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Existence and Uniqueness of the Weak Solutions for the Steady Incompressible Navier-Stokes Equations with Damping

Wuming Li*
School of Mathematical and information Science, Henan Polytechnic University,
Jiaozuo Henan , 454003, P.R. China

Xinxia Wang†
No. 62 Middle School of Beijing,
Beijing 100050, P.R. China

Quansen Jiu‡
Department of Mathematics, Capital Normal University,
Beijing 100048, P.R. China

Abstract

This paper is concerned with the boundary-value problem for the steady incompressible Navier-Stokes equations with damping. Two cases are considered here: 1) the Dirichlet's boundary condition; 2) the nonhomogeneous boundary condition. we obtain the existence and uniqueness of the weak solutions for the steady incompressible Navier-Stokes equations with damping using different methods for the above cases.

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1 Introduction

The boundary-value problem for the steady incompressible flows is written as

$$\begin{cases} -\gamma\triangle u + u\cdot\nabla u + \nabla p = f, & x\in\Omega \\ \nabla\cdot u = 0, & x\in\Omega \\ u = \phi, & x\in\partial\Omega \end{cases}$$

*Li is supported in part by National Natural Sciences Foundation of China (No.11071057), Applied Mathematics Provincial-Level key Discipline of Henan Province and Young Foundation of Henan Polytechnic University (No.Q2011-15). E-mail address: Liwum0626@126.com

†E-mail address: 99lilacs@163.com

‡Jiu is supported in part by National Natural Sciences Foundation of China (No.10871133) and project of Beijing Education Committee. E-mail address: jiuqs@mail.cnu.edu.cn

Here u, p stand for the velocity field and the pressure of the flows respectively, f is the external force, Ω is a bounded open domain of $\mathbb{R}^n$, n is the dimension of the space, $\gamma > 0$ is the constant.

Many mathematical studies have been made for the above boundary value problem. In the early 1930, F. K. G. Odqvist and Leray (see [7]) proved the existence of the solutions with restrictions on the Reynolds numbers. G. P. Galdi [4] also proved the existence of the solutions on the domains with many connected bifurcations.

Youdovich (1967) firstly obtained the uniqueness of the solutions for the steady-state Navier-Stokes problem in a bounded domain in [12]. R. Teman in [11] also had some researches. Especially, Galdi in [4] obtained existence, uniqueness and regularity of solutions for the steady Navier-Stokes equations in bounded domain, unbounded domain and exterior domain for spatial dimension n=2, 3, 4, respectively. Struwe (1995) have shown the existence of regular solutions in dimension $n \geq 5$ without restrictions on the size of the data in [10].

Damping is very common in nature, which rises from the motion of flows and can describe many physical phenomena, such as porous media flow, drag, or friction effects, and some dissipative mechanisms. The boundary-value problem for the steady incompressible Navier-Stokes equations with damping can be written as

$$(*1)\begin{cases} -\gamma\triangle u + u\cdot\nabla u + \alpha|u|^{\beta-1}u + \nabla p = f, & x \in \Omega, \quad (1.1)\\ \nabla\cdot u = 0, & x\in\Omega, \quad (1.2)\\ u = \phi, & x \in \partial\Omega, \quad (1.3)\end{cases}$$

where $\alpha > 0$ and $\beta \geq 1$ are both positive constants.

Cai and Jiu in [3] studied the Cauchy problem of Navier-Stokes equations with damping and obtained the existence of the global strong solutions for $\beta \geq \frac{7}{2}$ and existence and uniqueness of the strong solutions for $\frac{7}{2} \leq \beta \leq 5$. In the present paper, we broaden the corresponding results of existence and uniqueness of the weak solutions in $W^{1,2}(\Omega)$ for steady Navier-Stokes equations to Navier-Stokes equations with damping.

Before ending this section, we introduce some notations of function spaces. let Ω be an arbitrary domain in $\mathbb{R}^n$, $n \geq 1$. If $\|\phi\|_r := (\int_\Omega |\phi(x)|^r dx)^{\frac{1}{r}} < \infty, 1 \leq r < \infty$, we say $\phi \in L^r(\Omega)$, under the norm defined above, $L^r(\Omega)$ be a Banach space. If $r = \infty$, we define $\|\phi\|_\infty = ess\ sup|\phi| < \infty$. Let $C^\infty_{0,\sigma}(\Omega)$ denotes the set of all C^∞ real vector-valued functions $\phi = (\phi^1, ..., \phi^n)$ with compact support in Ω such that $div\phi = 0$. Then the function space $L^r_\sigma(\Omega)$, $0 < r < \infty$, is defined as the closure of $C^\infty_{0,\sigma}(\Omega)$ with respect to $\|\cdot\|_r$. $H^{1,r}_{0,\sigma}$ is the closure of $C^\infty_{0,\sigma}(\Omega)$ with the norm $\|\phi\|_{H^{1,r}} = \|\phi\|_r + \|\nabla\phi\|_r$. Helmholtz decomposition can be defined as $L^r = L^r_\sigma \oplus G^r$, $1 < r < \infty$, where $G^r = \{\nabla p \in L^r; p \in L^r_{loc}(\overline{\Omega})\}$, $P : L^r(\Omega) \to L^r_\sigma(\Omega)$ denotes Helmholtz Projector decomposition. We define $W^{k,p}(\Omega)$ as the usual Sobolev space with the norm $\|\cdot\|_{k,p}$ and $W^{k,p}_{0,\sigma}(\Omega)$ is the closure of $C^\infty_{0,\sigma}(\Omega)$ with respect to $\|\cdot\|_{k,p}$. If $p = 2$, we usually write $H^k(\Omega) = W^{k,2}(\Omega)$. Let $\widehat{W}^{k,p}(\Omega)$ be the homogeneous Sobolev space such that $|u|_{k,p} = (\int_\Omega \sum\limits_{|\alpha|=k} |\partial^\alpha u|^p dx)^{\frac{1}{p}} < \infty$, $\widehat{W}^{k,p}_0(\Omega)$ is the closure of $C^\infty_0(\Omega)$ with the norm $|u|_{k,p}$.

The rest of the paper is organized as follows. In Section 2, we give the definition of the weak solutions and the main results of this paper. In Section 3, we obtain the existence and uniqueness of the weak solutions in $H^1_0(\Omega) \cap L^{\beta+1}(\Omega)$ to equations $(*1)$ for homogeneous problem ($\phi = 0$). In Section 4, we prove the weak solutions in $\widehat{W}^{1,2}(\Omega)$ to equation ($*1$) for nonhomogeneous problem exists and is unique.

2 Main results

Before stating the main results, we firstly give the definition of the weak solutions to (1.1)-(1.3).

