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# Interior Controllability of the $nD$ Semilinear Heat Equation

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**Abstract**

In this paper we prove the interior approximate controllability of the following Semilinear Heat Equation

$$\begin{cases} z_t(t,x) = \Delta z(t,x) + 1_\omega u(t,x) + f(t,z,u(t,x)) & \text{in} \quad (0,\tau]\times\Omega, \\ z = 0, & \text{on} \quad (0,\tau)\times\partial\Omega, \\ z(0,x) = z_0(x), & x\in\Omega, \end{cases}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^N (N \geq 1)$, $z_0 \in L^2(\Omega)$, $\omega$ is an open nonempty subset of $\Omega$, $1_\omega$ denotes the characteristic function of the set $\omega$,the distributed control $u$ belong to $\in L^2([0,\tau]; L^2(\Omega;))$ and the nonlinear function $f : [0,\tau]\times I\!R\times I\!R \to I\!R$ is smooth enough and there are $a,b,c \in I\!R$, with $c \neq -1$, such that

$$\sup_{(t,z,u)\in Q_\tau} |f(t,z,u) - az - cu - b| < \infty,$$

where $Q_\tau = [0,\tau]\times I\!R\times I\!R$. Under this condition we prove the following statement: For all open nonempty subset $\omega$ of $\Omega$ the system is approximately controllable on $[0,\tau]$. Moreover, we could exhibit a sequence of controls steering the nonlinear system (1.1) from an initial state $z_0$ to an $\epsilon$ neighborhood of the final state $z_1$ at time $\tau > 0$, which is very important from a practical and numerical point of view.

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# 1 Introduction

In this paper we prove the interior approximate controllability of the following Semilinear Heat Equation

$$\begin{cases} z_t(t,x) = \Delta z(t,x) + 1_\omega u(t,x) + f(t,z,u(t,x)) & \text{in} \quad (0,\tau]\times\Omega, \\ z = 0, & \text{on} \quad (0,\tau)\times\partial\Omega, \\ z(0,x) = z_0(x), & x\in\Omega, \end{cases} \tag{1.1}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^N (N \geq 1)$, $z_0 \in L^2(\Omega)$, $\omega$ is an open nonempty subset of $\Omega$, $1_\omega$ denotes the characteristic function of the set $\omega$,the distributed control $u$ belong to $\in L^2([0,\tau]; L^2(\Omega;))$ and the nonlinear function $f : [0,\tau]\times I\!R\times I\!R \to I\!R$ is smooth enough and there are $a, b, c \in I\!R$, with $c \neq -1$, such that

$$\sup_{(t,z,u)\in Q_\tau} |f(t,z,u) - az - cu - b| < \infty, \tag{1.2}$$

where $Q_\tau = [0,\tau]\times I\!R\times I\!R$. Under this condition we prove the following statement: For all open nonempty subset $\omega$ of $\Omega$ the system is approximately controllable on $[0,\tau]$. Moreover, we could exhibit a sequence of controls steering the nonlinear system (1.1) from an initial state $z_0$ to an $\epsilon$ neighborhood of the final state $z_1$ at time $\tau > 0$, which is very important from a practical and numerical point of view.

We note that, the interior approximate controllability of the linear heat equation

$$\begin{cases} z_t(t,x) = \Delta z(t,x) + 1_\omega u(t,x) & \text{in} \quad (0,\tau]\times\Omega, \\ z = 0, & \text{on} \quad (0,\tau)\times\partial\Omega, \\ z(0,x) = z_0(x), & x\in\Omega, \end{cases} \tag{1.3}$$

has been study by several authors, particularly by [16],[17],[18]; and in a general fashion in [14].

The approximate controllability of the heat equation under non linear perturbation $f(z)$ independents of $t$ and $u$ variable

$$\begin{cases} z_t(t,x) = \Delta z(t,x) + 1_\omega u(t,x) + f(z) & \text{in} \quad (0,\tau]\times\Omega, \\ z = 0, & \text{on} \quad (0,\tau)\times\partial\Omega, \\ z(0,x) = z_0(x), & x\in\Omega, \end{cases} \tag{1.4}$$

has been studied by several authors, particularly in [8], [9] and [10], depending on conditions impose to the nonlinear term $f(z)$. For instance, in [9] and [10] the approximate controllability of the system (1.4) is proved if $f(z)$ is sublinear at infinity, i.e.,

$$|f(z)| \leq d|z| + e. \tag{1.5}$$

Also, in the above reference, they mentioned that when $f$ is superlinear at the infinity, the approximate controllability of system (1.4) fail.

