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S-Asymptotically ω-Periodic Functions and Applications to Evolution Equations

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Abstract

In this paper, we first study further properties of S-asymptotically ω-periodic functions taking values in Banach spaces including a theorem of composition. Then we apply the results obtained to study the existence and uniqueness of S-asymptotically ω-periodic mild solutions to a nonautonomous semilinear differential equation.

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1 introduction

The aim of this paper is two-fold. First to investigate in Section 3, further properties of S-asymptotically ω-periodic functions taking values in an infinite dimensional Banach space X, that is functions $f : \mathbb{R}^+ \to X$ which are bounded, continuous and such that

$$\lim_{t\to\infty}(f(t+\omega) - f(t)) = 0, \quad \omega > 0.$$

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Then we apply the results obtained to study S-asymptotically ω-periodic mild solutions to the semilinear differential equation (Section 4)

$$\begin{cases} x'(t) = A(t)x(t) + F(t, x(t)) \text{ for } t \geq 0, \\ x(0) = x_0, \end{cases} \tag{1.1}$$

where $x_0 \in X$, and $A(t)$ generates an exponentially stable ω-periodic evolutionary family in X.

The results obtained here complement and generalize some results in the papers [1, 2, 3, 4, 5, 6, 8, 9, 10].

2 Preliminaries and Notation

Let X be a Banach space. $BC(\mathbb{R}^+, X)$ denotes the space of the continuous bounded functions from $\mathbb{R}^+$ into X; endowed with the norm $\|f\|_\infty := \sup_{t\geq 0} \|f(t)\|$, it is a Banach space. $C_0(\mathbb{R}^+, X)$ denotes the space of the continuous functions from $\mathbb{R}$ into X such that $\lim_{t\to\infty} f(t) = 0$; it is a Banach subspace of $BC(\mathbb{R}^+, X)$. When we fix a positive number ω, $P_\omega(X)$ denotes the space of the continuous ω-periodic functions from $\mathbb{R}^+$ into X; it is a Banach subspace of $BC(\mathbb{R}^+, X)$.

When X and Y are two Banach spaces, $\mathcal{L}(X, Y)$ denotes the space of the continuous linear mappings from X into Y. When $X = Y$, $I \in \mathcal{L}(X)$ denotes the identity mapping.

Definition 2.1. Let $f : \mathbb{R} \to X$ be a continuous function. We say that f is almost periodic if

$$\forall \varepsilon > 0, \exists \ell > 0, \forall \alpha \in \mathbb{R}, \exists \tau \in [\alpha, \alpha + \ell], \qquad \sup_{t\in\mathbb{R}} \|f(t+\tau) - (t)\| \leq \varepsilon.$$

We denote by $AP(X)$ the set of all almost periodic functions from $\mathbb{R}$ to X.

Definition 2.2. Let $f : \mathbb{R} \to X$ be a continuous function. We say that f is almost automorphic if for every sequence of real numbers $(s_n)_n$, there exists a subsequence $(t_n)_n$ such that for all $t \in \mathbb{R}$

$$\lim_{m\to\infty} \lim_{n\to\infty} f(t + t_n - t_m) = f(t).$$

We denote by $AA(X)$ the set of all almost automorphic functions from $\mathbb{R}$ to X. Recall that $AP(X) \subset AA(X)$.

Definition 2.3. Let $f \in BC(\mathbb{R}^+, X)$. We say that f is asymptotically almost periodic if $f = g + h$ where $g \in AP(X)$ and $h \in C_0(\mathbb{R}^+, X)$.

Definition 2.4. Let $f \in BC(\mathbb{R}^+, X)$. We say that f is asymptotically almost automorphic if $f = g + h$ where $g \in AA(X)$ and $h \in C_0(\mathbb{R}^+, X)$.

It is obvious that an asymptotically almost periodic function is asymptotically almost automorphic.

3 S-Asymptotically ω-Periodic Functions

Definition 3.1. A function $f \in BC(\mathbb{R}^+, X)$ is called S-asymptotically ω-periodic if there exists $\omega > 0$ such that $\lim_{t\to\infty}(f(t+\omega) - f(t)) = 0$. In this case we say that ω is an asymptotic period of f and that f is S-asymptotically ω-periodic.

We will denote by $SAP_\omega(X)$, the set of all S-asymptotically ω-periodic functions from $\mathbb{R}^+$ to X.