Definition 2.1. Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$. If there exists $u : \Omega \to R^n$ satisfying
(i)

$$u \in \widehat{W}^{1,2}(\Omega) \cap L^{\beta+1}(\Omega);$$

(ii)

$$\nabla \cdot u = 0;$$

(iii) u satisfies the boundary condition (1.3) in the sense of trace; If $\phi = 0$, then $u \in \widehat{W}_0^{1,2}(\Omega)$.
(iv) For $\forall \varphi \in C_{0,\sigma}^{\infty}$, we have

$$\gamma(\nabla u, \nabla\varphi) + (u \cdot \nabla u, \varphi) + (\alpha|u|^{\beta-1}u, \varphi) = (f, \varphi), \tag{2.1}$$

then u is a weak solution to (1.1)-(1.3) in $H_0^1(\Omega) \cap L^{\beta+1}(\Omega)$.

Remark. If Ω is a bounded domain, we have $\widehat{W}_0^{1,q}(\Omega) = W_0^{1,q}(\Omega)$; Furthermore, if Ω is locally Lipschitz, then $\widehat{W}^{1,q}(\Omega)$ and $W^{1,q}(\Omega)$ is homeomorphic. If $\phi = 0$, we obtain $u \in W_0^{1,2}(\Omega)$ in (i) identically. If $\phi \neq 0$ and Ω is locally Lipschitz, we identically have $u \in W^{1,2}(\Omega)$ in (i).

Theorem 2.2. *Let Ω be a bounded domain in $\mathbb{R}^n$ ($n \geq 2$), $f \in \widehat{W}_0^{-1,2}(\Omega)$ is an given exterior function. If $\phi = 0$ and $1 \leq \beta < 11$, then there exists at least a weak solution to BVP (1.1)-(1.3) satisfying*

$$\gamma|u|_{1,2} + \alpha\|u\|_{\beta+1}^{\beta+1} \leq c\|f\|_{H^{-1}}, \tag{2.2}$$

$$\|p\|_2 \leq c(|f|_{-1,2} + |u|_{1,2} + |u|_{1,2}^2 + \|u\|_{\beta+1}). \tag{2.3}$$

Theorem 2.3. *Let Ω be a bounded domain in $\mathbb{R}^3$ and locally Lipschitz, $\phi = 0, f \in \widehat{W}_0^{-1,2}(\Omega)$. If γ is big enough and $\frac{5}{3} \leq \beta \leq 5$, then the weak solution to BVP (1.1)-(1.3) is unique.*

Theorem 2.4. *Suppose that $\phi = curl\ \zeta$ and $\zeta \in H^2(\Omega)$, $\partial_i\zeta \in L^6(\Omega)$, $\zeta \in L^{\infty}(\Omega)$. If $1 \leq \beta \leq 5$, then there exists at least a weak solution $u \in H^2(\Omega)$ to BVP (1.1)-(1.3).*

Theorem 2.5. *Let Ω be a bounded domain in $\mathbb{R}^3$ and locally Lipschitz, $f \in W^{-1,2}(\Omega)$. If γ is big enough and $\frac{5}{3} \leq \beta \leq 5$, then the weak solution to BVP (1.1)-(1.3) is unique.*

3 Existence and Uniqueness with Homogeneous Boundary Data

§3.1. Existence of weak solutions

In this part , we consider the existence of the weak solutions for problem (1.1)-(1.3) with Dirichlet's boundary condition, here Ω is a bounded domain in $\mathbb{R}^3$. The following Lemma will be needed later, of which proof is referred to [9].

Lemma 3.1. *Let Ω be a bounded locally Lipschitzian domain in $\mathbb{R}^n$, $n \geq 2$, $\Omega_0 \subseteq \Omega$, $\Omega_0 \neq \emptyset$. If for any $F \in W^{-1,q'}(\Omega)$, $\upsilon \in W^{1,q}_{0,\sigma}(\Omega)$,*

$$[F, \upsilon] =< F, \upsilon >= 0,$$

where

$$1 < q < \infty, q' = \frac{q}{q-1},$$

then there exists a unique $P \in L^{q'}(\Omega)$ satisfying

$$[F, \upsilon] =< F, \upsilon >= \int_\Omega P div \upsilon dx$$

and $\int_\Omega P dx = 0$.

Lemma 3.2. *Let Ω be an arbitrary domain in $\mathbb{R}^n$, $n = 2, 3$, and let $f \in W_0^{-1,2}(\Omega')$, for any bounded domain Ω', with $\overline{\Omega'} \subset \Omega$. Then a vector field $V \in W^{1,2}_{loc}(\Omega)$ satisfies (2.1) for all $\varphi \in C^\infty_{0,\sigma}(\Omega)$ if and only if there is $P \in L^2_{loc}(\Omega)$ satisfying the identity*

$$\gamma(\nabla V, \nabla \psi) + (V \cdot \nabla V, \psi) + (\alpha |V|^{\beta-1} V, \psi) = (P, \nabla \cdot \psi) + < f, \psi > \tag{3.1}$$

for all $\psi \in C^\infty_0(\Omega)$. If, moreover, Ω is bounded and lipschitzian and

$$f \in \widehat{W}_0^{-1,2}(\Omega), V \in \widehat{W}^{1,2}(\Omega),$$

then

$$P \in L^2(\Omega) \text{ with } \int_\Omega P dx = 0,$$

and (3.1) holds for all $\psi \in \widehat{W}^{1,2}_0(\Omega)$.

Proof. Clearly, (3.1) implies (2.1). Thus assume that Ω is locally lipschitzian, the functional

$$F(\psi) := \gamma(\nabla V, \nabla \psi) + (V \cdot \nabla V, \psi) + (\alpha |V|^{\beta-1} V, \psi) - < f, \psi > \tag{3.2}$$

is linear and bounded in $\psi \in W^{1,2}_0(\Omega)$ and vanishes when $\psi \in \widehat{W}^{1,2}_{0,\sigma}(\Omega)$. By virtue of Corollary 3.5.1 in [4] and Lemma 3.1, there exists $P \in L^2(\Omega)$ such that

$$F(\psi) = (P, \nabla \cdot \psi), \tag{3.3}$$

for all $\psi \in \widehat{W}_0^{1,2}(\Omega)(=W_0^{1,2}(\Omega))$, thus satisfying (3.1). If Ω is an arbitrary domain, we use Corollary 3.5.2 in [4] to deduce the existence of $P \in L^2_{loc}(\Omega)$ satisfying (3.3) for all $\psi \in C_0^\infty(\Omega)$. The proof is complete.

Lemma 3.3. *Let F be a continuous function of $\mathbb{R}^m$, $m \geq 1$, into itself such that some $k > 0$*

$$F(\xi)\cdot\xi > 0$$

for all $\xi \in \mathbb{R}^m$ with $|\xi| = k$. Then there exists $\xi_0 \in \mathbb{R}^m$ with $|\xi_0| \leq k$ such that $F(\xi_0) = 0$.

Proof. The proof of this Lemma is referred to [4].

Proof of Theorem 2.2. We employ the Galerkin method to prove the theorem and it is divided into the following three steps.