In this paper we use different technique for the linear part (see [14]) and Schauder fixed point Theorem for the semilinear system. Moreover, we find a sequence of control steering the semilinear system (1.1) from an initial state $z_0$ to a $\epsilon$-neighborhood of the final state $z_1$

on time $\tau > 0$.
Now, we shall describe the strategy of this work:
First, we observe that the hypothesis (1.2) is equivalent to the existence of $a, c \in I\!R$, with $c \neq -1$, such that

$$\sup_{(t,z,u)\in Q_\tau} |f(t,z,u) - az - cu| < \infty, \tag{1.6}$$

where $Q_\tau = [0,\tau] \times I\!R \times I\!R$.
Second, we prove that the linear system

$$\begin{cases} z_t(t,x) = \Delta z(t,x) + 1_\omega u(t,x) + az + cu(t,x) & \text{in} \quad (0,\tau] \times \Omega, \\ z = 0, & \text{on} \quad (0,\tau) \times \partial\Omega, \\ z(0,x) = z_0(x), & x \in \Omega, \end{cases} \tag{1.7}$$

is approximately controllable.
After that, we write the system(1.1) as follows

$$\begin{cases} z_t(t,x) = \Delta z(t,x) + 1_\omega u(t,x) + az + cu(t,x) + g(t,z,u) & \text{in} \quad (0,\tau] \times \Omega, \\ z = 0, & \text{on} \quad (0,\tau) \times \partial\Omega, \\ z(0,x) = z_0(x), & x \in \Omega, \end{cases} \tag{1.8}$$

where $g(t,z,u) = f(t,z,u) - az - cu$ is an smooth and bounded function.
The technique we use here to prove the controllability of the linear equation (1.7) is based in the following results:

Theorem 1.1. *(see Theorem 1.23 from [2], pg. 20) Suppose $\Omega \subset I\!R^n$ is open, non-empty and connected set, and $f$ is real analytic function in $\Omega$ with $f = 0$ on a non-empty open subset $\omega$ of $\Omega$. Then, $f = 0$ in $\Omega$.*

Lemma 1.1. *(see Lemma 3.14 from [6], pg. 62) Let $\{\alpha_j\}_{j\geq 1}$ and $\{\beta_{i,j} : i = 1,2,\ldots,m\}_{j\geq 1}$ be two sequences of real numbers such that: $\alpha_1 > \alpha_2 > \alpha_3 \cdots$. Then*

$$\sum_{j=1}^{\infty} e^{\alpha_j t}\beta_{i,j} = 0, \quad \forall t \in [0,t_1], \quad i = 1,2,\cdots,m$$

*iff*

$$\beta_{i,j} = 0, \quad i = 1,2,\cdots,m; j = 1,2,\cdots,\infty.$$

Finally, the approximate controllability of the system (1.8) follows from the controllability of (1.7), the compactness of the semigroup generated by the Laplacean operator $\Delta$ and the uniform boundedness of the nonlinear term $g$ by applying Schauder fixed point Theorem.

## 2 Abstract Formulation of the Problem.

In this section we choose a Hilbert space where system (1.1) can be written as an abstract differential equation; to this end, we consider the following notations:

Let us consider the Hilbert space $Z = L^2(\Omega)$ and $0 < \lambda_1 < \lambda_2 < ... < \lambda_j \longrightarrow \infty$ the eigenvalues of $-\Delta$, each one with finite multiplicity $\gamma_j$ equal to the dimension of the corresponding eigenspace. Then we have the following well known properties
(i) There exists a complete orthonormal set $\{\phi_{j,k}\}$ of eigenvectors of $A = -\Delta$.
(ii) For all $z \in D(A)$ we have

$$Az = \sum_{j=1}^{\infty} \lambda_j \sum_{k=1}^{\gamma_j} < \xi, \phi_{j,k} > \phi_{j,k} = \sum_{j=1}^{\infty} \lambda_j E_j z, \tag{2.9}$$