Remark 3.2. If ω is an asymptotic period of f, then $n\omega$ is also an asymptotic period of f for every $n = 1, 2, ...$

Proof. The proof is easy by using the principle of mathematical induction. □

The following result is due to Henriquez-Pierri-Táboas; Proposition 3.5 in [3].

Theorem 3.3. *Endowed with the norm $\|\cdot\|_\infty$, $SAP_\omega(X)$ is a Banach space.*

Remark 3.4. We give a very short proof of this result. We consider the translation operator $\tau_\omega : BC(\mathbb{R}^+, X) \to BC(\mathbb{R}^+, X)$ defined by $\tau_\omega f := [t \mapsto f(t+\omega)]$. τ_ω is clearly linear and it is continuous since $[\omega, \infty) \subset R^+$. We note that $SAP_\omega(X) = (\tau_\omega - I)^{-1}(C_0(\mathbb{R}^+, X))$. And then, since $(\tau_\omega - I)$ is linear continuous and since $C_0(\mathbb{R}^+, X)$ is a closed vector subspace of $BC(\mathbb{R}^+, X)$, $SAP_\omega(X)$ is a closed vector subspace of the Banach space $BC(\mathbb{R}^+, X)$.

Now we recall another notion which is related to the S-asymptotically ω-periodicity.

Definition 3.5. Let $f \in BC(\mathbb{R}^+, X)$ and $\omega > 0$. We say that f is asymptotically ω-periodic if $f = g + h$ where $f \in P_\omega(X)$ and $h \in C_0(\mathbb{R}^+, X)$.

Denote by $AP_\omega(X)$ the set of all ω-periodic functions. Then we have

$$AP_\omega(X) \subset SAP_\omega(X).$$

The inclusion is strict. Indeed consider the function $f : \mathbb{R}^+ \to c_0$ where $c_0 = \{x = (x_n)_{n\in\mathbb{N}} : \lim_{n\to\infty} x_n = 0\}$ equipped with the norm $\|x\| = \sup_{n\in\mathbb{N}} |x(n)|$, and $f(t) = (2nt/(t^2+n^2)_{n\in\mathbb{N}}$. Then $f \in SAP_\omega(X)$ but $f \notin AP_\omega(X)$ (cf. [3] Example 3.1).

The following extends ([3], Proposition 3.4) to the asymptotically almost automorphic case.

Proposition 3.6. *Let f be a S-asymptotically ω-periodic function. If f is asymptotically almost automorphic, then f is asymptotically ω-periodic. In particular case if f asymptotically almost periodic, then f is asymptotically ω-periodic.*

Proof. Let f be a S-asymptotically ω-periodic and an asymptotically almost automorphic function. We can decompose f as $f = g + \phi$ where g is almost automorphic and $\phi \in C_0(\mathbb{R}^+, X)$. It suffices to prove that $g \in P_\omega(X)$. From $C_0(\mathbb{R}^+, X) \subset SAP_\omega(X)$, it follows that $g = f - \phi \in SAP_\omega(X)$, thus

$$\lim_{t\to\infty} g(t+\omega) - g(t) = 0. \tag{3.1}$$

Consider the sequence $(k)_k$. Since g is almost automorphic, we can extract a subsequence $(k_n)_n$ such that for all $t \in \mathbb{R}$

$$\lim_{m\to\infty}\lim_{n\to\infty} g(t+\omega+k_n-k_m)-g(t+k_n-k_m) = g(t+\omega)-g(t). \tag{3.2}$$

From (3.1) and $\lim_{n\to\infty} k_n - k_m = \infty$, it follows

$$\lim_{n\to\infty} g(t+\omega+k_n-k_m)-g(t+k_n-k_m) = 0, \forall t \in \mathbb{R}^+,$$

and from (3.2), we obtain $g(t+\omega)-g(t) = 0$ for all $t \in \mathbb{R}^+$. This implies that $g(t+\omega)-g(t) = 0$ for all $t \in \mathbb{R}$ (cf. [7] Theorem 2.1.8), thus $g \in P_\omega(X)$. This ends the proof. □

Theorem 3.7. *Let $\phi : X \to Y$ be a function which is uniformly continuous on the bounded subsets of X and such that ϕ maps bounded subsets of X into bounded subsets of Y. Then for all $f \in SAP_\omega(X)$, the composition $\phi \circ f := [t \to \phi(f(t))] \in SAP_\omega(X)$.*

