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ON ANALOGUES OF BÄCKLUND THEOREM IN AFFINE DIFFERENTIAL GEOMETRY OF SURFACES

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Abstract. Here we recall the well-known Chern–Terng theorem concerning affine minimal surfaces. After that we formulate some complementary (with transversal fields necessarily not parallel) affine Bäcklund theorem. Next, we describe some geometrical conditions which imply the local symmetry of both induced connections. Finally, we give some necessary and sufficient conditions under which the affine fundamental forms are proportional.

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Contents

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1. Introduction

The most classical Bäcklund theorem is the Bäcklund theorem for surfaces in Euclidean space

Theorem 1. *Let $f, \widehat{f} : M \to \mathbb{R}^3$, be a pair of surfaces in the Euclidean space $\mathbb{R}^3$ satisfying the following conditions*

i) *for every $p \in M$ $f(p) \neq \widehat{f}(p)$, the vector $\widehat{f}(p) - f(p)$ is tangent to $f(M)$ at $f(p)$ and is tangent to $\widehat{f}(M)$ at $\widehat{f}(p)$*

ii) *the length $L := |\widehat{f}(p) - f(p)|$ of $\widehat{f}(p) - f(p)$ is independent of p*

iii) *the angle σ between the Euclidean normals $\mathbf{n}$ and $\widehat{\mathbf{n}}$ (of $f(M)$ and $\widehat{f}(M)$ respectively) is constant and $\sin\sigma \neq 0$.*

Then both surfaces are of constant negative Gaussian curvature $\kappa = \widehat{\kappa} = -\dfrac{\sin^2\sigma}{L^2}\cdot$ The second fundamental forms h and $\widehat{h}$ of f and $\widehat{f}$ are proportional.

In this article we will present analogues of Bäcklund theorem in affine differential geometry of surfaces. We recall Chern–Terng theorem and prove some other affine Bäcklund theorem, concerning surfaces with locally symmetric induced connection.

Our aim was to generalize Bäcklund theorem to the situation, when in ambient space there is only the volume form, and we cannot measure length or angle. The volume form is parallel with respect to the standard linear connection D in $\mathbb{R}^3$. We study two immersions f and $\widehat{f}$, which are focal surfaces of a rectilinear congruence. Each of them is endowed with an equiaffine transversal vector field, ξ and $\widehat{\xi}$ respectively. Unlike the Euclidean normals, those transversal fields are not determined by the immersions. Of course, one may use the affine normal, and this particular case will be also considered. We will impose on (f, ξ) and $(\widehat{f}, \widehat{\xi})$ some conditions which guarantee that both induced connections ∇ and $\widehat{\nabla}$ are locally symmetric. Our idea was to consider the volume of the parallelepiped spanned by $\widehat{f} - f$ and both transversal fields. In Euclidean case this volume is a non-zero constant. The conjecture that condition of constant volume together with some other conditions about the values of conormal map enforce both Blaschke connections to be locally symmetric turned out to be true. Some partial result, with Blaschke normal of f tangent to $\widehat{f}$ and vice versa, is contained in [7]. However, in case of arbitrary equiaffine transversal fields ξ and $\widehat{\xi}$ one should admit also non-constant volume $\det(\widehat{f} - f, \xi, \widehat{\xi})$.

Our result seems to be a common generalization of the classical Bäcklund theorem (see for example [2] or [9]) and Minkowski space Bäcklund theorem ([1], [7]). It also includes the case of non-metrizable connections with $\dim \operatorname{im} R = 1$, studied by Opozda in [6]. The theorem is complementary to Chern and Terng analogue of Bäcklund's theorem in affine geometry [2], because in [2] the affine normals $\mathbb{R}\xi$ and $\mathbb{R}\widehat{\xi}$ were assumed to be parallel, hence $\det(\widehat{f} - f, \xi, \widehat{\xi}) = 0$.

2. Preliminaries

We recall the basic notions of affine differential geometry. More details can be found in [5]. Here we consider only two-dimensional manifolds immersed into affine space $\mathbb{R}^3$. The standard connection in $\mathbb{R}^3$ is denoted by D.

Let $f : M \to \mathbb{R}^3$ be an immersion of a two-dimensional manifold M into $\mathbb{R}^3$. Let $\xi : M \to \mathbb{R}^3$ be a transversal vector field. For each $p \in M$ we have the decomposition $\mathbb{R}^3 = f_*(T_pM) \oplus \mathbb{R}\xi_p$. The *induced connection* ∇, the *affine fundamental form* h (relative to the transversal vector field ξ), the *affine shape operator* S and the *transversal connection form* τ are defined by the following Gauss and Weingarten formulae

$$D_X f_*(Y) = f_*(\nabla_X Y) + h(X,Y)\,\xi, \;\; D_X \xi = -\,f_*(SX) + \tau(X)\,\xi. \qquad (1)$$

The volume element induced by (f, ξ) on M is

$$\theta(X,Y) = \det(f_*X, f_*Y, \xi). \qquad (2)$$

The determinant $\det_\theta h$ of a symmetric covariant tensor h of degree 2 relative to θ is, by definition, equal to $\det[h_{ij}]$, where $h_{ij} = h(X_i, X_j)$ and X_1, X_2 is a unimodular basis for θ: $\theta(X_1, X_2) = 1$. Let $(\mathbb{R}^3)^*$ be the dual space of the vector space $\mathbb{R}^3$. For immersion $f : M \to \mathbb{R}^3$ with transversal vector field $\xi : M \to \mathbb{R}^3$ the *conormal map* $\nu : M \to (\mathbb{R}^3)^*$ is defined as follows

$$\nu_p(f_*(X_p)) := 0 \;\text{ and }\; \nu_p(\xi_p) := 1 \;\text{ for }\; p \in M, \;\; X_p \in T_pM. \qquad (3)$$

The rank of the affine fundamental form is independent of the choice of transversal vector field. If h is nondegenerate, then we say that the surface is *nondegenerate*. If f is nondegenerate, then for each point $p \in M$ there exists a transversal vector field defined in a neighbourhood of p satisfying the conditions

I) $\nabla\theta = 0$

II) θ coincides with the volume element of the nondegenerate metric h.

Such a transversal vector field is unique up to a sign and is called the *affine normal field* or *Blaschke normal field*. The connection induced by the affine normal vector field is called the *Blaschke connection* and h is called the *affine metric*. The condition I) is equivalent to $\tau = 0$ and the condition II) is equivalent to $|\det_\theta h| = 1$. An *equiaffine* transversal field is a transversal field satisfying the condition $\tau = 0$.

Bäcklund theorem is usually formulated for two focal surfaces of some rectilinear congruence. A rectilinear congruence is a two-parametric family of straight lines. Under some additional assumption about the congruence one can find two families of ruled developable surfaces with rulings belonging to the congruence. Each line of the congruence is contained in one developable surface of each family and is tangent to the edge of regression of this developable surface at the point which is called the *focal point*. Except of some particular degenerate cases the set of all focal points forms two *focal surfaces*. We parametrize the focal surfaces in such a way that $f(p)$ and $\widehat{f}(p)$ belong to the same straight line of congruence. We may consider the mapping $f(p) \mapsto \widehat{f}(p)$ between the two focal surfaces. If this mapping preserves the asymptotic lines, a rectilinear congruence is called a *W-congruence*.

More details about rectilinear congruences one can find for example in [3].

3. A Necessary and Sufficient Condition for Rectilinear Congruence with Non-Degenerate Focal Surfaces to be a W-congruence

In this section we will study the condition that the affine fundamental forms h and $\widehat{h}$, of (f, ξ) and $(\widehat{f}, \widehat{\xi})$ respectively, are proportional. In Euclidean or Minkowski space Bäcklund theorem this condition is a part of the assertion, whereas in affine case it is an assumption.

Proposition 2. *Let $f : M \to \mathbb{R}^3$ and $\widehat{f} : M \to \mathbb{R}^3$ be non-degenerate immersions of a two-dimensional manifold M into affine space $\mathbb{R}^3$ such that for every $p \in M$ $f(p) \neq \widehat{f}(p)$, the vector $\widehat{f}(p) - f(p)$ is tangent to $f(M)$ at $f(p)$ and is tangent to $\widehat{f}(M)$ at $\widehat{f}(p)$.*

Let ξ and $\widehat{\xi}$ be some transversal vector fields for f and $\widehat{f}$ respectively. We denote by h and $\widehat{h}$ the corresponding affine fundamental forms, and by ν and $\widehat{\nu}$ the conormal maps. Then

i) *If* $\det(\widehat{f} - f, \xi, \widehat{\xi}) = 0$, *then* $1 - \nu(\widehat{\xi})\,\widehat{\nu}(\xi) = 0$.

ii) *If* $1 - \nu(\widehat{\xi})\,\widehat{\nu}(\xi) = 0$ *at some point* p *and* $f_*(T_pM) \neq \widehat{f}_*(T_pM)$, *then* $\det(\widehat{f}(p) - f(p), \xi_p, \widehat{\xi}_p) = 0$.

iii) *If* ξ *and* $\widehat{\xi}$ *are such that* $\det(\widehat{f} - f, \xi, \widehat{\xi}) \neq 0$ *and* $\widetilde{\xi} = \lambda\,\xi + f_*Z$, $\overline{\xi} = \mu\,\widehat{\xi} + \widehat{f}_*V$, *then*

$$1 - \widetilde{\nu}(\overline{\xi})\,\overline{\nu}(\widetilde{\xi}) = \frac{1 - \nu(\widehat{\xi})\,\widehat{\nu}(\xi)}{\lambda\,\mu \det(\widehat{f} - f, \xi, \widehat{\xi})} \det(\widehat{f} - f, \widetilde{\xi}, \overline{\xi}).$$

iv) *If moreover* $\det(\widehat{f} - f, \widetilde{\xi}, \overline{\xi}) \neq 0$, *then*

$$\left(\frac{1 - \widetilde{\nu}(\overline{\xi})\,\overline{\nu}(\widetilde{\xi})}{\det(\widehat{f} - f, \widetilde{\xi}, \overline{\xi})}\right)^4 \frac{1}{\det_{\widetilde{\theta}}\widetilde{h}\,\det_{\overline{\theta}}\overline{h}} = \left(\frac{1 - \nu(\widehat{\xi})\,\widehat{\nu}(\xi)}{\det(\widehat{f} - f, \xi, \widehat{\xi})}\right)^4 \frac{1}{\det_{\theta}h\,\det_{\widehat{\theta}}\widehat{h}}.$$

Proof: i) There exist nowhere vanishing vector fields X_1 and $\widehat{X}_1$ on M such that

$$\widehat{f} - f = f_*X_1 \tag{4}$$

and

$$\widehat{f} - f = \widehat{f}_*\widehat{X}_1. \tag{5}$$

Since f_*X_1 and ξ are linearly independent, from $\det(\widehat{f} - f, \xi, \widehat{\xi}) = 0$ it follows that $\widehat{\xi} = \alpha\, f_*X_1 + \beta\,\xi$ for some α and β. Here $\beta \neq 0$, because $f_*X_1 = \widehat{f}_*\widehat{X}_1$ is tangent to $\widehat{f}$. We have $\nu(\widehat{\xi}) = \beta$ and from $\widehat{\xi} = \alpha\,\widehat{f}_*\widehat{X}_1 + \beta\,\xi$ we obtain $1 = \beta\,\widehat{\nu}(\xi)$.

ii) Conversely, if $\nu(\widehat{\xi})\,\widehat{\nu}(\xi) = 1$, then $\widehat{\xi} = f_*T + A\,\xi$ and $\xi = \widehat{f}_*U + \frac{1}{A}\,\widehat{\xi}$ with some $A \neq 0$. It follows that $f_*T = \widehat{\xi} - A\,\xi = -\widehat{f}_*(A\,U)$. Therefore f_*T is tangent to f and is tangent to $\widehat{f}$. By assumption $f_*T_pM \neq \widehat{f}_*T_pM$, hence $f_*T_pM \cap \widehat{f}_*T_pM = \mathbb{R} f_*X_{1p}$ and $\widehat{\xi} \in \operatorname{span}\{f_*X_{1p}, \xi_p\}$.

