\sigma_n) \quad &= \quad \partial\mathcal{V}_n(\mu_n,\sigma_n).((X-\mu_n),(Y-\sigma_n))\Big\}.\end{aligned} \tag{5.16}$$

For all $(X,Y)\in\Gamma$, one also has

$$\begin{aligned}\mathcal{V}_n(X,Y)-\mathcal{V}_n(\mu_n,\sigma_n) \quad = \quad & \frac{\partial\mathcal{V}_n(\mu_n,\sigma_n)}{\partial\mu}(X-\mu_n)\\ &+\frac{\partial\mathcal{V}_n(\mu_n,\sigma_n)}{\partial\sigma}(Y-\sigma_n).\end{aligned} \tag{5.17}$$

The derivation of the estimator $(\bar{\mu}_n,\bar{\sigma}_n)$ is done by a perturbation one of the components (μ_n,σ_n). For instance, a perturbation of the first component yields

$$\begin{aligned}\bar{\mu}_n \quad &= \quad \mu_n+x(n)\\ \bar{\sigma}_n \quad &= \quad \sigma_n.\end{aligned}$$

Plugging them into (5.17) yields

$$\mathcal{V}_n(\bar{\mu}_n,\bar{\sigma}_n)-\mathcal{V}_n(\mu_n,\sigma_n)=\frac{\partial\mathcal{V}_n(\mu_n,\sigma_n)}{\partial\mu}\times x(n), \tag{5.18}$$

where $x(n)$ can be computed from the equation

$$\frac{\partial\mathcal{V}_n(\mu_n,\sigma_n)}{\partial\mu}\times x(n)=D_n(\mu_N,\sigma_N,\mu_n,\sigma_n).$$

Solving this equation, one finds

$$x(n)=D_n(\mu_N,\sigma_N,\mu_n,\sigma_n)\Big/\left(\frac{\partial\mathcal{V}_n(\mu_n,\sigma_n)}{\partial\mu}\right). \tag{5.19}$$

The equality (5.18) is equivalent to

$$\mathcal{V}_n(\bar{\mu}_n,\bar{\sigma}_n)=\mathcal{V}_n(\mu_n,\sigma_n)+D_n(\mu_0,\sigma_0,\mu_n,\sigma_n). \tag{5.20}$$

Combining (5.20) with (5.15) gives

$$\mathcal{V}_n(\mu_0,\sigma_0) = \mathcal{V}_n(\bar{\mu}_n,\bar{\sigma}_n) + o_P(1), \tag{5.21}$$

where $\bar{\mu}_n = \mu_n + x(n)$ and $\bar{\sigma}_n = \sigma_n$.

The equality (5.21) establishes the equivalence between the central sequence and its estimated version. Another estimator ensuring this equivalence can be obtained similarly with the second component of (μ_n,σ_n). The reader can refer to [21] for more details.

5.4 Proof of Theorem 3.6

By the continuity of the function $(x,y) \mapsto \tau(x,y)$ and the convergence in probability of $(\bar{\mu}_n,\bar{\sigma}_n)$ to (μ_0,σ_0), using Proposition 3.5, one obtains that under H_0, the tests based respectively on

$$\bar{T}_n = I\Big[\frac{\mathcal{V}_n(\bar{\mu}_n,\bar{\sigma}_n)}{\tau(\bar{\mu}_n,\bar{\sigma}_n)} \geq c_\alpha\Big] \quad \text{and} \quad T_n = I\Big[\frac{\mathcal{V}_n(\mu_0,\sigma_0)}{\tau(\mu_0,\sigma_0)} \geq c_\alpha\Big]$$

are asymptotically equivalent. Consequently, under $H_1^{(n)}$ they have the same asymptotic power. Since the one based on T_n is optimal, so is $\bar{T}_n$.

Now, using the contiguity property and applying the Le Cam's third Lemma, one has that under $H_1^{(n)}$,

$$\mathcal{V}_n(\theta_0,\rho_0) \overset{\mathcal{D}}{\hookrightarrow} \mathcal{N}\Big(\tau^2(\mu_0,\sigma_0),\tau^2(\mu_0,\sigma_0)\Big).$$

It results from this that under $H_1^{(n)}$, the asymptotic power is

$$1-\Phi\Big[c_\alpha-\tau(\mu_0,\sigma_0)\Big].$$

References

[1] L. Bachelier (1900). Théorie de la spéculation. *Ann. Sci. Ecole Norm. Sup.***17**, 21–86.

[2] Y. Benghabrit and M. Hallin (1992). Optimal rank-based tests for against first-order superdiagonal bilinear dependence against diagonal bilinear dependence. *J. Statist. Plann. Inference*, **32**(1), 45–61.