Proof. Since the range of f is bounded, it follows that $\phi(f(\cdot))$ is bounded. Let $\epsilon > 0$ be given. Then there exists $\delta > 0$ such that $\|\phi(x)-\phi(y)\| < \epsilon$ for all $x, y \in f(\mathbb{R}^+)$ with $\|x-y\| < \delta$. Now we can find we can find $T = T(\delta) > 0$ such that $\|f(t+\omega)-f(t)\| < \delta$ for all $t > T$. Thus $\|\phi(f(t+\omega))-\phi(f(t))\| < \epsilon$ if $t > T$, which completes the proof. □

An example of such a function which satisfies the assumptions of Theorem 3.7 is a bilinear continuous function $B : X := U \times V \to Y$, where U and V are Banach spaces. From the inequality $\|B(u,v)\| \le c\|u\|\|v\|$, where $c \in (0,\infty)$, it is easy to see that B maps bounded subsets into bounded subsets. If M is a bounded subset of $U \times V$, there exists $c_1 \in (0,\infty)$ such that $\|u\| \le c_1$ and $\|v\| \le c_1$ for all $(u,v) \in M$. Then when $(u,v),(u_1,v_1) \in M$, we obtain $\|B(u,v)-B(u_1,v_1)\| \le c.c_1.(\|u-u_1\|+\|v-v_1\|)+c.\|u-u_1\|.\|v-v_1\|$, and so B is Lipschitzian on M and therefore it is uniformly continuous on M. Note that it is well-known that B is not uniformly continuous on $U \times V$. And so we obtain the following corollary.

Corollary 3.8. *Let X, Y and Z be three Banach spaces, and let $B : X \times Y \to Z$ be a bilinear continuous mapping. Then, when $f \in SAP_\omega(X)$ and $g \in SAP_\omega(Y)$, we have $B \circ (f,g) := [t \mapsto B(f(t),g(t))] \in SAP_\omega(Z)$.*

Proof. Note that the function $(f,g) := [t \mapsto (f(t),g(t))] \in SAP_\omega(X \times Y)$ since, by using the topology-product we have $\lim_{t\to\infty}(f(t+\omega),g(t+\omega))-(f(t),g(t)) = (\lim_{t\to\infty}(f(t+\omega)-f(t)),\lim_{t\to\infty}(g(t+\omega)-g(t)) = (0,0)$. And so we conclude by using Theorem 3.7 and the previous comments. □

For instance, if X^* is the topological dual space of a Banach space X, and if $\langle\cdot,\cdot\rangle$ denotes the duality bracket between X and X^*, when $f \in SAP_\omega(X)$ and when $f_* \in SAP_\omega(X^*)$, the function $\langle f_*, f\rangle := [t \mapsto \langle f_*(t), f(t)\rangle]$ belongs to $SAP_\omega(\mathbb{R})$. And in the special case $X = \mathbb{R}$ we obtain the following result.

Remark 3.9. $SAP_\omega(\mathbb{R})$ is a Banach algebra.

Since a linear continuous mapping $A \in \mathcal{L}(X,Y)$ is Lipschitzian, it satisfies the assumptions of Theorem 3.7, and consequently we obtain the following corollary.

Corollary 3.10. *Let X and Y be two Banach spaces, and let $A \in \mathcal{L}(X,Y)$. Then when $f \in SAP_\omega(X)$, we have $Af := [t \to Af(t)] \in SAP_\omega(Y)$.*

Remark 3.11. For a fixed $\omega > 0$, the bounded linear operator $\tau_\omega - I$, where I is the identity operator is not bijective since $Ker(\tau_\omega - I) = P_\omega(X)$ which is nonzero, however for $0 < \epsilon < 1$ the operator $(1-\epsilon)\tau_\omega - I$ is bijective, since $(1-\epsilon)\tau_\omega$ is a bounded linear operator with $\|(1-\epsilon)\tau_\omega\| < 1$. For this reason, if we consider

$$E_\omega^\epsilon := \{f \in BC(\mathbb{R}^+; X) : \lim_{t\to\infty}((1-\epsilon)f(t+\omega) - f(t)) = 0\},$$