Step 1. Approximate Solutions

Since $W^{1,2}_{0,\sigma}(\Omega)$ is separable and $C^\infty_{0,\sigma}(\Omega)$ is dense in $W^{1,2}_{0,\sigma}(\Omega)$, there exists a sequence $\{\psi_k\} \subset C^\infty_{0,\sigma}(\Omega)$ be the basis of $W^{1,2}_{0,\sigma}(\Omega)$ and

$$(\psi_k, \psi_{k'}) = \delta_{kk'} = \begin{cases} 1, k = k', \\ 0, k \neq k'. \end{cases}$$

For each $m \in \mathbb{N}$, we define the approximate solutions u_m as follows:

$$u_m = \sum_{k=1}^{m} \xi_{km}\psi_k, \tag{3.4}$$

and

$$\gamma(\nabla u_m, \nabla\psi_k) + (u_m\cdot\nabla u_m, \psi_k) + (\alpha|u_m|^{\beta-1}u_m, \psi_k) = (f, \psi_k). \tag{3.5}$$

the equations (3.5) form a nonlinear differential system for the coefficients ξ_{km}, $k = 1, 2, ...m$. Since $u_m \in C^\infty_{0,\sigma}(\Omega)$, by virtue of Lemma 8.2.1 in [4], we have

$$\sum_{k=1}^{m}(u_m\cdot\nabla u_m, \xi_{km}\psi_k) = (u_m\cdot\nabla u_m, u_m) = 0, \tag{3.6}$$

and

$$|\sum_{k=1}^{m}(f, \xi_{km}\psi_k)| \leq |f|_{-1,2}|u_m|_{1,2}. \tag{3.7}$$

Multiplying ξ_{km} on both sides of (3.5), $k = 1, 2, ...m$ and summing over the resulting equations, due to (3.6) and (3.7), we obtain

$$\gamma\|\nabla u_m\|_2^2 + \alpha\|u_m\|^{\beta+1}_{\beta+1} = (f, u_m) \leq \|f\|_{H^{-1}}\|\nabla u_m\|_2, \tag{3.8}$$

and

$$\gamma|u_m|_{1,2} + \alpha\|u_m\|^{\beta+1}_{\beta+1} \leq c\|f\|_{H^{-1}}. \tag{3.9}$$

Step 2. Existence of Approximate Solutions
Let

$$[F(u_m,u_m)] := \gamma(\nabla u_m,\nabla u_m)+\alpha(|u_m|^{\beta-1}u_m,u_m)-(f,u_m),$$

Correspondingly, one has

$$\begin{aligned}&[F(u_m,u_m)] \geq \gamma\|\nabla u_m\|_2^2+\alpha\|u_m\|_{\beta+1}^{\beta+1}-\|f\|_{H^{-1}}\|\nabla u_m\|_2\\&=(\gamma\|\nabla u_m\|_2-\|f\|_{H^{-1}})\|\nabla u_m\|_2+\alpha\|u_m\|_{\beta+1}^{\beta+1}\end{aligned}$$

when $\|\nabla u_m\|_2=|u_m|_{1,2}=K, K>\frac{1}{\gamma}\|f\|_{H^{-1}}$, by Lemma 3.3, there exist the approximate solutions to (3.5) for $m\in\mathbb{N}$. Since $\{u_m\}$ are uniformly bounded by (3.9), there exist a subsequence of $\{u_m\}$(without loss of generality, we denote them by $\{u_m\}$) and a field $u\in W^{1,2}_{0,\sigma}(\Omega)\cap L^{\beta+1}(\Omega)$ such that

$$u_m \rightharpoonup u, \quad in \quad W^{1,2}_{0,\sigma}(\Omega)(=H^1_0(\Omega))\cap L^{\beta+1}(\Omega) \tag{3.10}$$

It follows from Sobolev embedding theorem that

$$u_m\to u, \quad in \quad L^{6-\varepsilon} \tag{3.11}$$

where $\varepsilon>0$ is some constant. Now we consider the convergence of the four terms in (3.5). For the first term on the left, as $m\to\infty$, one has

$$(\nabla u_m,\nabla\psi_k)\to(\nabla u,\nabla\psi_k). \tag{3.12}$$

For the second term in (3.5), we have

$$\begin{aligned}&|(u_m\cdot\nabla u_m,\psi_k)-(u\cdot\nabla u,\psi_k)|\\&\leq|(u_m-u)\cdot\nabla u_m,\psi_k|+|(u\cdot\nabla(u_m-u),\psi_k)|\\&=I_m^{(1)}+I_m^{(2)},\end{aligned}$$

furthermore,

$$\begin{aligned}I_m^{(1)}&\leq\|\psi_k\|_6\|\|\nabla u_m\|_2\|u_m-u\|_3\\&\leq c\|\psi_k\|_{H^1}|u_m|_{1,2}\|u_m-u\|_3,\end{aligned}$$

by (3.11), we obtain

$$\lim_{m\to\infty}I_m^{(1)}=0;$$

and for $I_m^{(2)}$ we have

$$\begin{aligned}I_m^{(2)}&\leq|(u\cdot\nabla\psi_k,(u_m-u))|\\&\leq\|u\|_6\|\nabla\psi_k\|_2\|u_m-u\|_3\\&\leq C\|\nabla\psi_k\|_2\|u_m-u\|_3,\end{aligned}$$

therefore,

$$\lim_{m\to\infty}I_m^{(2)}=0,$$

we obtain

$$|(u_m \cdot \nabla u_m, \psi_k) - (u \cdot \nabla u, \psi_k)| \to 0, \quad m \to \infty \tag{3.13}$$

For the third term on the left in (3.5), we have

$$\begin{aligned}
&|(|u_m|^{\beta-1}u_m, \psi_k) - (|u|^{\beta-1}u, \psi_k)| \\
&= |(|u_m|^{\beta-1}u_m, \psi_k) - (|u_m|^{\beta-1}u, \psi_k) + (|u_m|^{\beta-1}u, \psi_k) - (|u|^{\beta-1}u, \psi_k)| \\
&\leq |(|u_m|^{\beta-1}(u_m - u), \psi_k)| + |((|u_m|^{\beta-1} - |u|^{\beta-1})u, \psi_k)| \\
&= I_m^{(3)} + I_m^{(4)}.
\end{aligned}$$

It follows from Hölder inequality and Sobolev embedding theorem that

$$\begin{aligned}
I_m^{(3)} &\leq C sup|\psi_k| \| |u_m|^{\beta-1} \|_{\frac{6}{5}} \|u_m - u\|_6 \\
&\leq C sup|\psi_k| \cdot \|u_m\|_{\beta+1}^{\beta-1} \|u_m - u\|_{L^{6-\varepsilon}},
\end{aligned}$$

which satisfies the equality $\frac{\beta-1}{\beta+1} + \frac{1}{6-\varepsilon} = 1$, i.e., $\beta = 11 - 2\varepsilon$. When $1 \leq \beta < 11$, one has

$$I_m^{(3)} \to 0, \quad m \to \infty.$$

After a complicated computation, we deduce that

$$\begin{aligned}
I_m^{(4)} &= |((|u_m|^{\beta-1} - |u|^{\beta-1})u, \psi_k)| \\
&\leq C|(|u|^{\beta-1}(u_m - u), \psi_k)| \\
&\leq C\|u_m - u\|_{L^{6-\varepsilon}} \|u\|_{\beta+1}^{\beta-1} \|\psi_k\|_{L^r}
\end{aligned}$$

where $\frac{1}{6-\varepsilon} + \frac{\beta-1}{\beta+1} + \frac{1}{r} = 1$, let $r = \infty$, one has $\frac{1}{6-\varepsilon} = \frac{2}{\beta+1}$, i.e., $\beta = 11 - 2\varepsilon$. When $1 \leq \beta < 11$, one has

$$I_m^{(4)} \to 0, \quad m \to \infty.$$

Therefore,

$$\lim_{m \to \infty} |(|u_m|^{\beta-1}u_m, \psi_k) - (|u|^{\beta-1}u, \psi_k)| = 0. \tag{3.14}$$