iii) Let $W = \det(\widehat{f} - f, \xi, \widehat{\xi})$, $A = \nu(\widehat{\xi})$ and $\widehat{A} = \widehat{\nu}(\xi)$. For every $p \in M$, $\dim f_*T_pM = 2$, $\dim \operatorname{span}\{\xi_p, \widehat{\xi}_p\} = 2$ and $f_*T_pM \neq \operatorname{span}\{\xi_p, \widehat{\xi}_p\}$, because $\xi_p \notin f_*T_pM$. Therefore $\dim\big(f_*T_pM \cap \operatorname{span}\{\xi_p, \widehat{\xi}_p\}\big) = 1$ and we can find the vector $X_{2p} \in T_pM$ such that $f_*X_{2p} \in \operatorname{span}\{\xi_p, \widehat{\xi}_p\}$ and $\det(f_*X_{1p}, f_*X_{2p}, \xi_p) = 1$. In this way we define the vector field X_2 such that

$$f_*X_2 = a_{11}\,\xi + a_{21}\,\widehat{\xi} \tag{6}$$

with some functions a_{11} and a_{21}, and

$$\det(f_* X_1, f_* X_2, \xi) = 1. \tag{7}$$

Similarly we may define the vector field $\widehat{X}_2$ such that

$$\widehat{f}_* \widehat{X}_2 = a_{12}\, \xi + a_{22}\, \widehat{\xi} \tag{8}$$

and

$$\det(\widehat{f}_* \widehat{X}_1, \widehat{f}_* \widehat{X}_2, \widehat{\xi}) = 1. \tag{9}$$

From (7), (4) and (6) it follows that $a_{21} = -\frac{1}{W}$ and from (9), (5) and (8) we obtain $a_{12} = \frac{1}{W}$. Since, by (6), $a_{11} + a_{21}\, \nu(\widehat{\xi}) = 0$, and by (8) $a_{12}\, \widehat{\nu}(\xi) + a_{22} = 0$, we have $a_{11} = \frac{A}{W}$ and $a_{22} = -\frac{\widehat{A}}{W}$. It follows that

$$\widehat{f}_* \widehat{X}_1 = f_* X_1, \quad \widehat{f}_* \widehat{X}_2 = \widehat{A}\, f_* X_2 + \frac{1 - A\widehat{A}}{W}\, \xi, \quad \widehat{\xi} = -W\, f_* X_2 + A\, \xi. \tag{10}$$

We have

$$\widetilde{\xi} = \lambda\, \xi + f_* Z, \qquad \overline{\xi} = \mu\, \widehat{\xi} + \widehat{f}_* V. \tag{11}$$

Let $Z = z^1\, X_1 + z^2\, X_2$ and $V = w^1\, \widehat{X}_1 + w^2\, \widehat{X}_2$. Let $\widetilde{W} := \det(\widehat{f} - f, \widetilde{\xi}, \overline{\xi})$.

$$\begin{aligned}
\widetilde{W} &= \det(\widehat{f} - f,\ \lambda\, \xi + z^1\, f_* X_1 + z^2\, f_* X_2,\ \mu\, \widehat{\xi} + w^1\, \widehat{f}_* \widehat{X}_1 + w^2\, \widehat{f}_* \widehat{X}_2) \\
&= \det(\widehat{f} - f,\ \lambda\, \xi + z^2\, f_* X_2,\ \mu\, \widehat{\xi} + w^2\, \widehat{f}_* \widehat{X}_2) \\
&= \det\Big(\widehat{f} - f,\ \lambda\, \xi + z^2 \left(\frac{A}{W}\, \xi - \frac{1}{W}\, \widehat{\xi}\right),\ \mu\, \widehat{\xi} + w^2 \left(\frac{1}{W}\, \xi - \frac{\widehat{A}}{W}\, \widehat{\xi}\right)\Big) \\
&= \det\Big(\widehat{f} - f,\ \left(\lambda + z^2\, \frac{A}{W}\right) \xi - \frac{z^2}{W}\, \widehat{\xi},\ \frac{w^2}{W}\, \xi + \left(\mu - w^2\, \frac{\widehat{A}}{W}\right) \widehat{\xi}\Big) \\
&= \left(\left(\lambda + z^2\, \frac{A}{W}\right)\left(\mu - w^2\, \frac{\widehat{A}}{W}\right) + \frac{z^2\, w^2}{W^2}\right) \det(\widehat{f} - f, \xi, \widehat{\xi}) \\
&= \lambda\, \mu\, W + z^2 A \mu - w^2 \widehat{A} \lambda + z^2 w^2 \frac{1 - A\widehat{A}}{W}.
\end{aligned}$$

To compute $\widetilde{\nu}(\overline{\xi})$ we have to write $\overline{\xi}$ in the basis $f_* X_1$, $f_* X_2$, $\widetilde{\xi}$.

$$\begin{aligned}
\overline{\xi} &= \mu\, \widehat{\xi} + w^1\, \widehat{f}_* \widehat{X}_1 + w^2\, \widehat{f}_* \widehat{X}_2 \\
&= \mu\, (-W\, f_* X_2 + A\, \xi) + w^1\, f_* X_1 + w^2 \left(\widehat{A}\, f_* X_2 + \frac{1 - A\widehat{A}}{W}\, \xi\right) \\
&= \left(\mu A + w^2\, \frac{1 - A\widehat{A}}{W}\right) \xi + f_* \left(w^1 X_1 + \left(w^2 \widehat{A} - \mu W\right) X_2\right) \\
&= \left(\mu A + w^2\, \frac{1 - A\widehat{A}}{W}\right) \left(\frac{1}{\lambda}\, \widetilde{\xi} - \frac{1}{\lambda}\, f_* Z\right) + f_* \left(w^1 X_1 + \left(w^2 \widehat{A} - \mu W\right) X_2\right).
\end{aligned}$$

It follows that

$$\widetilde{A} := \widetilde{\nu}(\overline{\xi}) = \frac{1}{\lambda}\Big(\mu A + w^2\,\frac{1 - A\widehat{A}}{W}\Big). \tag{12}$$

Similarly we obtain

$$\overline{A} := \overline{\nu}(\widetilde{\xi}) = \frac{1}{\mu}\Big(\lambda\widehat{A} - z^2\,\frac{1 - A\widehat{A}}{W}\Big). \tag{13}$$

Consequently

$$1 - \widetilde{A}\,\overline{A} = \frac{1 - A\widehat{A}}{\lambda\mu W}\,\widetilde{W}. \tag{14}$$

iv) Since $\det_{\widetilde{\theta}}\widetilde{h} = \frac{1}{\lambda^4}\det_{\theta} h$ and $\det_{\overline{\theta}}\overline{h} = \frac{1}{\mu^4}\det_{\widehat{\theta}}\widehat{h}$ [5], we obtain from iii)

$$\Big(\frac{1 - \widetilde{A}\,\overline{A}}{\widetilde{W}}\Big)^4 \frac{1}{\det_{\widetilde{\theta}}\widetilde{h}\,\det_{\overline{\theta}}\overline{h}} = \Big(\frac{1 - A\widehat{A}}{\lambda\mu W}\Big)^4 \frac{\lambda^4\mu^4}{\det_{\theta} h\,\det_{\widehat{\theta}}\widehat{h}} = \Big(\frac{1 - A\widehat{A}}{W}\Big)^4 \frac{1}{\det_{\theta} h\,\det_{\widehat{\theta}}\widehat{h}}.$$

From iv) of Proposition 2 it follows that

$$\psi(f, \widehat{f}) := \Big(\frac{1 - \nu(\widehat{\xi})\,\widehat{\nu}(\xi)}{\det(\widehat{f} - f, \xi, \widehat{\xi})}\Big)^4 \frac{1}{\det_{\theta} h\,\det_{\widehat{\theta}}\widehat{h}} \tag{15}$$

is a well defined function on M.

Throughout the paper we will make some assumption about the rank of the spherical representation of $\widehat{f} - f$. The following lemma explains the technical significance of this assumption: the forms $\omega^2{}_1$, $\omega^3{}_1$ constitute a local frame of T^*M.

Lemma 3. (cf [8] page 6 in the metric case) *Let* $\varphi : M \to \mathrm{GL}(3, \mathbb{R})$. *For* $p \in M$ *we denote by* v_{1p}, v_{2p}, v_{3p} *the columns of the matrix* $\varphi(p)$. *We consider the mappings* $v_1 : M \to \mathbb{R}^3 \setminus \{\mathbf{0}\}$ *and* $\pi \circ v_1 : M \to \mathbb{P}^2(\mathbb{R})$, *where* $\pi : \mathbb{R}^3 \setminus \{\mathbf{0}\} \to \mathbb{P}^2(\mathbb{R})$ *denotes the canonical projection. The forms* $\omega^i{}_1$ *are defined by the equality*

$$\mathrm{d}v_1 = \omega^1{}_1\, v_1 + \omega^2{}_1\, v_2 + \omega^3{}_1 v_3. \tag{16}$$

At each point of M *the following conditions are equivalent*

i) $\mathrm{rank}(\pi \circ v_1) = 2$

ii) $\omega^2{}_1 \wedge \omega^3{}_1 \neq 0$.

Proof: Let (v_i^1, v_i^2, v_i^3) be the coordinates of v_i. Assume for example that $v_1^3 \neq 0$. Then on $\mathbb{P}^2(\mathbb{R})$ we use the chart $(t^1 : t^2 : t^3) \mapsto \left(\frac{t^1}{t^3}, \frac{t^2}{t^3}\right)$. The composition of $\pi \circ v_1$ with this chart is $\left(\frac{v_1^1}{v_1^3}, \frac{v_1^2}{v_1^3}\right)$ and its rank equals two if and only if $\mathrm{d}\left(\frac{v_1^1}{v_1^3}\right) \wedge \mathrm{d}\left(\frac{v_1^2}{v_1^3}\right) \neq 0$. Let $Z(p)$ be the inverse matrix of $\varphi(p)$ and let $Z = (z_{ij})$. Using (16) we easily obtain $\mathrm{d}\left(\frac{v_1^1}{v_1^3}\right) = \dfrac{\det\varphi}{(v_1^3)^2}\left(z_{32}\omega^2{}_1 - z_{22}\omega^3{}_1\right)$, $\mathrm{d}\left(\frac{v_1^2}{v_1^3}\right) = \dfrac{\det\varphi}{(v_1^3)^2}\left(-z_{31}\omega^2{}_1 + z_{21}\omega^3{}_1\right)$ and

$$\begin{aligned}\mathrm{d}\left(\frac{v_1^1}{v_1^3}\right) \wedge \mathrm{d}\left(\frac{v_1^2}{v_1^3}\right) &= \frac{(\det\varphi)^2}{(v_1^3)^4}\begin{vmatrix} z_{21} & z_{22} \\ z_{31} & z_{32}\end{vmatrix}\,\omega^2{}_1 \wedge \omega^3{}_1 \\ &= \frac{(\det\varphi)^2}{(v_1^3)^4}\det Z\, v_1^3\,\omega^2{}_1 \wedge \omega^3{}_1 = \frac{\det\varphi}{(v_1^3)^3}\,\omega^2{}_1 \wedge \omega^3{}_1\end{aligned}$$

hence $\mathrm{d}\left(\frac{v_1^1}{v_1^3}\right) \wedge \mathrm{d}\left(\frac{v_1^2}{v_1^3}\right) \neq 0$ is equivalent to $\omega^2{}_1 \wedge \omega^3{}_1 \neq 0$. If at some point $v_1^3 = 0$, then we have to use another chart and one of the equalities $\mathrm{d}\left(\frac{v_1^1}{v_1^2}\right) \wedge \mathrm{d}\left(\frac{v_1^3}{v_1^2}\right) = -\dfrac{\det\varphi}{(v_1^2)^3}\,\omega^2{}_1 \wedge \omega^3{}_1$, $\mathrm{d}\left(\frac{v_1^2}{v_1^1}\right) \wedge \mathrm{d}\left(\frac{v_1^3}{v_1^1}\right) = \dfrac{\det\varphi}{(v_1^1)^3}\,\omega^2{}_1 \wedge \omega^3{}_1$.