where $< \cdot, \cdot >$ is the inner product in $Z$ and

$$E_n z = \sum_{k=1}^{\gamma_j} < z, \phi_{j,k} > \phi_{j,k}. \tag{2.10}$$

So, $\{E_j\}$ is a family of complete orthogonal projections in $Z$ and $z = \sum_{j=1}^{\infty} E_j z, \ \ z \in H$.
(iii) $-A$ generates an analytic semigroup $\{T(t)\}$ given by

$$T(t)z = \sum_{j=1}^{\infty} e^{-\lambda_j t} E_j z. \tag{2.11}$$

Consequently, systems (1.1), (1.7) and (1.8) can be written respectively as an abstract differential equations in $Z$:

$$z' = -Az + B_\omega u + f^e(t,z,u), \quad z \in Z \quad t \geq 0, \tag{2.12}$$

$$z' = -Az + B_\omega u + az + cu, \quad z \in Z \quad t \geq 0, \tag{2.13}$$

$$z' = -Az + B_\omega u + az + cu + g^e(t,z,u), \quad z \in Z \quad t \geq 0, \tag{2.14}$$

where $u \in L^2([0,\tau]; U)$, $U = Z$, $B_\omega : U \longrightarrow Z$, $B_\omega u = 1_\omega u$ is a bounded linear operator, $f^e : [0,\tau] \times Z \times U \to Z$, is defined by $f^e(t,z,u)(x) = f(t,z(x),u(x))$, $\forall x \in \Omega$ and $g^e(t,z,u) = f^e(t,z,u) - az - cu$. On the other hand, the hypothesis (1.2) implies that

$$\sup_{(t,z,u) \in Z_\tau} \|f^e(t,z,u) - az - cu\|_Z < \infty, \tag{2.15}$$

where $Z_\tau = [0,\tau] \times Z \times U$. Therefore, $g^e : [0,\tau] \times Z \times U \to Z$ is bounded and smooth enough.

## 3 Interior Controllability of the Linear Equation

In this section we shall prove the interior controllability of the linear system (2.13). But, before we shall give the definition of approximate controllability for this system. To this end, for all $z_0 \in Z$ and $u \in L^2(0,\tau; U)$ the initial value problem

$$\begin{cases} z' = -Az + B_\omega u(t) + az(t) + cu(t), & z \in Z, \\ z(0) = z_0, \end{cases} \tag{3.16}$$

where the control function $u$ belong to $L^2(0,\tau; U)$, admits only one mild solution given by

$$z(t) = e^{at} T(t) z_0 + \int_0^t e^{a(t-s)} T(t-s)(B_\omega + aI) u(s) ds, \ \ t \in [0,\tau]. \tag{3.17}$$

Definition 3.1. *(**Approximate Controllability**) The system (2.13) is said to be approximately controllable on* $[0,\tau]$ *if for every* $z_0$, $z_1 \in Z$, $\varepsilon > 0$ *there exists* $u \in L^2(0,\tau;U)$ *such that the solution* $z(t)$ *of (3.17) corresponding to* $u$ *verifies:*

$$\|z(\tau) - z_1\| < \varepsilon.$$

Definition 3.2. *For the system (2.13) we define the following concept: The controllability map (for* $\tau > 0$*)* $G_{\mathbf{a}} : L^2(0,\tau;U) \longrightarrow Z$ *is given by*

$$G_{\mathbf{a}}u = \int_0^\tau e^{as}T(s)(B_\omega + aI)u(s)ds. \tag{3.18}$$

whose adjoint operator $G^*_{\mathbf{a}} : Z \longrightarrow L^2(0,\tau;Z)$ is given by

$$(G^*_{\mathbf{a}}z)(s) = (B^*_\omega + aI)e^{as}T^*(s)z, \quad \forall s \in [0,\tau], \quad \forall z \in Z. \tag{3.19}$$

The following lemma holds in general for a linear bounded operator $G : W \to Z$ between Hilbert spaces $W$ and $Z$.

Lemma 3.1. *(see [6], [7], [1] and [14]) The equation (2.13) is approximately controllable on* $[0,\tau]$ *if, and only if, one of the following statements holds:*

*a)* $\overline{\mathrm{Rang}(G_{\mathbf{a}})} = Z$.