By (3.12), (3.13) and (3.14), it follows that the field u (belongs to $H_0^1(\Omega)$) satisfies the equation

$$\gamma(\nabla u, \nabla \psi_k) + (u \cdot \nabla u, \psi_k) + (\alpha |u|^{\beta-1}u, \psi_k) = (f, \psi_k) \tag{3.15}$$

for all $k = 1, 2, \ldots$. However, any $\varphi \in H^1(\Omega)$ can be approximated by linear combinations of ψ_k through suitable coefficients. Since every term in (3.15) defines a bounded linear functional in $\psi_k \in H^1(\Omega)$, we may conclude from (3.15) that the field u satisfies (1.4) for all $\forall \varphi \in H^1(\Omega)$. Existence is then established.

By (3.9), we also obtain

$$\gamma |u|_{1,2} + \alpha |u|_{\beta+1}^{\beta+1} \leq C\|f\|_{H^{-1}},$$

we have proved (2.2).

Step 3. Existence of pressure field

Assuming Ω is locally lipschitzian, from Lemma 3.2 follows the existence of a pressure field $p \in L^2(\Omega)$ satisfying (3.1) and

$$\int_\Omega p dx = 0. \tag{3.16}$$

Consider the problem

$$\begin{cases} \nabla \cdot \Psi = p, \\ \Psi \in W_0^{1,2}(\Omega), \\ \|\Psi\|_{1,2} \le C\|p\|_2. \end{cases} \tag{3.17}$$

Since P is in $L^2(\Omega)$ and satisfies (3.16), Problem (3.17) is solvable by virtue of Theorem 3.3.1 in [4]. By the inequality $|(u \cdot \nabla v, w)| \le c|u|_{1,2}|v|_{1,2}|w|_{1,2}$ (see Lemma 8.1.1 in [4]), let $\psi = \Psi$ in (3.1), after a straight computation, we obtain

$$\|p\|_2^2 \le C(|f|_{-1,2} + |u|_{1,2} + \|u\|_{\beta+1} + |u|_{1,2})\|p\|_2,$$

where $C = C(n, \Omega)$, which shows (2.3). Theorem 2.2 is therefore proved.

§3.2. **Uniqueness of weak solutions**

Proof of Theorem 2.3. Assume that under the same data, there exist two weak solutions u_1, u_2 of the equations satisfying

$$\gamma(\nabla u_1, \nabla\varphi) + (u_1 \cdot \nabla u_1, \varphi) + (\alpha|u_1|^{\beta-1}u_1, \varphi) = (f, \varphi),$$

and

$$\gamma(\nabla u_2, \nabla\varphi) + (u_2 \cdot \nabla u_2, \varphi) + (\alpha|u_2|^{\beta-1}u_2, \varphi) = (f, \varphi).$$

Then we have

$$\gamma(\nabla(u_1 - u_2), \nabla\varphi) + (u_1 \cdot \nabla u_1 - u_2 \cdot \nabla u_2, \varphi) + (\alpha|u_1|^{\beta-1}u_1 - \alpha|u_2|^{\beta-1}u_2, \varphi) = 0.$$

Let $\varphi = u_1 - u_2$, we obtain

$$\gamma(\nabla(u_1 - u_2), \nabla(u_1 - u_2)) + ((u_1 - u_2) \cdot \nabla u_1, u_1 - u_2) + (\alpha|u_1|^{\beta-1}u_1 - \alpha|u_2|^{\beta-1}u_2, u_1 - u_2) = 0.$$

Furthermore, we have

$$\gamma\| \nabla(u_1 - u_2)\|_2^2 + ((u_1 - u_2) \cdot \nabla u_1, u_1 - u_2)$$
$$+(\alpha|u_1|^{\beta-1}(u_1 - u_2), (u_1 - u_2)) + \alpha((|u_1|^{\beta-1} - |u_2|^{\beta-1})u_2, u_1 - u_2) = 0.$$

Therefore,

$$\begin{aligned} &\gamma\| \nabla(u_1 - u_2)\|_2^2 = -((u_1 - u_2) \cdot \nabla u_1, u_1 - u_2) \\ &-(\alpha|u_1|^{\beta-1}(u_1 - u_2), (u_1 - u_2)) \\ &-\alpha((|u_1|^{\beta-1} - |u_2|^{\beta-1})u_2, u_1 - u_2). \end{aligned} \tag{3.18}$$

For the first term on the right in (3.18),

$$\begin{aligned}
&|-((u_1-u_2)\cdot\nabla u_1, u_1-u_2)| \\
&\le \|\nabla u_1\|_2\cdot\|u_1-u_2\|_4^2 \\
&\le \frac{c_1}{\gamma}\|f\|_{H^{-1}}\|u_1-u_2\|_6^2 \\
&\le \frac{c_1}{\gamma}\|f\|_{H^{-1}}\|\nabla(u_1-u_2)\|_2^2. \qquad (3.19)
\end{aligned}$$

For the second term on the right in (3.18),

$$\begin{aligned}
&|-(\alpha|u_1|^{\beta-1}(u_1-u_2),(u_1-u_2))| \\
&= |\alpha|u_1|^{\beta-1}|u_1-u_2|^2| \\
&\le \alpha\||u_1|^{\beta-1}\|_{\frac{3}{2}}\||u_1-u_2|^2\|_3 \\
&\le \alpha\||u_1|\|^{\beta-1}_{\frac{3(\beta-1)}{2}}\|u_1-u_2\|_6^2 \\
&\le c\alpha\|u_1\|^{\beta-1}_{\beta+1}\|\nabla(u_1-u_2)\|_2^2 \\
&\le c_2\alpha\|f\|_{H^{-1}}\|\nabla(u_1-u_2)\|_2^2 \qquad (3.20)
\end{aligned}$$

For the above estimate, we have used $1\le\frac{3(\beta-1)}{2}\le\beta+1$, i.e., $\frac{5}{3}\le\beta\le 5$. For the third term on the right in (3.18), after a tedious computation we obtain

$$\begin{aligned}
&-\alpha((|u_1|^{\beta-1}-|u_2|^{\beta-1})u_2, u_1-u_2) \\
&\le c_3\alpha\beta\|f\|_{H^{-1}}\|\nabla(u_1-u_2)\|_2^2 \qquad (3.21)
\end{aligned}$$

Here we also have used $1\le\frac{3(\beta-1)}{2}\le\beta+1$, i.e., $\frac{5}{3}\le\beta\le 5$. Substituting (3.19), (3.20) and (3.21) into (3.18), we obtain

$$\gamma\|\nabla(u_1-u_2)\|_2^2-(\frac{c_1}{\gamma}\|f\|_{H^{-1}}+c_2\alpha\|f\|_{H^{-1}}+c_3\alpha\beta\|f\|_{H^{-1}})\|\nabla(u_1-u_2)\|_2^2\le 0.$$

When γ is big enough, one has $u_1=u_2$, *a.e.*. The Proof of Theorem 2.3 is complete.