then we have

$$\bigcap_{\epsilon>0} E_\omega^\epsilon \subset SAP_\omega(X).$$

Proof. Let $\epsilon > 0$ be given and take $f \in E_\omega^\epsilon$. Then

$$\|f(t+\omega) - f(t)\| \leq \|(1-\epsilon)f(t+\omega) - f(t)\| + \epsilon\|f(t+\omega)\|$$

$$\leq \|(1-\epsilon)f(t+\omega) - f(t)\| + \epsilon\|f\|_\infty.$$

Thus

$$\forall \epsilon > 0, \qquad \limsup_{t\to\infty} \|f(t+\omega) - f(t)\| \leq \epsilon\|f\|_\infty,$$

therefore

$$\lim_{t\to\infty} \|f(t+\omega) - f(t)\| = 0.$$

This completes the proof. □

For the sequel we consider asymptotically ω-periodic functions with parameters.

Definition 3.12. [3] A continuous function $f : [0,\infty) \times X \to X$ is said to be uniformly S-asymptotically ω-periodic on bounded sets if for every bounded set $K \subset X$, the set $\{f(t,x) : t \geq 0, x \in K\}$ is bounded and $\lim_{t\to\infty}(f(t,x) - f(t+\omega,x)) = 0$ uniformly on $x \in K$.

Definition 3.13. [3] A continuous function $f : [0,\infty) \times X \to X$ is said to be asymptotically uniformly continuous on bounded sets if for every $\epsilon > 0$ and every bounded set $K \subset X$, there exist $L_{\epsilon,K} \geq 0$ and $\delta_{\epsilon,K} > 0$ such that $\|f(t,x) - f(t,y)\| < \epsilon$ for all $t \geq L_{\epsilon,K}$ and all $x, y \in K$ with $\|x - y\| < \delta_{\epsilon,K}$.

Theorem 3.14. *[3] Let $f : [0,\infty) \times X \to X$ be a function which is uniformly S-asymptotically ω-periodic on bounded sets and asymptotically uniformly continuous on bounded sets. Let $u : [0,\infty)$ be an S-asymptotically ω-periodic function. Then the Nemytskii function $\phi(\cdot) := f(\cdot, u(\cdot))$ is S-asymptotically ω-periodic.*

4 Applications to Abstract Differential Equations

Now we consider the linear problem:

$$\begin{cases} x'(t) = A(t)x(t) + f(t) \text{ for } t \geq 0, \\ x(0) = x_0, \end{cases} \tag{4.1}$$

where $x_0 \in X$, $f \in BC(\mathbb{R}^+, X)$ and $A(t)$ generates a ω-periodic ($\omega > 0$) exponentially stable evolutionary process $(U(t,s))_{t \geq s}$ in X, that is, a two-parameter family of bounded linear operators that satisfies the following conditions:

1. $U(t,t) = I$ for all $t \in \mathbb{R}$,
2. $U(t,s)U(s,r) = U(t,r)$ for all $t \geq s \geq r$,
3. The map $(t,s) \mapsto U(t,s)x$ is continuous for every fixed $x \in X$,
4. $U(t+\omega, s+\omega) = U(t,s)$ for all $t \geq s$ (ω*-periodicity*),
5. There exist $K > 0$ and $a > 0$ such that $\|U(t,s)\| \leq Ke^{-a(t-s)}$ for $t \geq s$.

Definition 4.1. A continuous function $x : \mathbb{R}^+ \to X$ is called mild solution of (4.1) if

$$x(t) = U(t,0)x_0 + \int_0^t U(t,s)f(s)\,ds, \quad \text{for } t \geq 0. \tag{4.2}$$

Lemma 4.2. *Let $f \in SAP_\omega(X)$ and $(U(t,s))_{t \geq s}$ an ω-periodic exponentially stable evolutionary process. Then the function*

$$u(t) := \int_0^t U(t,s)f(s)ds$$

is also in $SAP_\omega(X)$.