Theorem 4. *Let f and $\widehat{f}$ be as in Proposition 2. Assume that the spherical representation of $\widehat{f} - f$, $M \ni p \mapsto \pi(\widehat{f}(p) - f(p)) \in \mathbb{P}^2(\mathbb{R})$, has rank 2 at every point of M. Then*

i) *$f_* T_p M \neq \widehat{f}_* T_p M$ for every $p \in M$*

ii) *$\widehat{f}(p) - f(p)$ is not an asymptotic vector*

iii) *the affine fundamental forms h and $\widehat{h}$ are conformal to each other if and only if $\psi(f, \widehat{f}) = 1$.*

Proof: We choose transversal fields ξ and $\widehat{\xi}$ satisfying $\det(\widehat{f} - f, \xi, \widehat{\xi}) \neq 0$. We retain the notation of Proposition 2 and Lemma 3. We take

$$v_1 = \widehat{v}_1 = \widehat{f} - f, \quad v_2 = f_* X_2, \quad \widehat{v}_2 = \widehat{f}_* \widehat{X}_2, \quad v_3 = \xi \text{ and } \widehat{v}_3 = \widehat{\xi}.$$

Together with f and $\widehat{f}$ we consider moving frames F and $\widehat{F}$ from M to $\mathrm{ASL}(3, \mathbb{R})$,

$$F = \begin{pmatrix} 1 & 0 \\ f & (v_1, v_2, v_3)\end{pmatrix}, \qquad \widehat{F} = \begin{pmatrix} 1 & 0 \\ \widehat{f} & (\widehat{v}_1, \widehat{v}_2, \widehat{v}_3)\end{pmatrix}.$$

We can now rewrite (4) and (10) as $\widehat{F} = Fa$ with

$$a = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & \widehat{A} & -W \\ 0 & 0 & \frac{1-A\widehat{A}}{W} & A \end{pmatrix}.$$

The pull-back of the Maurer-Cartan form ϑ on $\mathrm{ASL}(3,\mathbb{R})$ by F is

$$F^*\vartheta = F^{-1}\,\mathrm{d}F = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \vartheta^1 & \omega^1_{\ 1} & \omega^1_{\ 2} & \omega^1_{\ 3} \\ \vartheta^2 & \omega^2_{\ 1} & \omega^2_{\ 2} & \omega^2_{\ 3} \\ \vartheta^3 & \omega^3_{\ 1} & \omega^3_{\ 2} & \omega^3_{\ 3} \end{pmatrix}.$$

Then

$$\mathrm{d}f = \vartheta^1\,v_1 + \vartheta^2\,v_2 + \vartheta^3\,v_3, \qquad \mathrm{d}v_1 = \omega^1_{\ 1}\,v_1 + \omega^2_{\ 1}\,v_2 + \omega^3_{\ 1}\,v_3$$
$$\mathrm{d}v_2 = \omega^1_{\ 2}\,v_1 + \omega^2_{\ 2}\,v_2 + \omega^3_{\ 2}\,v_3, \qquad \mathrm{d}v_3 = \omega^1_{\ 3}\,v_1 + \omega^2_{\ 3}\,v_2 + \omega^3_{\ 3}\,v_3.$$

Since $\mathrm{d}\circ\mathrm{d} = 0$, the one-forms ϑ^i and $\omega^j_{\ k}$ satisfy the structure equations

$$\mathrm{d}\vartheta^s = -\sum_{k=1}^{3} \omega^s_{\ k} \wedge \vartheta^k, \quad s = 1,2,3 \tag{17}$$

and

$$\mathrm{d}\omega^i_{\ j} = -\sum_{k=1}^{3} \omega^i_{\ k} \wedge \omega^k_{\ j}, \qquad i,j = 1,2,3. \tag{18}$$

Similar equalities one can write for the dashed one-forms $\widehat{\vartheta}^i$ and $\widehat{\omega}^j_{\ k}$.

From

$$\widehat{F}^{-1}\mathrm{d}\widehat{F} = a^{-1}\left(F^{-1}\mathrm{d}F\right)a + a^{-1}\mathrm{d}a \tag{19}$$

we obtain

$$\widehat{\vartheta}^2 = A\,\vartheta^2 + W\,\vartheta^3 + A\,\omega^2_{\ 1} + W\,\omega^3_{\ 1} \tag{20}$$

$$\widehat{\vartheta}^3 = -\frac{1-A\widehat{A}}{W}\,\vartheta^2 + \widehat{A}\,\vartheta^3 - \frac{1-A\widehat{A}}{W}\,\omega^2_{\ 1} + \widehat{A}\,\omega^3_{\ 1}. \tag{21}$$

Since the frames (v_1, v_2, v_3) and $(\widehat{v}_1, \widehat{v}_2, \widehat{v}_3)$ are adapted to f and $\widehat{f}$ respectively, we have $\vartheta^3 = 0$ and $\widehat{\vartheta}^3 = 0$. From (21) we obtain

$$0 = -\frac{1-A\widehat{A}}{W}\left(\vartheta^2 + \omega^2_{\ 1}\right) + \widehat{A}\,\omega^3_{\ 1}. \tag{22}$$

Suppose that $1-A\widehat{A}=0$. Then (22) and $\omega^2_{\ 1}\wedge\omega^3_{\ 1}\neq 0$ imply $\widehat{A}=0$, which contradicts $1-A\widehat{A}=0$. Therefore $1-A\widehat{A}\neq 0$ and from (10) we obtain i).

From (22) and (20) it follows that

$$\vartheta^2=-\,\omega^2_{\ 1}+\frac{\widehat{A}W}{1-A\widehat{A}}\,\omega^3_{\ 1},\qquad \widehat{\vartheta}^2=\frac{W}{1-A\widehat{A}}\,\omega^3_{\ 1}. \tag{23}$$

From (19) we obtain also

$$\widehat{\omega}^2_{\ 1}=A\,\omega^2_{\ 1}+W\,\omega^3_{\ 1},\qquad \widehat{\omega}^3_{\ 1}=-\,\frac{1-A\widehat{A}}{W}\,\omega^2_{\ 1}+\widehat{A}\,\omega^3_{\ 1}. \tag{24}$$

Comparing (23) with (24) yields

$$\vartheta^2=\frac{W}{1-A\widehat{A}}\,\widehat{\omega}^3_{\ 1}. \tag{25}$$

Our next goal is to check that X_1 and $\widehat{X}_1$ are at each point linearly independent. We only need to show that $\vartheta^2\wedge\widehat{\vartheta}^2\neq 0$ and it suffices to use (23) to obtain

$$\vartheta^2\wedge\widehat{\vartheta}^2=-\,\frac{W}{1-A\widehat{A}}\,\omega^2_{\ 1}\wedge\omega^3_{\ 1}.$$

We may now find the matrices of h and $\widehat{h}$ in the basis X_1, $\widehat{X}_1$. Since for $k\in\{1,2\}$ $h(Y,X_k)=\omega^3_{\ k}(Y)$ and $\widehat{h}(Y,\widehat{X}_k)=\widehat{\omega}^3_{\ k}(Y)$, we obtain from (23) and (25)

$$h(\widehat{X}_1,X_1)=0\quad\text{ and }\quad \widehat{h}(X_1,\widehat{X}_1)=0. \tag{26}$$

It follows that $h(X_1,X_1)\neq 0$ and $\widehat{h}(\widehat{X}_1,\widehat{X}_1)\neq 0$, for otherwise f or $\widehat{f}$ would be degenerate. We thus get ii).

Let $h_{ij}=h(X_i,X_j)$ and $\widehat{h}_{ij}=\widehat{h}(\widehat{X}_i,\widehat{X}_j)$. Let $\widehat{X}_1=c_{11}\,X_1+c_{21}\,X_2$. Here

$$c_{21}=\vartheta^2(\widehat{X}_1)=\frac{W}{1-A\widehat{A}}\,\widehat{\omega}^3_{\ 1}(\widehat{X}_1)=\frac{W}{1-A\widehat{A}}\,\widehat{h}_{11}$$

and consequently

$$h(\widehat{X}_1,\widehat{X}_1)=h(\widehat{X}_1,c_{11}\,X_1+c_{21}\,X_2)=c_{21}\,h(\widehat{X}_1,X_2)=\frac{W\,\widehat{h}_{11}}{1-A\widehat{A}}\,\omega^3_{\ 2}(\widehat{X}_1).$$

In a similar way we obtain

$$\widehat{h}(X_1,X_1)=\frac{W\,h_{11}}{1-A\widehat{A}}\,\widehat{\omega}^3_{\ 2}(X_1).$$

Since $h(X_1, \widehat{X}_1) = 0$ and $\widehat{h}(X_1, \widehat{X}_1) = 0$, the affine fundamental form $\widehat{h}$ is conformal to h if and only if there exists a function λ such that $\widehat{h}(X_1, X_1) = \lambda\, h(X_1, X_1)$ and $\widehat{h}(\widehat{X}_1, \widehat{X}_1) = \lambda\, h(\widehat{X}_1, \widehat{X}_1)$, which is equivalent to

$$\begin{vmatrix} \frac{W\, h_{11}}{1-A\widehat{A}}\, \widehat{\omega}^3{}_2(X_1) & h_{11} \\ \widehat{h}_{11} & \frac{W\, \widehat{h}_{11}}{1-A\widehat{A}}\, \omega^3{}_2(\widehat{X}_1) \end{vmatrix} = 0. \tag{27}$$

The left-hand side of (27) equals 0 if and only if

$$\Big(\frac{W}{1 - A\widehat{A}}\Big)^2\, \widehat{\omega}^3{}_2(X_1)\, \omega^3{}_2(\widehat{X}_1) = 1 \tag{28}$$

because $h_{11}\, \widehat{h}_{11} \neq 0$. Let $H := \det_\theta h$ and $\widehat{H} := \det_{\widehat{\theta}} \widehat{h}$. We have

$$\begin{aligned} H\, \vartheta^1 \wedge \vartheta^2(X_1, X_2) &= H = h_{11}\, h_{22} - h_{12}\, h_{12} \\ &= \omega^3{}_1(X_1)\, \omega^3{}_2(X_2) - \omega^3{}_1(X_2)\, \omega^3{}_2(X_1) = \omega^3{}_1 \wedge \omega^3{}_2(X_1, X_2) \end{aligned}$$

hence

$$\omega^3{}_1 \wedge \omega^3{}_2 = H\, \vartheta^1 \wedge \vartheta^2. \tag{29}$$

Similarly

$$\widehat{\omega}^3{}_1 \wedge \widehat{\omega}^3{}_2 = \widehat{H}\, \widehat{\vartheta}^1 \wedge \widehat{\vartheta}^2. \tag{30}$$

Using (23) and (25) we obtain

$$\omega^3{}_1 \wedge \omega^3{}_2(\widehat{X}_1, \widehat{X}_2) = \frac{1 - A\widehat{A}}{W}\, \widehat{\vartheta}^2 \wedge \omega^3{}_2(\widehat{X}_1, \widehat{X}_2) = -\, \frac{1 - A\widehat{A}}{W}\, \omega^3{}_2(\widehat{X}_1) \tag{31}$$

$$\widehat{\omega}^3{}_1 \wedge \widehat{\omega}^3{}_2(X_1, X_2) = \frac{1 - A\widehat{A}}{W}\, \vartheta^2 \wedge \widehat{\omega}^3{}_2(X_1, X_2) = -\, \frac{1 - A\widehat{A}}{W}\, \widehat{\omega}^3{}_2(X_1). \tag{32}$$

Combining (31) with (29) and (32) with (30) gives

$$\omega^3{}_2(\widehat{X}_1) = -\, \frac{WH}{1 - A\widehat{A}}\, \vartheta^1 \wedge \vartheta^2(\widehat{X}_1, \widehat{X}_2) \tag{33}$$

and

$$\widehat{\omega}^3{}_2(X_1) = -\, \frac{W\widehat{H}}{1 - A\widehat{A}}\, \widehat{\vartheta}^1 \wedge \widehat{\vartheta}^2(X_1, X_2). \tag{34}$$

Condition (28) now becomes

$$\Big(\frac{W}{1 - A\widehat{A}}\Big)^4\, H\widehat{H}\, \vartheta^1 \wedge \vartheta^2(\widehat{X}_1, \widehat{X}_2)\, \widehat{\vartheta}^1 \wedge \widehat{\vartheta}^2(X_1, X_2) = 1. \tag{35}$$

But $\vartheta^1 \wedge \vartheta^2(\widehat{X}_1, \widehat{X}_2)\,\widehat{\vartheta}^1 \wedge \widehat{\vartheta}^2(X_1, X_2) = 1$, because the matrix $(\widehat{\vartheta}^i(X_j))$ is the inverse of $(\vartheta^k(\widehat{X}_l))$. We thus get iii).