*b)* $Ker(G^*_{\mathbf{a}}) = \{0\}$.

*c)* $\langle G_{\mathbf{a}}G^*_{\mathbf{a}}z, z\rangle > 0$, $z \neq 0$ *in* $Z$.

*d)* $\lim_{\alpha\to 0^+} \alpha(\alpha I + G_{\mathbf{a}}G^*_{\mathbf{a}})^{-1}z = 0$.

*e)* $\sup_{\alpha>0} \|\alpha(\alpha I + G_{\mathbf{a}}G^*_{\mathbf{a}})^{-1}\| \leq 1$.

*f)* $(B^*_\omega + aI)e^{at}T^*(t)z = 0, \quad \forall t \in [0,\tau], \quad \Rightarrow z = 0$.

*g) For all* $z \in Z$ *we have* $Gu_\alpha = z - \alpha(\alpha I + G_{\mathbf{a}}G^*_{\mathbf{a}})^{-1}z$, *where*

$$u_\alpha = G^*_{\mathbf{a}}(\alpha I + G_{\mathbf{a}}G^*_{\mathbf{a}})^{-1}z, \quad \alpha \in (0,1].$$

*So,* $\lim_{\alpha\to 0} G_{\mathbf{a}}u_\alpha = z$ *and the error* $E_\alpha z$ *of this approximation is given by*

$$E_\alpha z = \alpha(\alpha I + G_{\mathbf{a}}G^*_{\mathbf{a}})^{-1}z, \quad \alpha \in (0,1].$$

Remark 3.1. *The Lemma 3.1 implies that the family of linear operators* $\Gamma_\alpha : Z \to L^2(0,\tau;U)$, *defined for* $0 < \alpha \leq 1$ *by*

$$\Gamma_\alpha z = (B^*_\omega + aI)e^{a(\cdot)}T^*(\cdot)(\alpha I + G_{\mathbf{a}}G^*_{\mathbf{a}})^{-1}z = G^*_{\mathbf{a}}(\alpha I + G_{\mathbf{a}}G^*_{\mathbf{a}})^{-1}z, \tag{3.20}$$

*is an approximate inverse for the right of the operator* $G_{\mathbf{a}}$ *in the sense that*

$$\lim_{\alpha\to 0} G_{\mathbf{a}}\Gamma_\alpha = I. \tag{3.21}$$

THEOREM 3.1. *The system (2.13) is approximately controllable on* $[0,\tau]$. *Moreover, a sequence of controls steering the system (2.13) from initial state* $z_0$ *to an* $\epsilon$ *neighborhood of the final state* $z_1$ *at time* $\tau > 0$ *is given by*

$$u_\alpha(t) = (B^*_\omega + aI)e^{at}T^*(t)(\tau - t)(\alpha I + G_{\mathbf{a}}G^*_{\mathbf{a}})^{-1}(z_1 - T(\tau)z_0),$$

*and the error of this approximation* $E_\alpha$ *is given by*

$$E_\alpha = \alpha(\alpha I + G_{\mathbf{a}}G^*_{\mathbf{a}})^{-1}(z_1 - T(\tau)z_0).$$

**Proof** . It is enough to show that the restriction $G_{\mathbf{a},\omega} = G_{\mathbf{a}}|_{L^2(0,\tau;L^2(\omega))}$ of $G_{\mathbf{a}}$ to the space $L^2(0,\tau;L^2(\omega))$ has range dense. i.e., $\overline{\mathrm{Rang}(G_{\mathbf{a},\omega})} = Z$ or $Ker(G^*_{\mathbf{a},\omega}) = \{0\}$. Consequently, $G_{\mathbf{a},\omega} : L^2(0,\tau;L^2(\omega)) \to Z$ takes the following form

$$G_{\mathbf{a},\omega}u = \int_0^\tau e^{as}T(s)(1 + cI)B_\omega u(s)ds.$$

whose adjoint operator $G^*_{\mathbf{a},\omega} : Z \longrightarrow L^2(0,\tau;L^2(\omega))$ is given by

$$(G_{\mathbf{a},\omega}z)(s) = (1 + c)B^*_\omega e^{as}T^*(s)z, \quad \forall s \in [0,\tau], \quad \forall z \in Z.$$