Proof. For $t \geq 0$, one has

$$\begin{aligned} u(t+\omega) - u(t) &= \int_0^{t+\omega} U(t+\omega,s)f(s)\,ds - \int_0^t U(t,s)f(s)\,ds \\ &= \int_0^{\omega} U(t+\omega,s)f(s)\,ds + \int_{\omega}^{t+\omega} U(t+\omega,s)f(s)\,ds - \int_0^t U(t,s)f(s)\,ds \\ &= I_1(t) + I_2(t) \end{aligned}$$

where

$$I_1(t) = \int_0^{\omega} U(t+\omega,s)f(s)\,ds$$

and

$$I_2(t) = \int_{\omega}^{t+\omega} U(t+\omega,s)f(s)\,ds - \int_0^t U(t,s)f(s)\,ds.$$

Remark that

$$I_1(t) = U(t+\omega,\omega)\int_0^{\omega} U(\omega,s)f(s)\,ds = U(t+\omega,\omega)u(\omega)$$

and by using the fact $(U(t,s))_{t\geq s}$ is exponentially stable, we obtain

$$\|I_1(t)\| \leq Ke^{-at}\|u(\omega)\|,$$

which shows that $\lim_{t\to\infty} I_1(t) = 0$.

Now since $f \in SAP_\omega(X)$, we can find T sufficiently large such that

$$\|f(t+\omega) - f(t)\| < \epsilon, \quad \text{for } t > T.$$

Let's write

$$I_2(t) = \int_0^t (U(t+\omega, s+\omega)f(s+\omega) - U(t,s)f(s))\, ds$$

and since the evolution family is ω-periodic, we obtain

$$I_2(t) = \int_0^t U(t,s)(f(s+\omega) - f(s))\, ds.$$

Thus we get

$$\begin{aligned}
\|I_2(t)\| &\leq \int_0^T \|U(t,s)\|\|f(s+\omega) - f(s)\|\, ds + \int_T^t \|U(t,s)\|\|f(s+\omega) - f(s)\|\, ds \\
&\leq 2\|f\|_\infty \int_0^T \|U(t,s)\| ds + \epsilon \int_T^t \|U(t,s)\| ds \\
&\leq 2K\|f\|_\infty \int_0^T e^{-a(t-s)} ds + \epsilon K \int_T^t e^{-a(t-s)} ds \\
&\leq \frac{2K\|f\|_\infty}{a}(e^{-a(t-T)} - e^{-at}) + \frac{\epsilon K}{a}.
\end{aligned}$$

Thus $\lim_{t\to\infty} I_2(t) = 0$, this proves that $u \in SAP_\omega$. □

Theorem 4.3. *Let $f \in SAP_\omega(X)$ and $(U(t,s))_{t\geq s}$ an ω-periodic exponentially stable evolutionary process, then every mild solution of Eq.(4.1) is in $SAP_\omega(X)$.*

Proof. Since $A(t)$ generates a ω-periodic exponentially stable evolutionary process, then Eq.(4.1) has a mild solution x defined by (4.2). It remains to prove that it is in $SAP_\omega(X)$. This is immediate by using Lemma 4.2 and the fact that the two-parameter family is exponentially stable, thus $\lim_{t\to\infty} \|U(t,0)x_0\| = 0$, since $C_0(\mathbb{R}^+, X) \subset SAP_\omega(X)$, we deduce that $\lim_{t\to\infty} \|U(t+\omega,0)x_0 - U(t,0)x_0\| = 0$. □

Example 4.4. Consider the equation

$$x'(t) = a(t)x(t) + f(t), \quad t \geq 0 \tag{4.3}$$

where $f \in SAP_\omega(\mathbb{R})$ and $a \in P_\omega(\mathbb{R})$. We also assume that $\int_0^\omega a(t)\,dt < 0$. Then $U(t,s) := \exp(\int_s^t a(\sigma)\,d\sigma)$ is an ω-periodic exponentially stable evolutionary process, therefore the solution with initial data $x(0) = x_0$:

$$x(t) = \exp(\int_0^t a(\sigma)\,d\sigma)x_0 + \int_0^t \left(\exp(\int_s^t a(\sigma))\,d\sigma\right) f(s)ds,$$

is also in $SAP_\omega(\mathbb{R})$.

Now we consider semilinear problem

$$\begin{cases} x'(t) = A(t)x(t) + F(t,x(t)) \text{ for } t \geq 0, \\ x(0) = x_0, \end{cases} \tag{4.4}$$

where $x_0 \in X$.

We make the following assumptions.

$\mathbf{H_1}$ $A(t)$ generates a ω-periodic ($\omega > 0$) exponentially stable evolutionary process in X.