As a supplement we give here another similar criterion, applicable when we want to use parallel transversal fields ξ and $\widehat{\xi}$. The equality in iii) corresponds to (3.22) in [2].

Theorem 5. *Let f, $\widehat{f}$ be as in Proposition 2 and let X_1, $\widehat{X}_1$ satisfy (4) and (5). Assume that ξ and $\widehat{\xi}$, transversal fields for f and $\widehat{f}$ respectively, are parallel.*

We choose arbitrary X_2 such that X_1, X_2 is a local frame unimodular with respect to θ_ξ. Let $\widehat{X}_2$ be defined by the following two conditions: for every $p \in M$ $\widehat{f}_*(T_pM) \cap \operatorname{span}\{f_*(X_{2\,p}),\ \xi_p\} = \mathbb{R}\,\widehat{f}_*(\widehat{X}_{2\,p})$ and $\widehat{\theta}_{\widehat{\xi}}(\widehat{X}_1, \widehat{X}_2) = 1$. Then

- i) $\widehat{f}_*(\widehat{X}_2) = \lambda\, f_*(X_2) + \beta\,\xi, \quad \widehat{\xi} = \frac{1}{\lambda}\,\xi$ *for some functions* λ, β
- ii) λ, β *do not depend on* X_2 ($\widehat{X}_2$ *does*)
- iii) *if the spherical representation $\pi \circ (\widehat{f} - f)$ of $\widehat{f} - f$ has rank 2 at every point of M, then affine fundamental forms h and $\widehat{h}$ are proportional if and only if* $\det_\theta h \cdot \det_{\widehat{\theta}} \widehat{h} = \beta^4$.

Proof: By assumption, $\widehat{f}_*(\widehat{X}_2) = \lambda\, f_*(X_2) + \beta\,\xi$ and $\widehat{\xi} = \mu\,\xi$ for some functions λ, μ and β. From $\widehat{\theta}_{\widehat{\xi}}(\widehat{X}_1, \widehat{X}_2) = 1$ we obtain $\mu \cdot \lambda = 1$ and i) follows.

If we replace X_2 by $X_2 + t\,X_1$, then $\widehat{X}_2$ should be replaced by $\widehat{X}_2 + \lambda t\,\widehat{X}_1$. We have then $\widehat{f}_*(\widehat{X}_2 + \lambda t\,\widehat{X}_1) = \lambda f_*(X_2 + t\,X_1) + \beta\,\xi$.

Note that $\beta \neq 0$, because $\beta = 0$ would imply $\omega^3_{\ 1} = 0$, which contradicts the non-degeneracy of f.

Proof of iii) is similar to the proof of iii) in Theorem 4. We have now $\widehat{F} = F\,a$ with $a = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & \beta & \frac{1}{\lambda} \end{pmatrix}$ and from (19) we obtain in particular

$$\omega^3_{\ 1} = \beta\,\widehat{\vartheta}^2, \qquad \widehat{\omega}^3_{\ 1} = \beta\,\vartheta^2, \qquad \vartheta^2 \wedge \widehat{\vartheta}^2 = -\frac{1}{\beta}\,\omega^2_{\ 1} \wedge \omega^3_{\ 1} \neq 0. \tag{36}$$

The rest of the proof runs as before, with $\dfrac{1 - A\widehat{A}}{W}$ replaced by β.

We may also compute $\psi(f,\widehat{f})$ using ξ and $\widehat{\widehat{\xi}} = f_*(X_2) = \frac{1}{\lambda}\,\widehat{f}_*(\widehat{X}_2) - \beta\,\widehat{\xi}$ as linearly independent transversal fields for f and $\widehat{f}$ respectively, and apply Theorem 4. Note that $\nu(\widehat{\widehat{\xi}}) = 0$ and $\det_{\widehat{\widehat{\theta}}}\widehat{\widehat{h}} = \frac{1}{\beta^4}\det_{\widehat{\theta}}\widehat{h}$.

4. Chern–Terng Theorem

Theorem 6. [2] *Let* $\dim M = 2$ *and* $f,\widehat{f} : M \to \mathbb{R}^3$*, be a pair of non-degenerate immersions, satisfying the following conditions*

i) *for every* $p \in M$*:* $f(p) \neq \widehat{f}(p)$*, the vector* $\widehat{f}(p) - f(p)$ *is tangent to* $f(M)$ *at* $f(p)$ *and is tangent to* $\widehat{f}(M)$ *at* $\widehat{f}(p)$

ii) *the affine fundamental forms of* f *and* $\widehat{f}$ *are conformal to each other*

iii) *the affine normals of both surfaces at corresponding points* $f(p)$ *and* $\widehat{f}(p)$ *are parallel.*

Then the surfaces are both affine minimal.

Proof: We give here a proof which in some details will be different from that in [2], because we want to use local frames with the last vector field equal to corresponding affine normal vector field.

At first we consider the set of points where the rank of the spherical representation of $\widehat{f} - f$ equals 2. We use the same local frame as in Theorem 5. From assumption ii) and from Theorem 5 we have $H \cdot \widehat{H} = \beta^4$. Since ξ and $\widehat{\xi}$ are affine normal vector fields, $|H| = 1$ and $|\widehat{H}| = 1$. It follows that $|\beta| = 1$. If we replace $\widehat{\xi}$ by $-\,\widehat{\xi}$, then $\widehat{X}_2$ should be replaced by $-\,\widehat{X}_2$, λ by $-\,\lambda$ and β by $-\,\beta$. Therefore without loss of generality we may assume that $\beta = 1$. Moreover, $H = \widehat{H} =: \varepsilon_h$, because $H \cdot \widehat{H} > 0$.

From (19) we obtain $\widehat{\vartheta}^3 = -\,\beta\vartheta^2 - \beta\,\omega^2_{\ 1} + \lambda\,\omega^3_{\ 1}$ and $\widehat{\omega}^3_{\ 3} = -\,\frac{\beta}{\lambda}\,\omega^2_{\ 3} - \frac{d\lambda}{\lambda}$. Then $\widehat{\vartheta}^3 = 0$, $\widehat{\omega}^3_{\ 3} = 0$ together with $\beta = 1$ give

$$\vartheta^2 + \omega^2_{\ 1} = \lambda\,\omega^3_{\ 1}, \qquad \mathrm{d}\lambda + \omega^2_{\ 3} = 0 \tag{37}$$

which corresponds to $\gamma = 0$ and $\beta = 0$ in (3.8) of [2]. We will next assume that $\varepsilon_h + \lambda^2 \neq 0$ and prove the equality corresponding to $\alpha = 0$, that is

$$\vartheta^1 + \omega^1_{\ 1} = -\,\lambda\,\omega^3_{\ 2}. \tag{38}$$

Application of (19) gives

$$\widehat{\vartheta}^1 = \vartheta^1 + \omega^1_{\ 1},\ \ \widehat{\omega}^3_{\ 2} = -\lambda\beta\,\omega^2_{\ 2} + \lambda^2\,\omega^3_{\ 2} - \beta(\beta\,\omega^2_{\ 3} + \mathrm{d}\lambda) + \lambda\,\mathrm{d}\beta = \lambda\,\omega^1_{\ 1} + \lambda^2\,\omega^3_{\ 2}.$$

Let $\varphi = \vartheta^1 + \omega^1_{\ 1} + \lambda\,\omega^3_{\ 2}$. We have

$$\widehat{\vartheta}^1 = \varphi - \lambda\,\omega^3_{\ 2}, \qquad \widehat{\vartheta}^2 = \omega^3_{\ 1}, \qquad \widehat{\omega}^3_{\ 1} = \vartheta^2, \qquad \widehat{\omega}^3_{\ 2} = \lambda\,(\varphi - \vartheta^1).$$

Then

$$\begin{aligned} 0 &= \widehat{\omega}^3_{\ 1} \wedge \widehat{\vartheta}^1 + \widehat{\omega}^3_{\ 2} \wedge \widehat{\vartheta}^2 = \vartheta^2 \wedge (\varphi - \lambda\,\omega^3_{\ 2}) + \lambda\,(\varphi - \vartheta^1) \wedge \omega^3_{\ 1} \\ &= (\vartheta^2 - \lambda\,\omega^3_{\ 1}) \wedge \varphi + \lambda(\omega^3_{\ 1} \wedge \vartheta^1 + \omega^3_{\ 2} \wedge \vartheta^2) = (\vartheta^2 - \lambda\,\omega^3_{\ 1}) \wedge \varphi \end{aligned}$$

and

$$\begin{aligned} 0 &= \widehat{\omega}^3_{\ 1} \wedge \widehat{\omega}^3_{\ 2} - \varepsilon_h\,\widehat{\vartheta}^1 \wedge \widehat{\vartheta}^2 = \vartheta^2 \wedge \lambda\,(\varphi - \vartheta^1) - \varepsilon_h\,(\varphi - \lambda\,\omega^3_{\ 2}) \wedge \omega^3_{\ 1} \\ &= (\lambda\,\vartheta^2 + \varepsilon_h\,\omega^3_1) \wedge \varphi + \varepsilon_h\,\lambda(\varepsilon_h\,\vartheta^1 \wedge \vartheta^2 - \omega^3_{\ 1} \wedge \omega^3_{\ 2}) = (\lambda\,\vartheta^2 + \varepsilon_h\,\omega^3_1) \wedge \varphi. \end{aligned}$$

If $\varepsilon_h + \lambda^2 \neq 0$, then the one-forms $\vartheta^2 - \lambda\,\omega^3_{\ 1}$ and $\lambda\,\vartheta^2 + \varepsilon_h\,\omega^3_1$ are linearly independent, because $(\vartheta^2 - \lambda\,\omega^3_{\ 1}) \wedge (\lambda\,\vartheta^2 + \varepsilon_h\,\omega^3_1) = (\varepsilon_h + \lambda^2)\,\vartheta^2 \wedge \omega^3_{\ 1} \neq 0$ (recall that in the considered case $h_{11} \neq 0$). Consequently the equalities $(\vartheta^2 - \lambda\,\omega^3_{\ 1}) \wedge \varphi = 0$ and $(\lambda\,\vartheta^2 + \varepsilon_h\,\omega^3_1) \wedge \varphi = 0$ imply $\varphi = 0$.