4 Existence and Uniqueness with Nonhomogeneous Boundary Data

§4.1. Existence of weak solutions

In this part, we consider the existence of the weak solutions for problem (1.1)-(1.3) with nonhomogeneous boundary condition ($\phi\neq 0$). Here we assume that $\Omega\in C^2$ is a bounded open domain in $\mathbb{R}^3$, $f\in H^{-1}$ is a given exterior function. For the given boundary function ϕ, let

$$\phi = curl\,\zeta, \qquad (4.1)$$

where

$$\zeta\in H^2(\Omega),\ \partial_i\zeta\in L^6(\Omega),\ \zeta\in L^\infty(\Omega). \qquad (4.2)$$

In $\mathbb{R}^3$, *curl* is the rotation operator. When $n = 3$, *curl* is a linear differential operator and $div(curl\,\zeta) \equiv 0$.

Proof of Theorem 2.4. The proof is divided into the following three steps.

Step 1. Transform nonhomogeneous problem into homogeneous problem.
Assume that ψ is any vector field in $H^1(\Omega)$ and satisfying

$$\psi \in H^1(\Omega), div\psi = 0, \psi = \phi(x \in \partial\Omega) \tag{4.3}$$

Let $\widehat{u} = u - \psi$, substitute $u = \widehat{u} + \psi$ into (1.1) , one has

$$-\gamma \triangle \widehat{u} + \widehat{u} \triangledown \widehat{u} + \widehat{u} \triangledown \psi + \psi \triangledown \widehat{u} + \alpha(|\widehat{u} + \psi|^{\beta-1})(\widehat{u} + \psi) + \nabla p = \widehat{f}, \tag{4.4}$$

where $\widehat{f} := f + \gamma \triangle \psi - \psi \triangledown \psi$. Since $f + \gamma \triangledown \psi \in H^{-1}, \psi \triangledown \psi \in H^{-1}$, we have $\widehat{f} \in H^{-1}$. Taking inner product between (4.4) and v over Ω, where $v \in C^\infty_{0,\sigma}$, we can obtain

$$\gamma(\triangledown\widehat{u}, \triangledown v) + (\widehat{u} \triangledown \widehat{u}, v) + (\widehat{u} \triangledown \psi, v) + (\psi \triangledown \widehat{u}, v) + (\alpha|(\widehat{u} + \psi)|^{\beta-1}(\widehat{u} + \psi), v) = (\widehat{f}, v).$$

We transform $(*1)$ into the following systems

$$(*2)\begin{cases} \gamma(\triangledown\widehat{u}, \triangledown v) + (\widehat{u} \triangledown \widehat{u}, v) + (\widehat{u} \triangledown \psi, v) & \\ +(\psi \triangledown \widehat{u}, v) + (\alpha|(\widehat{u} + \psi)|^{\beta-1}(\widehat{u} + \psi), v) = (\widehat{f}, v), & x \in \Omega, \\ \nabla \cdot \widehat{u} = 0, & x \in \Omega, \\ \widehat{u} = 0, & x \in \partial\Omega, \end{cases}$$

Therefore solving nonhomogeneous problem (1.1)-(1.3) is equivalent to solving the homogeneous problem $(*2)$, the approaches to solving $(*2)$ are same to those in Section 3.

Step 2. The weak solutions to homogeneous problem $(*2)$.
By Lemma 3.3, we define

$$\begin{aligned} [F(\widehat{u}, \widehat{u})] &:= \gamma(\triangledown\widehat{u}, \triangledown\widehat{u}) + (\widehat{u} \triangledown \widehat{u}, \widehat{u}) + (\widehat{u} \triangledown \psi, \widehat{u}) + (\psi \triangledown \widehat{u}, \widehat{u}) \\ &+(\alpha|\widehat{u} + \psi|^{\beta-1}(\widehat{u} + \psi), \widehat{u}) - (\widehat{f}, \widehat{u}) \\ &= \gamma\|\triangledown\widehat{u}\|_2^2 + (\widehat{u} \triangledown \psi, \widehat{u}) + (\alpha|\widehat{u} + \psi|^{\beta-1}(\widehat{u} + \psi), \widehat{u}) - (\widehat{f}, \widehat{u}). \end{aligned}$$

We hope

$$\begin{aligned} [F(\widehat{u}, \widehat{u})] &\geq \gamma\|\triangledown\widehat{u}\|_2^2 - \frac{\gamma}{2}\|\triangledown\widehat{u}\|_2^2 - c\alpha\|\triangledown\widehat{u}\|_2 - \|\widehat{f}\|_{H^{-1}}\|\triangledown\widehat{u}\|_2 \\ &= \frac{\gamma}{2}\|\triangledown\widehat{u}\|_2^2 - c\alpha\|\triangledown\widehat{u}\|_2 - \|\widehat{f}\|_{H^{-1}}\|\triangledown\widehat{u}\|_2 \\ &= (\frac{\gamma}{2}\|\triangledown\widehat{u}\|_2 - c\alpha - \|\widehat{f}\|_{H^{-1}})\|\triangledown\widehat{u}\|_2 > 0, \end{aligned}$$

if only there exists a constant K, such that $K > \frac{2(c\alpha + \|\widehat{f}\|_{H^{-1}})}{\gamma}$. So when

$$\|\triangledown\widehat{u}\|_2 \leq K, \tag{4.5}$$

by Lemma 3.3, the nonhomogeneous problem (1.1)-(1.3) is solvable. Now we begin to prove (4.5). Taking inner product between (4.4) and $\widehat{u}$, we can obtain

$$\gamma(\nabla\widehat{u},\nabla\widehat{u})+(\widehat{u}\nabla\widehat{u},\widehat{u})+(\widehat{u}\nabla\psi,\widehat{u})+(\psi\nabla\widehat{u},\widehat{u})+(\alpha|\widehat{u}+\psi|^{\beta-1}(\widehat{u}+\psi),\widehat{u})=(\widehat{f},\widehat{u}),$$

therefore, we have

$$\gamma\|\nabla\widehat{u}\|_2^2+(\widehat{u}\nabla\psi,\widehat{u})+(\alpha|\widehat{u}+\psi|^{\beta-1}(\widehat{u}+\psi),\widehat{u})=(\widehat{f},\widehat{u})\le\|\widehat{f}\|_{H^1}\|\nabla\widehat{u}\|_2.$$

If

$$|(\widehat{u}\nabla\psi,\widehat{u})|+|(\alpha|\widehat{u}+\psi|^{\beta-1}(\widehat{u}+\psi),\widehat{u})|<c\|\widehat{f}\|_{H^1}\|\nabla\widehat{u}\|_2, \tag{4.6}$$

we can obtain that $\|\nabla\widehat{u}\|_2$ is uniformly bounded, and (4.5) holds.