To this end, we observe that $B_\omega = B^*_\omega$ and $T^*(t) = T(t)$. Suppose that

$$(1 + c)B^*_\omega e^{at}T^*(t)z = 0, \ \ \forall t \in [0,\tau].$$

Then, since $1 + c \neq 0$, this is equivalents to

$$B^*_\omega T^*(t)z = 0, \ \ \forall t \in [0,\tau].$$

On the other hand,

$$\begin{aligned} B^*_\omega T^*(t)z \quad &= \quad \sum_{j=1}^{\infty} e^{-\lambda_j t}B^*_\omega E_j z = \sum_{j=1}^{\infty} e^{-\lambda_j t}\sum_{k=1}^{\gamma_j} < z, \phi_{j,k} > 1_\omega \phi_{j,k} = 0. \\ &\Longleftrightarrow \quad \sum_{j=1}^{\infty} e^{-\lambda_j t}\sum_{k=1}^{\gamma_j} < z, \phi_{j,k} > 1_\omega \phi_{j,k}(x) = 0, \quad \forall x \in \omega. \end{aligned}$$

Hence, from Lemma 1.1, we obtain that

$$E_j z(x) = \sum_{k=1}^{\gamma_j} < z, \phi_{j,k} > \phi_{j,k}(x) = 0, \quad \forall x \in \omega, \quad j = 1,2,3,\ldots.$$

Since $\phi_{j,k}$ are analytic functions on $\Omega$, from Theorem 1.1, we obtain that

$$E_j z(x) = \sum_{k=1}^{\gamma_j} < z, \phi_{j,k} > \phi_{j,k}(x) = 0, \quad \forall x \in \Omega, \quad j = 1,2,3,\ldots.$$

Therefore, $E_j z = 0, \quad j = 1,2,3,\ldots,$ which implies that $z = 0$. So, $\overline{\mathrm{Rang}(G_{\mathbf{a},\omega})} = Z$, and consequently $\overline{\mathrm{Rang}(G_{\mathbf{a}})} = Z$. Hence, the system (2.13) is approximately controllable on $[0,\tau]$, and the remainder of the proof follows from Lemma 3.1. □

## 4 Controllability of the Semilinear System

In this section we shall prove the main result of this paper, the interior controllability of the semilinear $nD$ heat equation given by (1.1), which is equivalent to prove the approximate controllability of the system (2.14). To this end, for all $z_0 \in Z$ and $u \in L^2(0,\tau;U)$ the initial value problem

$$\begin{cases} z' = -Az + B_\omega u + az + cu + g^e(t,z,u), & z \in Z \quad t \geq 0 \\ z(0) = z_0 \end{cases} \tag{4.22}$$

where the control function $u$ belong to $L^2(0,\tau;U)$ admits only one mild solution given by

$$\begin{aligned} z_u(t) &= e^{at}T(t)z_0 + \int_0^t e^{a(t-s)}T(t-s)(B_\omega + cI)u(s)ds \\ &+ \int_0^t e^{a(t-s)}T(t-s)g^e(s,z_u(s),(s))ds, \ \ t \in [0,\tau]. \end{aligned} \tag{4.23}$$

Definition 4.1. *(**Approximate Controllability**) The system (2.14) is said to be approximately controllable on* $[0,\tau]$ *if for every* $z_0$, $z_1 \in Z$, $\varepsilon > 0$ *there exists* $u \in L^2(0,\tau;U)$ *such that the solution* $z(t)$ *of (4.23) corresponding to u verifies:*

$$\|z(\tau) - z_1\| < \varepsilon.$$

Definition 4.2. *For the system (2.14) we define the following concept: The nonlinear controllability map (for* $\tau > 0$*)* $G_g : L^2(0,\tau;U) \longrightarrow Z$ *is given by*

$$G_g u = \int_0^\tau e^{as}T(s)(B_\omega + cI)u(s)ds + \int_0^\tau e^{as}T(s)g^e(s,z_u(s),(s))ds = G(u) + H(u), \tag{4.24}$$

*where* $H : L^2(0,\tau;U) \longrightarrow Z$ *is the nonlinear operator given by*

$$H(u) = \int_0^\tau e^{as}T(s)g^e(s,z_u(s),(s))ds, \ \ u \in L^2(0,\tau;U) \tag{4.25}$$

The following lemma is trivial:

Lemma 4.1. *The equation (2.14) is approximately controllable on* $[0,\tau]$ *if and only if* $\overline{\mathrm{Rang}(G_g)} = Z$.