It follows that

$$\begin{aligned} 0 &= \mathrm{d}\widehat{\omega}^3_{\ 3} = -\,\widehat{\omega}^3_{\ 1} \wedge \widehat{\omega}^1_{\ 3} - \widehat{\omega}^3_{\ 2} \wedge \widehat{\omega}^2_{\ 3} = -\,\vartheta^2 \wedge \frac{1}{\lambda}\,\omega^1_{\ 3} + \lambda\,\vartheta^1 \wedge \frac{1}{\lambda^2}\,\omega^2_{\ 3} \\ &= \frac{1}{\lambda}\,(-\,\vartheta^2 \wedge \omega^1_{\ 3} + \vartheta^1 \wedge \omega^2_{\ 3}) \end{aligned}$$

which implies $\operatorname{tr} S = 0$, and

$$\begin{aligned} 0 &= \mathrm{d}\omega^3_{\ 3} = -\,\omega^3_{\ 1} \wedge \omega^1_{\ 3} - \omega^3_{\ 2} \wedge \omega^2_{\ 3} = -\,\widehat{\vartheta}^2 \wedge \lambda\,\widehat{\omega}^1_{\ 3} + \frac{1}{\lambda}\,\widehat{\vartheta}^1 \wedge \lambda^2\,\widehat{\omega}^2_{\ 3} \\ &= \lambda\,(-\,\widehat{\vartheta}^2 \wedge \widehat{\omega}^1_{\ 3} + \widehat{\vartheta}^1 \wedge \widehat{\omega}^2_{\ 3}) \end{aligned}$$

hence also $\operatorname{tr} \widehat{S} = 0$.

We thus get $\operatorname{tr} S = 0$ and $\operatorname{tr} \widehat{S} = 0$ on the set of points where $\operatorname{rank}(\pi \circ (\widehat{f} - f)) = 2$ and $\varepsilon_h + \lambda^2 \neq 0$, and also on its closure, by continuity.

Assume now that $\varepsilon_h + \lambda^2 = 0$ on some open set, contained in the set where $\operatorname{rank}(\pi \circ (\widehat{f} - f)) = 2$ holds. In this case $\mathrm{d}\lambda = 0$, hence $\omega^2_{\ 3} = 0$. We have

$$\begin{aligned} 0 &= \mathrm{d}\omega^2_{\ 3} = -\,\omega^2_{\ 1} \wedge \omega^1_{\ 3} - \omega^2_{\ 2} \wedge \omega^2_{\ 3} = -\,\omega^2_{\ 1} \wedge \omega^1_{\ 3} \\ 0 &= \mathrm{d}\omega^3_{\ 3} = -\,\omega^3_{\ 1} \wedge \omega^1_{\ 3} - \omega^3_{\ 2} \wedge \omega^2_{\ 3} = -\,\omega^3_{\ 1} \wedge \omega^1_{\ 3} \end{aligned}$$

and it follows that

$$\vartheta^2 \wedge \omega^1_{\ 3} - \vartheta^1 \wedge \omega^2_{\ 3} = \vartheta^2 \wedge \omega^1_{\ 3} = (-\,\omega^2_{\ 1} + \lambda\,\omega^3_{\ 1}) \wedge \omega^1_{\ 3} = 0 \tag{39}$$

and

$$\widehat{\vartheta}^2 \wedge \widehat{\omega}^1_{\ 3} - \widehat{\vartheta}^1 \wedge \widehat{\omega}^2_{\ 3} = \omega^3_{\ 1} \wedge \frac{1}{\lambda}\,\omega^1_{\ 3} - \widehat{\vartheta}^1 \wedge \frac{1}{\lambda^2}\,\omega^2_{\ 3} = 0. \tag{40}$$

Finally, we consider the interior of the set where rank $(\pi \circ (\widehat{f} - f)) < 2$. Since $\omega^3_{\ 1} \neq 0$, rank $(\pi \circ (\widehat{f} - f)) \neq 0$. By Lemma 3, $\omega^2_{\ 1} \wedge \omega^3_{\ 1} = 0$. We will show that also in this case proportionality of h and $\widehat{h}$ implies $|\beta| = 1$, $\mathrm{d}\lambda = 0$ and $\omega^2_{\ 3} = 0$ as in the preceding case.

From (19) we get $\beta\,\vartheta^2 + \beta\,\omega^2_{\ 1} = \lambda\,\omega^3_{\ 1}$. Then $\omega^2_{\ 1} \wedge \omega^3_{\ 1} = 0$ and $\beta \neq 0$ imply $\omega^3_{\ 1} \wedge \vartheta^2 = 0$, in particular $h_{11} = h(X_1, X_1) = \omega^3_{\ 1} \wedge \vartheta^2(X_1, X_2) = 0$. Since ξ is an affine normal vector field

$$1 = |H| = |h_{11}\,h_{22} - h_{12}^2| = |h_{12}|^2$$

hence $h(X_1, X_2) = h_{12} = \varepsilon_1 \in \{1, -1\}$ and we see that

$$\omega^3_{\ 1} = \varepsilon_1\,\vartheta^2 \quad \text{and} \quad \omega^3_{\ 2} = \varepsilon_1\,\vartheta^1 + h_{22}\,\vartheta^2. \tag{41}$$

From (41) and (19) we have $\varepsilon_1\,\vartheta^2 = \omega^3_{\ 1} = \beta\,\widehat{\vartheta}^2$ and it follows that $\widehat{X}_1 = c_{11}\,X_1$ for some function c_{11}. Then

$$\widehat{h}_{11} = \widehat{h}(\widehat{X}_1, \widehat{X}_1) = c_{11}^2\,\widehat{h}(X_1, X_1) = 0,$$

because $h_{11} = 0$ and $\widehat{h}$ is proportional to h. Now from $|\widehat{H}| = 1$ we easily obtain

$$\widehat{\omega}^3_{\ 1} = \varepsilon_2\,\widehat{\vartheta}^2 \tag{42}$$

and consequently

$$\vartheta^2 = \varepsilon_1\,\omega^3_{\ 1} = \varepsilon_1\,\beta\,\widehat{\vartheta}^2 = \varepsilon_1\,\beta\,\varepsilon_2\,\widehat{\omega}^3_{\ 1} = \varepsilon_1\,\varepsilon_2\,\beta^2\,\vartheta^2$$

hence $|\beta| = 1$ and $\varepsilon_1 = \varepsilon_2$. Without loss of generality we may assume that $\beta = 1$.

Differentiating both sides of $\omega^3_{\ 1} = \varepsilon_1\,\vartheta^2$, using fundamental equations and the equality $\omega^2_{\ 2} = -\,\omega^1_{\ 1}$ we obtain

$$\omega^3_{\ 1} \wedge \omega^1_{\ 1} + \omega^3_{\ 2} \wedge \omega^2_{\ 1} = \varepsilon_1\,\omega^2_{\ 1} \wedge \vartheta^1 + \varepsilon_1\,\omega^2_{\ 2} \wedge \vartheta^2, \quad \omega^3_{\ 2} \wedge \omega^2_{\ 1} = \varepsilon_1\,\omega^2_{\ 1} \wedge \vartheta^1. \tag{43}$$

We have also

$$\omega^2_{\ 1} \wedge \vartheta^2 = \omega^2_{\ 1} \wedge (-\,\omega^2_{\ 1} + \lambda\,\omega^3_{\ 1}) = 0 \tag{44}$$

$$\omega^3_{\ 2} \wedge \omega^2_{\ 1} = (\varepsilon_1\,\vartheta^1 + h_{22}\,\vartheta^2) \wedge \omega^2_{\ 1} = -\,\varepsilon_1\,\omega^2_{\ 1} \wedge \vartheta^1. \tag{45}$$

Comparing (43) with (45) we see that $\omega^2_{\ 1} \wedge \vartheta^1 = 0$, which together with (44) implies $\omega^2_{\ 1} = 0$. We have now $\vartheta^2 = \lambda\,\omega^3_{\ 1}$ and $\vartheta^2 = \varepsilon_1\,\omega^3_{\ 1}$, therefore $\lambda = \varepsilon_1 =$ const.

5. Bäcklund Theorem Concerning Locally Symmetric Surfaces

Theorem 7. *Let $f : M \to \mathbb{R}^3$ and $\widehat{f} : M \to \mathbb{R}^3$ be non-degenerate immersions of a two-dimensional connected manifold M into affine space $\mathbb{R}^3$, endowed with equiaffine transversal vector fields ξ and $\widehat{\xi}$ respectively.*

We denote by h and $\widehat{h}$ the corresponding affine fundamental forms, and by ν and $\widehat{\nu}$ the conormal maps.

If f, $\widehat{f}$, ξ and $\widehat{\xi}$ satisfy the following conditions

1° *for every $p \in M$ $f(p) \neq \widehat{f}(p)$, the vector $\widehat{f}(p) - f(p)$ is tangent to $f(M)$ at $f(p)$ and is tangent to $\widehat{f}(M)$ at $\widehat{f}(p)$*

2° *the spherical representation of $\widehat{f} - f$, $M \ni p \mapsto \pi(\widehat{f}(p) - f(p)) \in \mathbb{P}^2(\mathbb{R})$, has rank 2 at every point of M*

3° $\det(\widehat{f} - f, \xi, \widehat{\xi}) \neq 0$ *everywhere*

4° *the functions $\nu(\widehat{\xi})$ and $\widehat{\nu}(\xi)$ are constant and $\nu(\widehat{\xi}) \neq 0$ or $\widehat{\nu}(\xi) \neq 0$*

5° $\big(\det(\widehat{f} - f, \xi, \widehat{\xi})\big)^4 \cdot \det_\theta h \cdot \det_{\widehat{\theta}} \widehat{h} = \big(1 - \nu(\widehat{\xi})\,\widehat{\nu}(\xi)\big)^4$

6° *for every $Y \in TM$* $\det(f_*(Y), \xi, \widehat{\xi}) = \det(\widehat{f}_*(Y), \xi, \widehat{\xi})$

7° $\mathrm{d}(\det(\widehat{f} - f, \xi, \widehat{\xi})) \wedge \mathrm{d}(\det_\theta h) = 0$

then affine fundamental forms h and $\widehat{h}$ are conformal to each other, the connections ∇ and $\widehat{\nabla}$ induced by (f, ξ) and $(\widehat{f}, \widehat{\xi})$ respectively, are locally symmetric and $\dim \mathrm{im} R = \dim \mathrm{im} \widehat{R}$.

Proof: We continue analysis from the proof of Theorem 4 with the same notation. From 3° and 5° we conclude that $1 - A\widehat{A} \neq 0$ and

$$\psi(f, \widehat{f}) = \Big(\frac{1 - A\widehat{A}}{W}\Big)^4 \frac{1}{H\,\widehat{H}} = 1 \tag{46}$$

hence h and $\widehat{h}$ are conformal to each other, by Theorem 4.

It remains to prove that $\nabla R = 0$ and $\widehat{\nabla}\widehat{R} = 0$.