Step 3. Control the two terms on the left in (4.6) by $\|\nabla\widehat{u}\|_2$.

(1). For the first term on the left in (4.6), we need the following lemma.

Lemma 4.1. *For $\forall\ \gamma>0$, there exists a function $\psi=\psi(\gamma)$ satisfying*

$$\psi\in H^1(\Omega), div\psi=0, \psi=\phi, x\in\partial\,\Omega$$

such that

$$|(v\nabla\psi,v)|\le|v|_{1,2},\ \forall v\in H_0^1(\Omega). \tag{4.7}$$

To prove Lemma 4.1, we need the following two lemmas.

Lemma 4.2. *Let $\rho(x)=d(x,\partial\Omega)$ denote the distance between x and $\partial\Omega$. For $\forall\ \epsilon>0$, there exists a function $\theta_\epsilon\in C^2(\overline{\Omega})$ satisfying*

$$(*3)\begin{cases}\theta_\epsilon(x)=1, & \rho(x)\le\delta(\epsilon),\delta(\epsilon)=exp(-\frac{1}{\epsilon})\\ \theta_\epsilon(x)=0, & \rho(x)\ge2\delta(\epsilon),\\ |D_k\theta_\epsilon(x)|\le\frac{\epsilon}{\rho(x)}, & \rho(x)\le2\delta(\epsilon),k=1,2...,n\end{cases}$$

Proof. The proof of this Lemma is referred to [11].

We introduce a function $\lambda\to\xi_\epsilon(\lambda)$ satisfying

$$\xi_\epsilon(\lambda)=\begin{cases}1, & \lambda<\delta(\epsilon)^2\\ \epsilon\ log(\frac{\delta(\epsilon)}{\lambda}), & \delta(\epsilon)^2<\lambda<\delta(\epsilon)\\ 0, & \lambda>\delta(\epsilon)\end{cases}$$

and let $\chi_\epsilon(x)=\xi_\epsilon(\rho(x))$.

Lemma 4.3. *There exists a positive constant c_1 depending only Ω such that*

$$\|\frac{1}{\rho}v\|_{L^2(\Omega)}<c_1\|v\|_{H_0^1(\Omega)},\quad\forall v\in H_0^1(\Omega). \tag{4.8}$$

Proof. In term of the finite covering theorem and local coordinates near the boundary, we transform the given problem into the same problem on $\Omega = \{x = (x_n, x'), x_n > 0, x' = (x_1, ...; x_{n-1}) \in R^{n-1}\}$. For this case, $\rho(x) = \chi_n$, we only prove

$$\int_\Omega \frac{v(x)^2}{x_n^2} dx \le c_1 \int_\Omega |D_n v(x)|^2 dx, \quad \forall v \in C_0^\infty(\Omega). \tag{4.9}$$

First, we consider the Hardy inequality in $\mathbb{R}^1$:

$$\int_0^{+\infty} |\frac{v(s)}{s}|^2 ds \le 2 \int_0^{+\infty} |v'(s)|^2 ds, \forall v \in C_0^\infty(0, +\infty). \tag{4.10}$$

We assume that $s = e^\sigma, t = e^\tau$ and

$$\frac{v(s)}{s} = \frac{1}{s} \int_0^s w(t) dt, \ v' = w,$$

therefore we obtain

$$\int_0^{+\infty} \frac{|v(s)|^2}{|s|^2} dx = \int_{-\infty}^{+\infty} e^{-\sigma} (\int_0^{e^\sigma} w(t) dt)^2 d\sigma$$
$$= \int_{-\infty}^{+\infty} (\int_{-\infty}^{+\infty} Y(\sigma - \tau) e^{-(\sigma-\tau)/2} w(e^\tau) e^{\tau/2} d\tau)^2 d\sigma,$$

here, Y denote Heaviside function and when $\sigma > 0$, $Y(\sigma) = 1$; when $\sigma < 0$, $Y(\sigma) = 0$. Using convolution inequality, we can obtain

$$(\int_{-\infty}^{+\infty} Y(\sigma) e^{-\sigma/2} d\sigma)^2 \cdot \int_{-\infty}^{+\infty} |w(e^\tau)|^2 e^\tau d\tau = 4 \int_0^{+\infty} |w(t)|^2 dt.$$

So we have proved (4.9). Consequently, (4.8) holds. The proof of the Lemma 4.3 is complete.

Proof of Lemma 4.1. Let $\psi = \phi$, when $x \in (\rho(x) < \delta(\epsilon))$, using (4.1) and (4.2), we can deduce that (4.3) is solvable (for more details, please see [11]). Since

$$\psi_j(x) = 0, \quad (\rho(x) > 2\delta(\epsilon))$$

and

$$|\psi_j(x)| \le c_2 (\frac{\epsilon}{\rho(x)} |\zeta(x)| + |D\zeta(x)|), (\rho(x) \le 2\delta(\epsilon)) \tag{4.11}$$

here $|D\zeta(x)| = (\sum_{i,j=1}^n |D_i \zeta_j(x)|^2)^{1/2}$, furthermore, we assume that for $\forall j \in \mathbb{N}$, $\zeta_j \in L^\infty(\Omega)$, then

$$|\psi_j(x)| \le c_3 (\frac{\epsilon}{\rho(x)} + |D\rho(x)|), \quad (\rho(x) \le 2\delta(\epsilon)).$$

So we have

$$|v_j \psi_j|_{L^2} \le c_4 \{\epsilon |\frac{v_i}{\rho}|_{L^2} + (\int_{\rho \le 2\delta(\epsilon)} v_i^2 |D\zeta|^2 dx)^{1/2}\} \tag{4.12}$$

Using Hölder inequality, we can deduce

$$(\int_{\rho\le 2\delta(\epsilon)} v_i^2|D\zeta|^2dx)^{1/2} \le \mu(\epsilon)|v_i|_{L^3},$$

where $1/3 = 1/2-1/6$, $\mu(\epsilon) = (\int_{\rho(x)\le 2\delta(\epsilon)}|D\zeta(x)|^6dx)^{1/6}$. Since $D_i\zeta_j \in L^6(\Omega)$, $1 \le i, j \le 3$, one has $\mu(\epsilon) \to 0$ $(\epsilon \to 0)$. When $n \ge 3$, $|u|_{L^{n/(n-2)}}(\Omega) \le c(\Omega)\|u\|_{H_0^1(\Omega)}$, by Lemma 4.3 and (4.12), we can deduce

$$|v_i\psi_j|_{L^2(\Omega)} \le c_5(\epsilon|v|_{1,2}+\mu(\epsilon)|v|_{L^3}) \le c_6(\epsilon+\mu(\epsilon))|v|_{1,2}, \quad 1 \le i, j \le n. \tag{4.13}$$