Definition 4.3. *The following equation will be called the controllability equations associated to the non linear equation (2.14)*

$$u_\alpha = \Gamma_\alpha(z - H(u_\alpha)) = G_{\mathbf{a}}^*(\alpha I + G_{\mathbf{a}}G_{\mathbf{a}}^*)^{-1}(z - H(u_\alpha)), \quad (0 < \alpha \leq 1). \tag{4.26}$$

Now, we are ready to present and prove the main result of this paper, which is interior approximate controllability of the semilinear $nD$ heat equation (1.1)

THEOREM 4.1. *The system (2.14) is approximately controllable on* $[0,\tau]$. *Moreover, a sequence of controls steering the system (2.14) from initial state* $z_0$ *to an* $\epsilon$ *neighborhood of the final state* $z_1$ *at time* $\tau > 0$ *is given by*

$$u_\alpha(t) = (B^*_\omega + cI)e^{a(\tau-t)}T^*(\tau-t)(\alpha I + G_{\mathbf{a}}G^*_{\mathbf{a}})^{-1}(z_1 - T(\tau)z_0 - H(u_\alpha)),$$

*and the error of this approximation* $E_\alpha$ *is given by*

$$E_\alpha = \alpha(\alpha I + G_{\mathbf{a}}G^*_{\mathbf{a}})^{-1}(z_1 - T(\tau)z_0 - H(u_\alpha)).$$

**Proof** For each $z \in Z$ fixed we shall consider the following family of nonlinear operators $K_\alpha : L^2(0,\tau;U) \to L^2(0,\tau;U)$ given by

$$K_\alpha(u) = \Gamma_\alpha(z - H(u)) = G^*_{\mathbf{a}}(\alpha I + G_{\mathbf{a}}G^*_{\mathbf{a}})^{-1}(z - H(u)), \quad (0 < \alpha \le 1). \tag{4.27}$$

First, we shall prove that for all $\alpha \in (0,1]$ the operator $K_\alpha$ has a fixed point $u_\alpha$. In fact, since the semigroup $\{T(t)\}_{t\ge 0}$ given by (2.11) is compact (see [4],[5]), then using the result from [3], the smoothness and the bounddeness of the non linear term $g^e$ we obtain that the operator $H$ is compact and $\overline{\text{Rang}(H)}$ is compact.

On the other hand, since $g^e$ is bounded and $\|T(t)\| \le Re^{Wt}, \quad t \ge 0$, there exists a constant $R > 0$ such that

$$\|H(u)\| \le M, \ \ \forall u \in L^2(0,\tau;U).$$

Then,

$$\|K_\alpha(u)\| \le \|\Gamma_\alpha\|(\|z\| + M), \ \ \forall u \in L^2(0,\tau;U).$$

Therefore, the operator $K_\alpha$ maps the ball $B_r(0) \subset L^2(0,\tau;U)$ of center zero and radio $r \ge \|\Gamma_\alpha\|(\|z\| + M)$ into itself. Hence, applying the Schauder fixed point Theorem we get that the operator $K_\alpha$ has a fixed point $u_\alpha \in B_r(0) \subset L^2(0,\tau;U)$.
Since $\overline{\text{Rang}(H)}$ is compact, without loss of generality, we can assume that the sequence $H(u_\alpha)$ converges to $y \in Z$. So, if

$$u_\alpha = \Gamma_\alpha(z - H(u)) = G^*_{\mathbf{a}}(\alpha I + G_{\mathbf{a}}G^*_{\mathbf{a}})^{-1}(z - H(u_\alpha)).$$

Then,

$$\begin{aligned} G_{\mathbf{a}}u_\alpha &= G_{\mathbf{a}}\Gamma_\alpha(z - H(u)) = G_{\mathbf{a}}G^*_{\mathbf{a}}(\alpha I + G_{\mathbf{a}}G^*_{\mathbf{a}})^{-1}(z - H(u_\alpha)) \\ &= (\alpha I + G_{\mathbf{a}}G^*_{\mathbf{a}} - \alpha I)(\alpha I + G_{\mathbf{a}}G^*_{\mathbf{a}})^{-1}(z - H(u_\alpha)) \\ &= z - H(u_\alpha) - \alpha(\alpha I + G_{\mathbf{a}}G^*_{\mathbf{a}})^{-1}(z - H(u_\alpha)) \end{aligned}$$