From (19) with constant A, $\widehat{A}$ we obtain in particular

$$\begin{aligned}
\widehat{\vartheta}^1 &= \vartheta^1 + \omega^1_{\ 1} \\
\widehat{\omega}^3_{\ 2} &= -\frac{\widehat{A}(1 - A\widehat{A})}{W}\,\omega^2_{\ 2} - \frac{(1 - A\widehat{A})^2}{W^2}\,\omega^2_{\ 3} + (\widehat{A})^2\,\omega^3_{\ 2} \\
&\quad + \frac{\widehat{A}(1 - A\widehat{A})}{W}\,\omega^3_{\ 3} - \frac{\widehat{A}(1 - A\widehat{A})}{W^2}\,\mathrm{d}W \\
\widehat{\omega}^3_{\ 3} &= (1 - A\widehat{A})\,\omega^2_{\ 2} - \widehat{A}\,W\,\omega^3_{\ 2} - \frac{A(1 - A\widehat{A})}{W}\,\omega^2_{\ 3} + A\widehat{A}\,\omega^3_{\ 3} + \frac{1 - A\widehat{A}}{W}\,\mathrm{d}W.
\end{aligned} \tag{47}$$

For equiaffine vector fields ξ and $\widehat{\xi}$ we have $\omega^3_{\ 3} = 0$ and $\widehat{\omega}^3_{\ 3} = 0$, therefore (47) yields

$$\omega^2_{\ 2} = \frac{\widehat{A}W}{1 - A\widehat{A}}\,\omega^3_{\ 2} + \frac{A}{W}\,\omega^2_{\ 3} - \frac{\mathrm{d}W}{W} \tag{48}$$

and substituting (48) into (47) we obtain

$$\widehat{\omega}^3_{\ 2} = -\frac{1 - A\widehat{A}}{W^2}\,\omega^2_{\ 3}. \tag{49}$$

From (19) we have also

$$\begin{aligned}
\widehat{\omega}^2_{\ 3} &= -AW\,\omega^2_{\ 2} + A^2\,\omega^2_{\ 3} - W^2\,\omega^3_{\ 2} + AW\,\omega^3_{\ 3} - A\,\mathrm{d}W \\
&= -AW\left(\omega^2_{\ 2} - \frac{A}{W}\,\omega^2_{\ 3} + \frac{\mathrm{d}W}{W}\right) - W^2\,\omega^3_{\ 2} = -AW\,\frac{\widehat{A}W}{1 - A\widehat{A}}\,\omega^3_{\ 2} - W^2\,\omega^3_{\ 2}
\end{aligned}$$

and it follows that

$$\widehat{\omega}^2_{\ 3} = \frac{-W^2}{1 - A\widehat{A}}\,\omega^3_{\ 2}. \tag{50}$$

The structural equation (17) with $\vartheta^3 = 0$ and $\mathrm{d}\vartheta^3 = 0$ becomes

$$0 = \omega^3_{\ 1} \wedge \vartheta^1 + \omega^3_{\ 2} \wedge \vartheta^2. \tag{51}$$

Let $\vartheta^1 = s\,\omega^2_{\ 1} + t\,\omega^3_{\ 1}$ and $\omega^3_{\ 2} = u\,\omega^2_{\ 1} + v\,\omega^3_{\ 1}$ with some functions s, t, u and v. Applying (51), (23) and $\omega^2_{\ 1} \wedge \omega^3_{\ 1} \neq 0$ yields

$$s = \frac{\widehat{A}Wu}{1 - A\widehat{A}} + v. \tag{52}$$

From (29) we obtain

$$\omega^3_{\ 1} \wedge (u\,\omega^2_{\ 1} + v\,\omega^3_{\ 1}) = H\,(s\,\omega^2_{\ 1} + t\,\omega^3_{\ 1}) \wedge \Big(-\,\omega^2_{\ 1} + \frac{\widehat{A}W}{1 - A\widehat{A}}\,\omega^3_{\ 1}\Big)$$

which implies

$$t = -\,\frac{u}{H} - \frac{\widehat{A}Ws}{1 - A\widehat{A}} = -\Big(\frac{1}{H} + \frac{\widehat{A}^2\,W^2}{(1 - A\widehat{A})^2}\Big)u - \frac{\widehat{A}W}{1 - A\widehat{A}}\,v. \tag{53}$$

Consequently

$$\vartheta^1 = \Big(\frac{\widehat{A}Wu}{1 - A\widehat{A}} + v\Big)\,\omega^2_{\ 1} - \Big(\frac{u}{H} + \frac{\widehat{A}^2\,W^2\,u}{(1 - A\widehat{A})^2} + \frac{\widehat{A}W\,v}{1 - A\widehat{A}}\Big)\,\omega^3_{\ 1}. \tag{54}$$

We use now the assumption 6°. Since

$$\begin{gathered}
\widehat{f}_*(Y) - f_*(Y) = D_Y(\widehat{f} - f) = D_Y f_* X_1 = \omega^1_{\ 1}(Y)\,f_*X_1 + \omega^2_{\ 1}(Y)\,f_*X_2 \\
+\omega^3_{\ 1}(Y)\,\xi = \omega^1_{\ 1}(Y)\,f_*X_1 + \omega^2_{\ 1}(Y)\Big(\frac{A}{W}\,\xi - \frac{1}{W}\,\widehat{\xi}\Big) + \omega^3_{\ 1}(Y)\,\xi \\
\det(\widehat{f}_*(Y) - f_*(Y), \xi, \widehat{\xi}) = \omega^1_{\ 1}(Y)\det(f_*(X_1), \xi, \widehat{\xi}) = \omega^1_{\ 1}(Y)\,W
\end{gathered}$$

the equality

$$\det(f_*(Y), \xi, \widehat{\xi}) - \det(\widehat{f}_*(Y), \xi, \widehat{\xi}) = 0$$

gives $\omega^1_{\ 1} = 0$ and consequently $\omega^2_{\ 2} = 0$, because differentiating the equality (7) we obtain $\omega^1_{\ 1} + \omega^2_{\ 2} + \omega^3_{\ 3} = 0$. Similarly from (9) we obtain $\widehat{\omega}^1_{\ 1} + \widehat{\omega}^2_{\ 2} + \widehat{\omega}^3_{\ 3} = 0$ and from (19) it follows that $\widehat{\omega}^1_{\ 1} = \omega^1_{\ 1}$, therefore $\widehat{\omega}^1_{\ 1} = 0$, $\widehat{\omega}^2_{\ 2} = 0$ and $\widehat{\vartheta}^1 = \vartheta^1$.

Let $\widehat{\omega}^3_{\ 2} = x\,\omega^2_{\ 1} + y\,\omega^3_{\ 1}$ with some functions x, y. Then from the structural equation $\widehat{\omega}^3_{\ 1} \wedge \widehat{\vartheta}^1 + \widehat{\omega}^3_{\ 2} \wedge \widehat{\vartheta}^2 = 0$, (24), (54) and (23) we obtain

$$x = -\,\frac{(1 - A\widehat{A})^2}{W^2H}\,u.$$

Using (30) we obtain

$$y = \frac{1 - A\widehat{A}}{WH}\,\widehat{A}u - \frac{\widehat{H}W^3}{(1 - A\widehat{A})^3}\,\widehat{A}u - \frac{\widehat{H}W^2}{(1 - A\widehat{A})^2}\,v.$$

But $W^4 H\widehat{H} = (1 - A\widehat{A})^4$, by 4°, hence $y = -\dfrac{(1 - A\widehat{A})^2}{HW^2}\, v$ and

$$\widehat{\omega}^3_{\ 2} = -\frac{(1 - A\widehat{A})^2}{HW^2}\, \omega^3_{\ 2}. \tag{55}$$

Comparing (55) with (49) we obtain

$$\omega^2_{\ 3} = \frac{1 - A\widehat{A}}{H}\, \omega^3_{\ 2} \tag{56}$$

and from (48) with $\omega^2_{\ 2} = 0$

$$\mathrm{d}W = \Big(\frac{\widehat{A}W^2}{1 - A\widehat{A}} + \frac{A(1 - A\widehat{A})}{H}\Big)\omega^3_{\ 2}. \tag{57}$$

It follows that $\omega^2_{\ 3} \wedge \omega^3_{\ 2} = 0$. From the fundamental equation

$$0 = \mathrm{d}\omega^2_{\ 2} = -\,\omega^2_{\ 1} \wedge \omega^1_{\ 2} - \omega^2_{\ 2} \wedge \omega^2_{\ 2} - \omega^2_{\ 3} \wedge \omega^3_{\ 2}$$

we obtain $\omega^2_{\ 1} \wedge \omega^1_{\ 2} = 0$, which means that

$$\omega^1_{\ 2} = \alpha\, \omega^2_{\ 1} \tag{58}$$

for some function α. Similarly $\omega^1_{\ 3} = \beta\, \omega^3_{\ 1}$, which follows from

$$0 = \mathrm{d}\omega^3_{\ 3} = -\,\omega^3_{\ 1} \wedge \omega^1_{\ 3} - \omega^3_{\ 2} \wedge \omega^2_{\ 3} - \omega^3_{\ 3} \wedge \omega^3_{\ 3}.$$

In the same way we obtain $\widehat{\omega}^2_{\ 1} \wedge \widehat{\omega}^1_{\ 2} = 0$. From (19) we have

$$\widehat{\omega}^1_{\ 2} = \widehat{A}\,\omega^1_{\ 2} + \frac{1 - A\widehat{A}}{W}\,\omega^1_{\ 3}, \qquad \widehat{\omega}^1_{\ 3} = -\,W\,\omega^1_{\ 2} + A\,\omega^1_{\ 3}. \tag{59}$$

Using (24) and (59) we obtain

$$\begin{aligned} \widehat{\omega}^2_{\ 1} \wedge \widehat{\omega}^1_{\ 2} &= (A\,\omega^2_{\ 1} + W\,\omega^3_{\ 1}) \wedge \Big(\widehat{A}\,\alpha\omega^2_{\ 1} + \frac{1 - A\widehat{A}}{W}\,\beta\omega^3_{\ 1}\Big) \\ &= \Big(\frac{A(1 - A\widehat{A})}{W}\,\beta - \widehat{A}W\,\alpha\Big)\omega^2_{\ 1} \wedge \omega^3_{\ 1}. \end{aligned} \tag{60}$$

At first we consider the case $A \neq 0$. It follows that

$$\beta = \frac{\widehat{A}W^2}{A(1 - A\widehat{A})}\,\alpha \tag{61}$$

and

$$\omega^1_{\ 3} = \frac{\alpha \widehat{A} W^2}{A(1 - A\widehat{A})}\, \omega^3_{\ 1}. \tag{62}$$

We have now

$$\widehat{\omega}^1_{\ 2} = \frac{\widehat{A}}{A}\, \alpha\, \widehat{\omega}^2_{\ 1} \tag{63}$$

and

$$\widehat{\omega}^1_{\ 3} = \frac{\alpha W^2}{1 - A\widehat{A}}\, \widehat{\omega}^3_{\ 1}. \tag{64}$$

We can already find the curvature tensors of ∇ and $\widehat{\nabla}$. We have

$$\nabla_Y X_1 = \omega^1_{\ 1}(Y)\, X_1 + \omega^2_{\ 1}(Y)\, X_2 = \omega^2_{\ 1}(Y)\, X_2 \tag{65}$$

and

$$\nabla_Y X_2 = \omega^1_{\ 2}(Y)\, X_1 + \omega^2_{\ 2}(Y)\, X_2 = \alpha\, \omega^2_{\ 1}(Y)\, X_1. \tag{66}$$

The Gauss equation

$$\mathrm{d}\omega^k_{\ l} + \omega^k_{\ 1} \wedge \omega^1_{\ l} + \omega^k_{\ 2} \wedge \omega^2_{\ l} = -\, \omega^k_{\ 3} \wedge \omega^3_{\ l}, \quad k, l \in \{1, 2\}$$

now leads to

$$R(X, Y)X_1 = -\, \omega^2_{\ 3} \wedge \omega^3_{\ 1}(X, Y)\, X_2 \tag{67}$$

and

$$R(X, Y)X_2 = -\, \omega^1_{\ 3} \wedge \omega^3_{\ 2}(X, Y)\, X_1. \tag{68}$$

In particular

$$R(X_1, X_2)\, X_1 = (1 - A\widehat{A})\, X_2 \tag{69}$$

and

$$R(X_1, X_2)\, X_2 = -\, \alpha \frac{\widehat{A} W^2 H}{A(1 - A\widehat{A})}\, X_1. \tag{70}$$

The Ricci tensor is

$$\mathrm{Ric}(X_1, X_1) = -\,(1 - A\widehat{A}),\ \ \mathrm{Ric}(X_1, X_2) = 0,\ \ \mathrm{Ric}(X_2, X_2) = -\, \frac{\alpha \widehat{A} W^2 H}{A(1 - A\widehat{A})}.$$