Now we investigate (4.7). Since for $\forall v \in C_{0,\sigma}^\infty$, we have

$$(v\cdot\nabla\psi, v) = -(v\cdot\nabla v, \psi),$$

$$|(v\cdot\nabla v,\psi)| \le |v|_{1,2}(\sum_{i,j=1}^{n}|v_i\psi_j|) \le c_7(\epsilon+\mu(\epsilon))|v|_{1,2}^2 \tag{4.14}$$

If ϵ is small enough such that $c_7(\epsilon+\mu(\epsilon)) \le \frac{\gamma}{2}$, then for $\forall v \in C_{0,\sigma}^\infty(\Omega)$, (4.7) holds. Since $C_{0,\sigma}^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, (4.7) also holds for $\forall v \in H_0^1(\Omega)$. The proof of the Lemma 4.1. is complete. So we obtain $|(\widehat{u}\nabla\psi,\widehat{u})| \le \frac{\gamma}{2}\|\nabla\widehat{u}\|_2^2 = \frac{\gamma}{2}|\widehat{u}|_{1,2}^2$.

(2). For the second term on the left in (4.6), we have

$$\begin{aligned}
&|(\alpha|\widehat{u}+\psi|^{\beta-1}(\widehat{u}+\psi),\widehat{u})| \le \alpha\||\widehat{u}+\psi|^\beta\|_{\frac{\beta+1}{\beta}}\|\widehat{u}\|_{\beta+1}\\
&\le c_1\alpha\||\widehat{u}+\psi|\|_{\beta+1}^\beta\|\widehat{u}\|_6\\
&\le c_1\alpha(\|\widehat{u}\|_{\beta+1}+\|\psi\|_{\beta+1})^\beta\|\widehat{u}\|_6\\
&\le c_1\alpha(\|\widehat{u}\|_6+c_2\|\psi\|_6)^\beta\|\widehat{u}\|_6\\
&\le c\alpha\|\widehat{u}\|_6 \le c\alpha\|\nabla\widehat{u}\|_2,
\end{aligned}$$

here we have used $\beta+1 \le 6$, then $1 \le \beta \le 5$.

By (1) and (2), if

$$\frac{\gamma}{2}\|\nabla\widehat{u}\|_2^2 + c\alpha\|\nabla\widehat{u}\|_2 \le c\|\widehat{f}\|_{H^{-1}}\|\nabla\widehat{u}\|_2,$$

that is $\frac{\gamma}{2}\|\nabla\widehat{u}\|_2^2 \le c(\|\widehat{f}\|_{H^{-1}}\|-\alpha)\|\nabla\widehat{u}\|_2$, i.e., $\|\nabla\widehat{u}\|_2 \le 2c(\|\widehat{f}\|_{H^{-1}}\|-\alpha)/\gamma$. According to the theory in Section 3, the equations $(*2)$ is solvable. The proof of Theorem 2.4 is complete.

§4.2. Uniqueness of weak solutions

In this part, we consider the uniqueness of the weak solutions for problem (1.1)-(1.3) with nonhomogeneous boundary condition ($\phi \ne 0$) in a bounded domain Ω.

Proof of Theorem 2.5. Assume that u_0 and u_1 are the two weak solutions of the equations $(*1)$, let $\widehat{u_0} = u_0-\phi$, $\widehat{u_1} = u_1-\phi$, $\widehat{u} = \widehat{u_0}-\widehat{u_1}$, then we have

$$\gamma(\nabla\widehat{u_0},\nabla v)+(\widehat{u_0}\cdot\nabla\widehat{u_0}, v)+(\widehat{u_0}\cdot\nabla\phi, v)+(\phi\cdot\nabla\widehat{u_0}, v)+\alpha(|\widehat{u_0}+\phi|^{\beta-1}(\widehat{u_0}+\phi), v) = (\widehat{f}, v).$$

$$\gamma(\nabla\widehat{u}_1,\nabla v)+(\widehat{u}_1\cdot\nabla\widehat{u}_1,v)+(\widehat{u}_1\cdot\nabla\phi,v)+(\phi\cdot\nabla\widehat{u}_1,v)+\alpha(|\widehat{u}_1+\phi|^{\beta-1}(\widehat{u}_1+\phi),v)=(\widehat{f},v).$$

for $\forall v\in C^{\infty}_{0,\sigma}$ and $\widehat{f}=f+\gamma\triangle\phi-\phi\nabla\phi$. Let $v=\widehat{u}=\widehat{u}_0-\widehat{u}_1$, we obtain

$$\gamma|\widehat{u}|^2_{1,2}+(\widehat{u}\cdot\nabla\widehat{u}_0,\widehat{u})+b(\widehat{u},\phi,\widehat{u})+\alpha(|\widehat{u}_0+\phi|^{\beta-1}(\widehat{u}_0+\phi)-|\widehat{u}_1+\phi|^{\beta-1}(\widehat{u}_1+\phi),\widehat{u})=0,$$

that is

$$\gamma|\widehat{u}|^2_{1,2}=-b(\widehat{u},\widehat{u}_0,\widehat{u})-b(\widehat{u},\phi,\widehat{u})-\alpha(|\widehat{u}_0+\phi|^{\beta-1}(\widehat{u}_0+\phi)-|\widehat{u}_1+\phi|^{\beta-1}(\widehat{u}_1+\phi),\widehat{u}),$$

where $b(u,v,w)=(u\cdot\nabla v,w)$. By Lemma 4.1, one has

$$\gamma|\widehat{u}|^2_{1,2}\le c(n)|\widehat{u}_0|_{1,2}|\widehat{u}|_{1,2}+\frac{\gamma}{2}|\widehat{u}|_{1,2}+J_1. \tag{4.15}$$

Let $J_1=\alpha(|\widehat{u}_0+\phi|^{\beta-1}(\widehat{u}_0+\phi)-|\widehat{u}_1+\phi|^{\beta-1}(\widehat{u}_1+\phi),\widehat{u})$, we deduce that

$$\begin{aligned}
&J_1=\alpha(|\widehat{u}_0+\phi|^{\beta-1}(\widehat{u}_0+\phi)-|\widehat{u}_1+\phi|^{\beta-1}(\widehat{u}_1+\phi),\widehat{u})\\
&=\alpha(|\widehat{u}_0+\phi|^{\beta-1}(\widehat{u}_0+\phi)-|\widehat{u}_0+\phi|^{\beta-1}(\widehat{u}_1+\phi)\\
&+|\widehat{u}_0+\phi|^{\beta-1}(\widehat{u}_1+\phi)-|\widehat{u}_1+\phi|^{\beta-1}(\widehat{u}_1+\phi),\widehat{u})\\
&=\alpha(|\widehat{u}_0+\phi|^{\beta-1}\cdot\widehat{u},\widehat{u})+\alpha((|\widehat{u}_0+\phi|^{\beta-1}-|\widehat{u}_1+\phi|^{\beta-1})(\widehat{u}_1+\phi),\widehat{u})\\
&=J_2+J_3.
\end{aligned}$$