Hence,

$$G_{\mathbf{a}}u_\alpha + H(u_\alpha) = z - \alpha(\alpha I + G_{\mathbf{a}}G^*_{\mathbf{a}})^{-1}(z - H(u_\alpha)).$$

To conclude the proof of this Theorem, it enough to prove that

$$\lim_{\alpha\to 0}\{-\alpha(\alpha I + G_{\mathbf{a}}G^*_{\mathbf{a}})^{-1}(z - H(u_\alpha))\} = 0$$

From Lemma 3.1 pat d) we get that

$$\begin{aligned}\lim_{\alpha\to 0}\{-\alpha(\alpha I+G_{\mathbf{a}}G_{\mathbf{a}}^{*})^{-1}(z-H(u_{\alpha}))\} &= -\lim_{\alpha\to 0}\{-\alpha(\alpha I+G_{\mathbf{a}}G_{\mathbf{a}}^{*})^{-1}H(u_{\alpha})\}\\ =-\lim_{\alpha\to 0}-\alpha(\alpha I+G_{\mathbf{a}}G_{\mathbf{a}}^{*})^{-1}y &- \lim_{\alpha\to 0}-\alpha(\alpha I+G_{\mathbf{a}}G_{\mathbf{a}}^{*})^{-1}(H(u_{\alpha})-y)\\ &= \lim_{\alpha\to 0}-\alpha(\alpha I+G_{\mathbf{a}}G_{\mathbf{a}}^{*})^{-1}(H(u_{\alpha})-y).\end{aligned}$$

On the other hand, from Lemma 3.1 pat e) we get that

$$\|\alpha(\alpha I+G_{\mathbf{a}}G_{\mathbf{a}}^{*})^{-1}(H(u_{\alpha})-y)\| \le \|(H(u_{\alpha})-y)\|.$$

Therefore, since $H(u_{\alpha})$ converges to $y$, we get that

$$\lim_{\alpha\to 0}\{-\alpha(\alpha I+G_{\mathbf{a}}G_{\mathbf{a}}^{*})^{-1}(H(u_{\alpha})-y)\}=0.$$

Consequently,

$$\lim_{\alpha\to 0}\{-\alpha(\alpha I+G_{\mathbf{a}}G_{\mathbf{a}}^{*})^{-1}(z-H(u_{\alpha}))\}=0$$

So, putting $z=z_1-T(\tau)z_0$ and using (4.23), we obtain the nice result:

$$\begin{aligned}z_1 &= \lim_{\alpha\to 0^+}\{e^{a\tau}T(\tau)z_0+\int_0^{\tau}e^{(\tau-s)}T(\tau-s)(B_{\omega}+cI)u_{\alpha}(s)ds\\ &+ \int_0^{\tau}e^{(\tau-s)}T(\tau-s)g^{e}(s,z_{u_{\alpha}}(s),u_{\alpha}(s))ds\}\end{aligned}$$

□

# 5 Final Remark

Our technique is simple and can be apply to those system involving compact semigroups like some control system governed by diffusion processes. For example, the Benjamin -Bona-Mohany Equation, the strongly damped wave equations, beam equations, etc.

EXAMPLE 5.1. *The original Benjamin -Bona-Mohany Equation is a non-linear one, in [15] the authors proved the approximate controllability of the linear part of this equation, which is the fundamental base for the study of the controllability of the non linear BBM equation. So, our next work is concerned with the controllability of non linear BBM equation*

$$\begin{cases} z_t-a\Delta z_t-b\Delta z=1_{\omega}u(t,x)+f(t,z,u(t)), & t\in(0,\tau), \qquad x\in\Omega,\\ z(t,x)=0, \quad t\ge 0, & x\in\partial\Omega,\end{cases} \tag{5.28}$$