Applying (65), (66), (69) and (70) we obtain

$$(\nabla_Y R)(X_1, X_2)X_1 = (1 - A\widehat{A})\alpha\Big(\frac{\widehat{A} W^2 H}{A(1 - A\widehat{A})^2} + 1\Big)\omega^2_{\ 1}(Y)\, X_1$$

and

$$(\nabla_Y R)(X_1, X_2)\, X_2$$
$$= -\,Y\Big(\frac{\alpha\widehat{A}W^2H}{A(1-A\widehat{A})}\Big)X_1 - (1-A\widehat{A})\alpha\Big(\frac{\widehat{A}W^2H}{A(1-A\widehat{A})^2}+1\Big)\omega^2{}_1(Y)\, X_2.$$

For $\widehat{\nabla}$ we obtain

$$\widehat{\nabla}_Y \widehat{X}_1 = \widehat{\omega}^2{}_1(Y)\, \widehat{X}_2, \qquad \widehat{\nabla}_Y \widehat{X}_2 = \frac{\widehat{A}\,\alpha}{A}\, \widehat{\omega}^2{}_1(Y)\, \widehat{X}_1$$

$$\begin{aligned}
\widehat{R}(\widehat{X}_1,\widehat{X}_2)\widehat{X}_1 &= -\,\widehat{\omega}^2{}_3 \wedge \widehat{\omega}^3{}_1(\widehat{X}_1,\widehat{X}_2)\,\widehat{X}_2 = \frac{-\,HW^4}{(1-A\widehat{A})^3}\,\widehat{\omega}^3{}_2 \wedge \widehat{\omega}^3{}_1(\widehat{X}_1,\widehat{X}_2)\,\widehat{X}_2\\
&= \frac{W^4H\widehat{H}}{(1-A\widehat{A})^3}\,\widehat{X}_2 = (1-A\widehat{A})\,\widehat{X}_2,\\
\widehat{R}(\widehat{X}_1,\widehat{X}_2)\widehat{X}_2 &= -\,\widehat{\omega}^1{}_3 \wedge \widehat{\omega}^3{}_2(\widehat{X}_1,\widehat{X}_2)\,\widehat{X}_1 = \frac{-\,\alpha W^2}{1-A\widehat{A}}\,\widehat{\omega}^3{}_1 \wedge \widehat{\omega}^3{}_2(\widehat{X}_1,\widehat{X}_2)\,\widehat{X}_1\\
&= \frac{-\alpha W^2\widehat{H}}{1-A\widehat{A}}\,\widehat{X}_1 = -\,\frac{\alpha\,(1-A\widehat{A})^3}{W^2H}\,\widehat{X}_1
\end{aligned}$$

$$\widehat{\mathrm{Ric}}(\widehat{X}_1,\widehat{X}_1) = -\,(1-A\widehat{A}),\ \ \widehat{\mathrm{Ric}}(\widehat{X}_1,\widehat{X}_2) = 0,\ \ \widehat{\mathrm{Ric}}(\widehat{X}_2,\widehat{X}_2) = \frac{-\,\alpha\,(1-A\widehat{A})^3}{W^2H}$$

$$\begin{aligned}
(\widehat{\nabla}_Y\widehat{R})(\widehat{X}_1,\widehat{X}_2)\widehat{X}_1 =& \frac{\alpha\,(1-A\widehat{A})^3}{W^2H}\left(\frac{\widehat{A}W^2H}{A(1-A\widehat{A})^2}+1\right)\widehat{\omega}^2{}_1(Y)\,\widehat{X}_1\\
(\widehat{\nabla}_Y\widehat{R})(\widehat{X}_1,\widehat{X}_2)\widehat{X}_2 =& -\ (1-A\widehat{A})^3 Y\left(\frac{\alpha}{W^2H}\right)\widehat{X}_1\\
&-\frac{\alpha\,(1-A\widehat{A})^3}{W^2H}\left(\frac{\widehat{A}W^2H}{A(1-A\widehat{A})^2}+1\right)\widehat{\omega}^2{}_1(Y)\,\widehat{X}_2.
\end{aligned}$$

Next we want to use the assumption 7°: $\mathrm{d}W \wedge \mathrm{d}H = 0$.

Diferentiating (56) we obtain

$$\mathrm{d}\omega^2{}_3 = -\,\frac{1-A\widehat{A}}{H^2}\,\mathrm{d}H \wedge \omega^3{}_2 + \frac{1-A\widehat{A}}{H}\,\mathrm{d}\omega^3{}_2.$$

From the fundamental equations and from (58) and (62) we get

$$\mathrm{d}\omega^2_{\ 3} = -\,\omega^2_{\ 1}\wedge\omega^1_{\ 3} = -\,\frac{\alpha\widehat{A}W^2}{A(1-A\widehat{A})}\,\omega^2_{\ 1}\wedge\omega^3_{\ 1},\ \ \mathrm{d}\omega^3_{\ 2} = -\,\omega^3_{\ 1}\wedge\omega^1_{\ 2} = \alpha\,\omega^2_{\ 1}\wedge\omega^3_{\ 1}.$$

It follows that

$$\frac{\mathrm{d}H}{H}\wedge\omega^3_{\ 2} = \alpha\Big(\frac{\widehat{A}W^2H}{A(1-A\widehat{A})^2}+1\Big)\omega^2_{\ 1}\wedge\omega^3_{\ 1} \tag{71}$$

and consequently, by (57)

$$\mathrm{d}H\wedge\mathrm{d}W = A(1-A\widehat{A})\,\alpha\Big(\frac{\widehat{A}W^2H}{A(1-A\widehat{A})^2}+1\Big)^2\omega^2_{\ 1}\wedge\omega^3_{\ 1}. \tag{72}$$

If $\widehat{A}=0$ (and still $A\neq 0$), then (72) and $\mathrm{d}H\wedge\mathrm{d}W=0$ imply $\alpha\equiv 0$.

If $\widehat{A}\neq 0$ we may compute $\mathrm{d}\alpha$ in the following way.
Differentiating (58) and (62) we obtain

$$\mathrm{d}\omega^1_{\ 2} = \mathrm{d}\alpha\wedge\omega^2_{\ 1} + \alpha\,\mathrm{d}\omega^2_{\ 1}$$

$$\mathrm{d}\omega^1_{\ 3} = \frac{\widehat{A}W^2}{A(1-A\widehat{A})}\,\mathrm{d}\alpha\wedge\omega^3_{\ 1} + \frac{2\alpha\widehat{A}W}{A(1-A\widehat{A})}\,\mathrm{d}W\wedge\omega^3_{\ 1} + \frac{\alpha\widehat{A}W^2}{A(1-A\widehat{A})}\,\mathrm{d}\omega^3_{\ 1}$$

and next, after using the fundamental equations, (57) and $\omega^3_{\ 2} = u\,\omega^2_{\ 1} + v\,\omega^3_{\ 1}$

$$\mathrm{d}\alpha\wedge\omega^2_{\ 1} = \alpha\Big(\frac{\widehat{A}W^2H}{A(1-A\widehat{A})^2}+1\Big)\frac{1-A\widehat{A}}{H}\,u\,\omega^2_{\ 1}\wedge\omega^3_{\ 1}$$

$$\mathrm{d}\alpha\wedge\omega^3_{\ 1} = -\,\alpha\Big(\frac{\widehat{A}W^2H}{A(1-A\widehat{A})^2}+1\Big)\frac{1-A\widehat{A}}{H}\Big(\frac{2A}{W}\,u + \frac{A(1-A\widehat{A})}{\widehat{A}W^2}\,v\Big)\omega^2_{\ 1}\wedge\omega^3_{\ 1}.$$

It follows that

$$\mathrm{d}\alpha = -\,\alpha\Big(\frac{\widehat{A}W^2H}{A(1-A\widehat{A})^2}+1\Big)\frac{1-A\widehat{A}}{H}\Big(\Big(\frac{2A}{W}\,u + \frac{A(1-A\widehat{A})}{\widehat{A}W^2}\,v\Big)\omega^2_{\ 1} + u\,\omega^3_{\ 1}\Big). \tag{73}$$

From (72) and $\mathrm{d}H\wedge\mathrm{d}W=0$ it follows that $\alpha\left(\frac{\widehat{A}W^2H}{A(1-A\widehat{A})^2}+1\right)\equiv 0$ on M. Then from (73) we conclude that α is constant, because M is connected.

Now we consider the case $A=0$. Then, by assumption 4°, $\widehat{A}\neq 0$. We return to (60) and obtain $\alpha\equiv 0$.

Thus in each case $\alpha = \text{const}$.

If $\alpha = 0$, then $\mathrm{im} R_p = \mathbb{R}\,(X_2)_p$, $\mathrm{im}\widehat{R}_p = \mathbb{R}\,(\widehat{X}_2)_p$, $\dim \mathrm{im} R = \dim \mathrm{im} \widehat{R} = 1$ and $\operatorname{sign} \mathrm{Ric} = \operatorname{sign} \widehat{\mathrm{Ric}} = -\operatorname{sign}(1 - A\widehat{A})$.

Let $\alpha \neq 0$. Then $\frac{\widehat{A}W^2 H}{A(1-A\widehat{A})^2} + 1 \equiv 0$, which implies

$$H = -\frac{A(1-A\widehat{A})^2}{\widehat{A}W^2}.$$

From (57) it follows that W is constant. This clearly forces H to be constant.

In both cases ($\alpha = 0$, $\alpha \neq 0$) we obtain $\nabla R = 0$ and $\widehat{\nabla}\widehat{R} = 0$.

We shall show that the case of $\alpha \neq 0$ corresponds to the situation described in the classical Bäcklund theorem or in the Bäcklund theorem for surfaces in Minkowski space.

Theorem 8. *If f, $\widehat{f}$, ξ, $\widehat{\xi}$ satisfy the assumptions of Theorem 7 and the induced connections ∇, $\widehat{\nabla}$ satisfy the condition $\dim \mathrm{im} R = \dim \mathrm{im} \widehat{R} = 2$, then $\det(\widehat{f} - f, \xi, \widehat{\xi})$ and $\det_\theta h$ are constant, $\mathbb{R}\xi$ and $\mathbb{R}\widehat{\xi}$ are the corresponding affine normals and there exists a scalar or pseudoscalar product on $\mathbb{R}^3$ such that ξ and $\widehat{\xi}$ are orthogonal to the corresponding surfaces with constant, non-zero, length. Moreover the length of $\widehat{f} - f$ is constant, the angle between ξ and $\widehat{\xi}$ is constant and f and $\widehat{f}$ have the same constant sectional curvature.*