$\mathbf{H_2}$ F is uniformly S-asymptotically ω-periodic on bounded sets.

$\mathbf{H_3}$ F satisfies a Lipschitz condition in second variable uniformly with respect to the first variable, i.e. there exists $L > 0$ such that

$$||F(t,x) - F(t,y)|| \leq L||x - y||, \quad x, y \in X, \quad t \geq 0.$$

Theorem 4.5. *Under* $\mathbf{H_1} - \mathbf{H_3}$*, Eq.*(4.4) *possesses a unique mild solution in* $SAP_\omega(X)$ *if* $L < \frac{a}{K}$.

Proof. Consider the mapping Γ defined on $SAP_\omega(X)$ by

$$\Gamma u(t) := U(t,0)x_0 + \int_0^t U(t,s)F(s,u(s))ds.$$

Γ is well-defined by the above results. Now let u, $v \in SAP_\omega(X)$. Then we have

$$||\Gamma u(t) - \Gamma v(t)|| = \int_0^t ||U(t,s)||_{\mathcal{L}(X,X)} ||F(s,u(s)) - F(s,v(s))||\,ds$$

$$\leq L||u - v||_\infty \int_0^t Ke^{-a(t-s))}\,ds,$$

thus

$$||\Gamma u - \Gamma v||_\infty \leq \frac{LK}{a}||u - v||_\infty \text{ with } \frac{LK}{a} < 1.$$

The results follows in virtue of the contraction mapping principle. □

Remark 4.6. Theorem 4.5 contains the case of semilinear equations where the linear part is the infinitesimal generator of a semigroup which is exponentially stable. Consider the following equation:

$$\begin{cases} x'(t) = Ax(t) + F(t, x(t)) \text{ for } t \geq 0, \\ x(0) = x_0, \end{cases} \tag{4.5}$$

where $x_0 \in X$ and $A : D(A) \to X$ is the infinitesimal generator of a semigroup $(S(t))_{t\geq 0}$. If the semigroup is exponentially stable: $\|S(t)\| \leq Ke^{-at}$ for all $t \geq 0$ and F satisfies Under $\mathbf{H_2}$ and $\mathbf{H_3}$, Eq.(4.5) possesses a unique mild solution in $SAP_\omega(X)$ if $L < \dfrac{a}{K}$.

This last result is a corollary of Theorem 4.5 by setting $U(t, s) = T(t - s)$ for $t \geq s$.

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References

[1] H. Gao, K. Wang, F. Wei, X. Ding, *Massera-type theorem and asymptotically periodic logistic equations*, Nonlinear Analysis, R.W.A., **7** (2006), 1268-1283.

[2] R. C. Grimmer, *Asymptotically almost periodic solutions of differential equations*, SIAM J. Appl. Math. **17** (1969), 109-115.

[3] H. R. Henríquez, M. Pierri, P. Táboas, *On S-asymptotically ω-periodic functions on Banach spaces and applications*, J. Math. Anal. Appl. **343** (2008), 1119-1130.

[4] H. R. Henríquez, M. Pierri, P. Táboas, *Existence of S-asymptotically ω-periodic solutions for abstract neutral equations*, Bull. Austr. Math. Soc. **78** (2008), 365-382.

[5] Z. C. Liang, *Asymptotically periodic solutions of a class of second order nonlinear differential equations*, Proc. Amer. Math. Soc., **99** (4), (1987), 693-699.

[6] C. Lizama, G. M. N'Guérékata, *Bounded mild solutions for semilinear integro differential equations in Banach spaces*, Integral Equations and Operator Theory, **68** (2010), 207-227.

[7] G. M. N'Guérékata, *Almost Automorphic and Almost Periodic Functions in Abstract Spaces*, Kluwer Academic, New York, London, Moscow, 2001.

[8] S. Nicola, M. Pierri, *A note on S-asymptotically periodic functions*, Nonlinear Analysis, R.W.A., **10** (2009), 2937-2938.

[9] W. R. Utz, P. Waltman, *Asymptotic almost periodicity of solutions of a system of differential equations*, Proc. Amer. Math. Soc., **18** (1967), 597-601.

[10] J. S. Wong, T. A. Burton, *Some properties of solutions of $u''(t) + a(t)f(u)g(u') = 0$, II*, Monatsh. Math. **69** (1965), 368-374.