*where $a\ge 0$ and $b>0$ are constants, $\Omega$ is a domain in $I\!R^N$, $\omega$ is an open nonempty subset of $\Omega$, $1_{\omega}$ denotes the characteristic function of the set $\omega$, the distributed control $u\in L^2(0,\tau;L^2(\Omega))$ and $f(t,z,u(t))$ is a nonlinear perturbation.*

Example 5.2. *We believe that this technique can be applied to prove the interior controllability of the strongly damped wave equation with Dirichlet boundary conditions*

$$\begin{cases} w_{tt} + \eta(-\Delta)^{1/2} w_t + \gamma(-\Delta) w = 1_\omega u(t,x) + f(t,z,u(t)), & in \quad (0,\tau)\times\Omega, \\ w = 0, & in \quad (0,\tau)\times\partial\Omega, \\ w(0,x) = w_0(x), \quad w_t(0,x) = w_1(x), & in \quad \Omega, \end{cases}$$

*in the space* $Z_{1/2} = D((-\Delta)^{1/2}) \times L^2(\Omega)$, *where* $\Omega$ *is a bounded domain in* $\mathbb{R}^n$, $\omega$ *is an open nonempty subset of* $\Omega$, $1_\omega$ *denotes the characteristic function of the set* $\omega$, *the distributed control* $u \in L^2(0,\tau; L^2(\Omega))$ *and* $\eta$, $\gamma$ *are positive numbers.*

Example 5.3. *Another example where this technique my be applied is a partial differential equations modeling the structural damped vibrations of a string or a beam:*

$$\begin{cases} y_{tt} - 2\beta\Delta y_t + \Delta^2 y = 1_\omega u(t,x) + f(t,z,u(t)), & on \quad (0,\tau)\times\Omega, \\ y = \Delta y = 0, & on \quad (0,\tau)\times\partial\Omega, \\ y(0,x) = y_0(x), \quad y_t(0,x) = y_1(x), & in \quad \Omega, \end{cases} \tag{5.29}$$

*where* $\Omega$ *is a bounded domain in* $\mathbb{R}^n$, $\omega$ *is an open nonempty subset of* $\Omega$, $1_\omega$ *denotes the characteristic function of the set* $\omega$, *the distributed control* $u \in L^2(0,\tau; L^2(\Omega))$ *and* $y_0, y_1 \in L^2(\Omega)$.

Moreover, Our result can be formulated in a more general setting. Indeed, we can consider the following semilinear evolution equation in a general Hilbert space $Z$

$$\begin{cases} \acute{z} = -Az + Bu(t) + f^e(t,z,u), \quad z \in Z, \quad t \in (0,\tau], \\ z(0) = z_0, \end{cases} \tag{5.30}$$

where, $A : D(A) \subset Z \to Z$ is an unbounded linear operator in $Z$ with the following spectral decomposition:

$$Az = \sum_{j=1}^{\infty} \lambda_j \sum_{k=1}^{\gamma_j} < z, \phi_{j,k} > \phi_{j,k},$$

with the eigenvalues $0 < \lambda_1 < \lambda_2 < \cdots < \cdots \lambda_n \to \infty$ of $A$ having finite multiplicity $\gamma_j$ equal to the dimension of the corresponding eigenspaces, and $\{\phi_{j,k}\}$ is a complete orthonormal set of eigenfunctions of $A$. The operator $-A$ generates a strongly continuous compact semigroup $\{T_A(t)\}_{t\geq 0}$ given by

$$T_A(t)z = \sum_{j=1}^{\infty} e^{-\lambda_j t} \sum_{k=1}^{\gamma_j} < z, \phi_{j,k} > \phi_{j,k}.$$

The control $u \in L^2(0,\tau; U)$, with $U = Z$, $B : Z \to Z$ is a linear and bounded operator(linear and continuous) and the function $f^e : [0,\tau] \times Z \times U \to Z$ is smooth enough and

$$\sup_{(t,z,u)\in Z_\tau} \|f^e(t,z,u) - az - cu\|_Z < \infty, \tag{5.31}$$

where $Z_\tau = [0,\tau] \times Z \times U$. In this case the characteristic function set is a particular operator $B$, and the following theorem is a generalization of Theorem 4.1.

Theorem 5.1. *If vectors* $B^*\phi_{j,k}$ *are linearly independent in* $Z$, *then the system (5.30) is approximately controllable on* $[0,\tau]$.

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