[3] Y. Benghabrit and M. Hallin (1996). Locally asymptotically optimal tests for autoregressive against bilinear serial dependence. *Statist. Sinica*, **6**(1), 147–169.

[4] Y. Benghabrit and M. Hallin (1998). Locally asymptotically optimal tests for AR(p) against diagonal bilinear dependence. *J. Statist. Plann. Inference*, **68**(1), 47–63.

[5] F. Black and M. Scholes (1973). The pricing of options and corporate liabilities. *J. Polit. Econom.* **81**, 637–659.

[6] D. Cassart, M. Hallin and D. Paindaveine (2001). Optimal detection of Fechner-asymmetry. *J. Statist. Plann. Inference.*, **138**(8), 2499–2525.

[7] F. Chebana and N. Laïb (2008). A locally asymptotically powerful test for nonlinear autoregressive models. *C. R. Math. Acad. Sci. Paris*, **346**(11-12), 671–676.

[8] J. -J. Droesbeke and J. Fine (1996). *Inférence non paramétrique. Les statistiques de rangs*. Editions de l'Université de Bruxelles.

[9] E. Fama and K. French (1988a) Permanent and Temporary Components of Stock Prices. Journal of Political Economy. **96**, 246–273.

[10] E. Fama and K. French (1988b). Dividend Yields and Expected Stock Returns. Journal of Financial Economics. **22**, 3–27.

[11] W. J. Hall and D. J. Mathiason (1990). On large-sample estimation and testing in parametric models.. *Econometric Theory.*, **58**(1), 77-97.

[12] M. J. Hinich (1982). Testing for Gaussianity and linearity of a stationary time series. *J. Time Ser. Anal.*, **3**(3), 169–176.

[13] V. Hjellvik and D. Tjøstheim (1995). Nonparametric tests of linearity for time series. *Biometrika*, **82**(2), 351–368.

[14] S. Y. Hwang and I. V. Basawa (2001). Nonlinear time series contiguous to AR(1) processes and a related efficient test for linearity. *Statist. Probab. Lett.*, **52**(4), 381–390.

[15] S. Y. Hwang and I. V. Basawa (2003). Estimation for nonlinear autoregressive models generated by beta-ARCH processes. *Sankhyā*, **65**(4), 744–762.

[16] D. M. Keenan (1985). A Tukey nonadditivity-type test for time series nonlinearity. *Biometrika*, **72**(1), 39–44.

[17] J. P. Kreiss (1987). On adaptive estimation in stationary ARMA processes. *The Annals of Statistics.*, **15**(1), 112–133.

[18] L. Le Cam (1960). Locally asymptotically normal families of distributions. Certain approximations to families of distributions and their use in the theory of estimation and testing hypotheses. *Univ. california Publ. Statist.*, **3**, 37–98.

[19] L. Le Cam (1986). Asymptotic Methods in Statistical Decision Theory, *Springer-Verlag*.

[20] O. Linton (1993). Adaptive estimation in *ARCH* models. *Econometric Theory.*, **9**(4), 539–569.

[21] T. Lounis (2011). Asymptotically optimal tests when parameters are estimated. *Albanian J. Math.*, **5**(4), 193–214.

[22] T. Lounis (2013). Local asymptotically optimal test in *Arch* model. *Rev. Roumaine Math. Pures Appl.*, **4**, pp 333–392.

[23] T. Lounis (2015). phd thesis Inference in ARCH Models: Asymptotically optimal local tests. http://www.theses.fr/2015LORR0222

[24] T. Lounis (2017). Optimal tests in AR(m) time series model. *Comm. Statist. Simulations and Methods.,* **46**(2), pp 1583–1610.

[25] J. Ngatchou-Wandji (2005). Checking nonlinear heteroscedastic time series models. *Journal of Statistical Planning and Inference.*, **133**(1), pp 33–68.

[26] G. Rajna (1991). L'évaluation des options. *Press universitaires de France.*

[27] P. A. Samuelson (1965). Rational theory of warrant pricing. *Industrial Management Review* **6**(2), pp 13–31.

[28] G. Saporta (2006). Probabilités Analyse des données et Statistique. *Edition TECHNIP, Second Edition, Paris. France.*

[29] T. Subba Rao and M. M. Gabr (1980). A test for linearity of stationary time series. *J. Time Ser. Anal.*, **1**(2), pp 145–158.

[30] A. R. Swensen (1985). The asymptotic distribution of the likelihood ratio for autoregressive time series with a regression trend. *J. Multivariate Anal.***16**(1), pp 54–70.

[31] T. Hastie, R. Tibshirani and J.Friedman (2001). The elemenst of Statistical Learning. *Second Edition, Springer.*