Now we begin to estimate J_2 and J_3 respectively.

$$\begin{aligned}
J_2&\le\alpha\||\widehat{u}_0+\phi|^{\frac{\beta-1}{2}}|\widehat{u}_0-\widehat{u}_1|\|_2^2\\
&\le\|(\widehat{u}_0+\phi)^{\frac{\beta-1}{2}}\|_3^2\|\widehat{u}\|_6^2\\
&\le\|(\widehat{u}_0+\phi)\|^{\beta-1}_{\frac{3(\beta-1)}{2}}|\widehat{u}|^2_{1,2}\\
&\le c(\|\widehat{u}_0\|_{\frac{3(\beta-1)}{2}}+\|\phi\|_{\frac{3(\beta-1)}{2}})^{\beta-1}|\widehat{u}|^2_{1,2}\\
&\le c(\|\widehat{u}_0\|_6+\|\phi\|_{\frac{3(\beta-1)}{2}})^{\beta-1}|\widehat{u}|^2_{1,2}\\
&\le c(\frac{\|\widehat{f}\|_{H^{-1}}+\alpha}{\gamma}+\|\phi\|_6)^{\beta-1}|\widehat{u}|^{1,2}_{1,2},
\end{aligned}$$

where $\frac{3(\beta-1)}{2}\le 6$, we obtain $1\le\beta\le 5$. After a tedious computation , we have

$$\begin{aligned}
&J_3=\alpha((|\widehat{u}_0+\phi|^{\beta-1}-|\widehat{u}_1+\phi|^{\beta-1})(\widehat{u}_1+\phi),\widehat{u})\\
&\le c\alpha\beta\||\widehat{u}_0+\phi|^{\beta-1}+|\widehat{u}_1+\phi|^{\beta-1}\|_{\frac{3}{2}}\|\widehat{u}^2\|_3+\alpha\|(\widehat{u}_0+\phi)^{\beta-1}\|_{\frac{3}{2}}\|\widehat{u}\|_6^2\\
&\le c\alpha\beta(\|\widehat{u}_0\|^{\beta-1}_{\frac{3(\beta-1)}{2}}+\|\phi\|^{\beta-1}_{\frac{3(\beta-1)}{2}}+\|\widehat{u}_1\|^{\beta-1}_{\frac{3(\beta-1)}{2}})\|\widehat{u}\|_6^2\\
&+\alpha(\|\widehat{u}_0\|^{\beta-1}_{\frac{3(\beta-1)}{2}}+\|\phi\|^{\beta-1}_{\frac{3(\beta-1)}{2}})\|\widehat{u}\|_6^2\\
&=c\alpha\beta(\|\widehat{u}_0\|^{\beta-1}_{\frac{3(\beta-1)}{2}}+\|\widehat{u}_1\|^{\beta-1}_{\frac{3(\beta-1)}{2}}+\|\phi\|^{\beta-1}_{\frac{3(\beta-1)}{2}})|\widehat{u}|^2_{1,2},
\end{aligned}$$

where we have used $1 \le \frac{3(\beta-1)}{2} \le 6$, i.e., $\frac{5}{3} \le \beta \le 5$. Substituting the above results into (4.15), we can obtain

$$\gamma|\widehat{u}|_{1,2}^2 \le c(n)|\widehat{u_0}|_{1,2}|\widehat{u}|_{1,2}^2 + \frac{\gamma}{2}|\widehat{u}|_{1,2}^2 + c(\frac{\|\widehat{f}\|_{H^{-1}} + \alpha}{\gamma} + \|\phi\|_6^{\beta-1})|\widehat{u}|_{1,2}^2$$
$$+c\alpha\beta(\frac{\|\widehat{f}\|_{H^{-1}} + \alpha}{\gamma} + \|\phi\|_6^{\beta-1})|\widehat{u}|_{1,2}^2.$$

Furthermore, we deduce that

$$(\frac{\gamma}{2} - c(n)|\widehat{u_0}|_{1,2} - c(\frac{\|\widehat{f}\|_{H^{-1}} + \alpha}{\gamma} + \|\phi\|_6^{\beta-1}) - c\alpha\beta(\frac{\|\widehat{f}\|_{H^{-1}} + \alpha}{\gamma} + \|\phi\|_6^{\beta-1}))|\widehat{u}|_{1,2}^2 \le 0,$$

when γ is big enough, we have $\widehat{u} = 0, a.e.$. The proof of Theorem 2.5 is complete.

References

[1] D.Bresch, B.Desjardins, Existence of global weak solutions for a 2D viscous shallow water equations and covergence to the quasigeostrophic model. *Comm. Math. Phys.* **238**(1-2) (2003), 221-223.

[2] D. Bresch, B. Desjardins, Chi-Kun Lin, On some compressible fluid models: Korteweg, lubrication, and shallow water systems. *Comm. Partial Differential Equations.* **28** (3-4) (2003), 843-868.

[3] X. J. Cai, Q. S. Jiu, Weak and strong solutions for the incompressible Navier-Stokes equations with damping. *J. Math. Anal. Appl.* **343** (2008), 799-809.

[4] Galdi. G. P, An Introduction to the Mathematical Theory of the Navier-Stokes Equations. *Vol. Linearized Steady Problems.* Springer-Verlag. NewYork. Berlin. etc. 1994.

[5] L.Hsiao, *Quasilinear Hyperbolic Systems and Dissipative Mechanisims*, World Scientific, 1997.

[6] F. M. Huang, R. H. Pan, Convergence rate for compressible Euler equations with damping and vaccum. *Arch. Ration. Mech. Anal.* **166** (2003), 359-367.

[7] LERAY, J. Etude de Diverses Équations Intégrales on Linéaires et de Quelques Problémes que Pose I'Hydrodynamique. *J. Math. Pures Appl.* 12, 1-82 [Introduction to VIII, VIII.3, VIII.4, Introduction to IX, IX. 4, Notes for X], 1933.

[8] K. Masuda, Weak solutions of Navier-Stokes equations. *Tohoku Math. J.* **36**(1984), 623-646.

[9] H. Sohr, *The Navier-Stokes Equations: an elementary funcational analytic approach*, Birkhäuser, 2001.

[10] Struwe, M. Regular Solutions of the Stationary Navier-Stokes Equations on R^5, *Math. Annalen.* 302, 719-741[Introduction to VIII], 1995.

[11] R. Teman , Navier-Stokes equations theory and numerical analysis. *North-Holland-Amsterdam.* New York, Oxford, 1984.

[12] Youdovich, V.I. An Example of Loss of Stability and Generation of Secondary Flow in a Closed Vessel. *Mat .Sb.* 74, 306-329; English Transl: *Math. USSR Sbornik.* 3, 1967, 519-533 [Introduction to VIII, VIII.2].