Proof: We define $G_p \in (\mathbb{R}^3)^*$ by the equalities

$$\begin{aligned}
&G_p(f_*(X_1)_p, f_*(X_1)_p) := -\,\delta(1 - A\widehat{A}), \qquad && G_p(\xi_p, \xi_p) := \delta\alpha\frac{\widehat{A}}{A}\,W^2\\
&G_p(f_*(X_1)_p, f_*(X_2)_p) := 0 && \\
&G_p(f_*(X_2)_p, f_*(X_2)_p) := \delta\alpha(1 - A\widehat{A}), && G_p(f_*X_p, \xi_p) := 0
\end{aligned}$$

with some $\delta \in \{1, -1\}$. We have

$$\begin{aligned}
SY &= -\,\omega^1_{\ 3}(Y)\,X_1 - \omega^2_{\ 3}(Y)\,X_2\\
&= -\frac{\alpha\,\widehat{A}W^2}{A(1-A\widehat{A})}\,\omega^3_{\ 1}(Y)X_1 - \frac{1 - A\widehat{A}}{H}\,\omega^3_{\ 2}(Y)\,X_2\\
&= -\frac{\alpha\,\widehat{A}W^2}{A(1-A\widehat{A})}\,\omega^3_{\ 1}(Y)X_1 + \frac{\widehat{A}W^2}{A(1-A\widehat{A})}\,\omega^3_{\ 2}(Y)\,X_2\\
&= \frac{\widehat{A}W^2}{A(1-A\widehat{A})}\big(-\,\alpha\,\omega^3_{\ 1}(Y)\,X_1 + \omega^3_{\ 2}(Y)\,X_2\big)
\end{aligned}$$

and

$$
\begin{aligned}
G(f_*X, f_*SY) =& \frac{\widehat{A}W^2}{A(1-A\widehat{A})}\, G\big(\omega^1(X)\, f_*(X_1) + \omega^2(X)\, f_*(X_2), \\
& -\alpha\, \omega^3{}_1(Y) f_*(X_1) + \omega^3{}_2(Y)\, f_*(X_2)\big) \\
=& \frac{\widehat{A}W^2}{A(1-A\widehat{A})}\, \big(\omega^1(X)\omega^3{}_1(Y) + \omega^2(X)\omega^3{}_2(Y)\big)\, \alpha\, \delta\, (1-A\widehat{A}) \\
=& \delta\, \alpha\, \frac{\widehat{A}}{A}\, W^2\, h(Y,X).
\end{aligned}
$$

Now it is easy to check that $DG = 0$, hence we have well defined scalar product on $\mathbb{R}^3$, which also will be denoted by G. The Riemannian or pseudo-Riemannian metric g induced on M by f, $g(X,Y) = G(f_*(X), f_*(Y))$, has the sectional curvature

$$
\begin{aligned}
\kappa &= \frac{g(R(X_1,X_2)X_2, X_1)}{g(X_1,X_1)\, g(X_2,X_2) - g(X_1,X_2)\, g(X_1,X_2)} \\
&= \frac{g(\alpha\,(1-A\widehat{A})\, X_1, X_1)}{-\alpha\, \delta^2\, (1-A\widehat{A})^2} = \frac{-\delta\, \alpha\, (1-A\widehat{A})^2}{-\alpha\, \delta^2\, (1-A\widehat{A})^2} = \delta
\end{aligned}
$$

and the same curvature has the metric $\widehat{g}$ induced by $\widehat{f}$

$$
\begin{aligned}
\widehat{\kappa} &= \frac{\widehat{g}(\widehat{R}(\widehat{X}_1,\widehat{X}_2)\widehat{X}_2, \widehat{X}_1)}{\widehat{g}(\widehat{X}_1,\widehat{X}_1)\, \widehat{g}(\widehat{X}_2,\widehat{X}_2) - \widehat{g}(\widehat{X}_1,\widehat{X}_2)\, \widehat{g}(\widehat{X}_1,\widehat{X}_2)} = \frac{\widehat{g}\left(-\frac{\alpha\,(1-A\widehat{A})^3}{W^2H}\, \widehat{X}_1, \widehat{X}_1\right)}{-\alpha\, \delta^2\, (1-A\widehat{A})^2 \frac{\widehat{A}}{A}} \\
&= \frac{A(1-A\widehat{A})}{\widehat{A}W^2H}\, \widehat{g}(\widehat{X}_1,\widehat{X}_1) = -\frac{A(1-A\widehat{A})}{\widehat{A}W^2H}\, \delta\, (1-A\widehat{A}) = \delta
\end{aligned}
$$

because

$$
\begin{aligned}
\widehat{g}(\widehat{X}_1,\widehat{X}_1) &= G(\widehat{f}_*(\widehat{X}_1), \widehat{f}_*(\widehat{X}_1)) = G(f_*(X_1), f_*(X_1)) = -\delta\, (1-A\widehat{A}) \\
\widehat{g}(\widehat{X}_1,\widehat{X}_2) &= G(\widehat{f}_*(\widehat{X}_1), \widehat{f}_*(\widehat{X}_2)) = G\left(f_*(X_1), \widehat{A}\, f_*(X_2) + \frac{1-A\widehat{A}}{W}\, \xi\right) = 0 \\
\widehat{g}(\widehat{X}_2,\widehat{X}_2) &= G(\widehat{f}_*(\widehat{X}_2), \widehat{f}_*(\widehat{X}_2)) \\
&= G\left(\widehat{A}\, f_*(X_2) + \frac{1-A\widehat{A}}{W}\, \xi, \widehat{A}\, f_*(X_2) + \frac{1-A\widehat{A}}{W}\, \xi\right) \\
&= \widehat{A}^2\, \delta\, \alpha\, (1-A\widehat{A}) + \frac{(1-A\widehat{A})^2}{W^2}\, \delta\, \alpha\, \frac{\widehat{A}}{A}\, W^2 = \delta\, \alpha\, (1-A\widehat{A})\, \frac{\widehat{A}}{A}
\end{aligned}
$$

and $\dfrac{A(1-A\widehat{A})^2}{\widehat{A}W^2H} = -1$.

We compute

$$\begin{aligned} G(\widehat{f}-f,\widehat{f}-f) &= G(f_*(X_1),f_*(X_1)) = -\delta(1-A\widehat{A})\\ G(\xi,\widehat{\xi}) &= G(\xi,-W f_*(X_2)+A\,\xi) = A\,G(\xi,\xi) = \delta\,\alpha\,\widehat{A}\,W^2\\ G(\widehat{\xi},\widehat{\xi}) &= G(-W f_*(X_2)+A\,\xi,-W f_*(X_2)+A\,\xi)\\ &= W^2\,\delta\,\alpha\,(1-A\widehat{A}) + A^2\,\delta\,\alpha\,\frac{\widehat{A}}{A}\,W^2 = \delta\,\alpha\,W^2. \end{aligned}$$

There are five possibilities and we will consider the corresponding cases separately.

i) *Euclidean case*

If $0 < A\widehat{A} < 1$ and $\alpha < 0$, then we take $\delta = -1$ and obtain positively definite G. Then the square of the length L of $\widehat{f}-f$ is equal to the positive constant $1-A\widehat{A}$ and the angle $\measuredangle(\xi,\widehat{\xi})$ between ξ and $\widehat{\xi}$ is constant too, with

$$\cos\measuredangle(\xi,\widehat{\xi}) = \mathrm{sign}\widehat{A}\cdot\sqrt{A\widehat{A}}.$$

Note that

$$-\frac{\sin^2(\measuredangle(\xi,\widehat{\xi}))}{L^2} = -\frac{1-\cos^2(\measuredangle(\xi,\widehat{\xi}))}{L^2} = -\frac{1-A\widehat{A}}{1-A\widehat{A}} = -1 = \delta = \kappa = \widehat{\kappa}.$$

ii) *Lorentzian case with timelike congruence $\widehat{f}-f$ and timelike focal surfaces f and $\widehat{f}$*

If $0 < A\widehat{A} < 1$ and $\alpha > 0$, then we take $\delta = 1$. We obtain $G(\widehat{f}-f,\widehat{f}-f) = -(1-A\widehat{A}) =: -L^2$. The plane spanned by ξ_p and $\widehat{\xi}_p$ is spacelike, hence

$$\cos\measuredangle(\xi,\widehat{\xi}) = \frac{G(\xi,\widehat{\xi})}{\sqrt{G(\xi,\xi)}\sqrt{G(\widehat{\xi},\widehat{\xi})}} = \mathrm{sign}\widehat{A}\cdot\sqrt{A\widehat{A}}.$$

We obtain

$$\frac{\sin^2(\measuredangle(\xi,\widehat{\xi}))}{L^2} = 1 = \delta = \kappa = \widehat{\kappa}.$$

This case corresponds to (A) of Theorem 2.2 in [1].

iii) *Lorentzian case with spacelike congruence $\widehat{f} - f$ and timelike focal surfaces f and $\widehat{f}$*

If $A\widehat{A} > 1$ and $\alpha > 0$, then we take $\delta = 1$ and obtain $G(\widehat{f} - f, \widehat{f} - f) = -(1 - A\widehat{A}) =: L^2$. Both ξ_p and $\widehat{\xi}_p$ are spacelike, but the plane $\operatorname{span}\{\xi_p, \widehat{\xi}_p\} = \operatorname{span}\{f_*(X_{2p}), \xi_p\}$ is timelike. The hyperbolic angle $\measuredangle(\xi, \widehat{\xi})$ between two space-like vectors satisfies the equality

$$\cosh^2(\measuredangle(\xi, \widehat{\xi})) = \frac{\left(G(\xi, \widehat{\xi})\right)^2}{G(\xi, \xi)G(\widehat{\xi}, \widehat{\xi})}$$

which follows from the definition given in [4]. We obtain $\cosh^2(\measuredangle(\xi, \widehat{\xi})) = A\widehat{A}$ and

$$\frac{\sinh^2(\measuredangle(\xi, \widehat{\xi}))}{L^2} = \frac{\cosh^2(\measuredangle(\xi, \widehat{\xi})) - 1}{L^2} = \frac{A\widehat{A} - 1}{L^2} = 1 = \delta = \kappa = \widehat{\kappa}.$$

This case corresponds to (B) of Theorem 2.2 in [1].

iv) *Lorentzian case with spacelike congruence $\widehat{f} - f$ and spacelike focal surfaces f and $\widehat{f}$*

If $A\widehat{A} > 1$ and $\alpha < 0$, then we take $\delta = 1$. We have $G(\widehat{f} - f, \widehat{f} - f) = -(1 - A\widehat{A}) =: L^2$ as before, the hyperbolic angle between two timelike vectors satisfies the same equality as above and we obtain again

$$\frac{\sinh^2(\measuredangle(\xi, \widehat{\xi}))}{L^2} = 1 = \delta = \kappa = \widehat{\kappa}.$$

This result is in contradiction with that of Theorem 2.1 in [1], where the curvature was claimed to be negative. (It seems that in [1] there is a mistake in going from (2.18) to (2.19), probably $\mathrm{d}\omega_{13}$ and $\mathrm{d}\omega_{23}$ were incorrect. Moreover, (2.9) on page 43 is in contradiction with $K = -\det h_{ij}$ on page 44.)

v) *Lorentzian case with spacelike congruence $\widehat{f} - f$ and focal surfaces f and $\widehat{f}$ of different kinds*

If $A\widehat{A} < 0$, then we take $\delta = -1$. Now $G(\widehat{f} - f, \widehat{f} - f) = 1 - A\widehat{A} =: L^2$ is positive, whereas $G(\xi, \xi)$ and $G(\widehat{\xi}, \widehat{\xi})$ have opposite signs, because $\frac{\widehat{A}}{A} < 0$. According to the definition of the hyperbolic angle between timelike vector and spacelike vector, given in [4], $\measuredangle(\xi, \widehat{\xi})$ satisfies now the equality

$$\sinh^2(\measuredangle(\xi, \widehat{\xi})) = -\frac{\left(G(\xi, \widehat{\xi})\right)^2}{G(\xi, \xi)G(\widehat{\xi}, \widehat{\xi})}.$$

We obtain $\sinh^2(\measuredangle(\xi,\widehat{\xi})) = -A\widehat{A}$ and

$$-\frac{\cosh^2(\measuredangle(\xi,\widehat{\xi}))}{L^2} = -\frac{1+\sinh^2(\measuredangle(\xi,\widehat{\xi}))}{L^2} = -\frac{1-A\widehat{A}}{1-A\widehat{A}} = -1 = \delta = \kappa = \widehat{\kappa}.$$

The Bäcklund theorem for surfaces of different kinds in Minkowski space can be found in [7].

Remark. In case when both $\nu(\widehat{\xi})$ and $\widehat{\nu}(\xi)$ both equal zero we obtain $W = \mathrm{const}$, $\mathrm{d}W = 0$, the assumption 7° is satisfied, but we get therefrom no information about relation between α and β. This case may be characterized by the following proposition.

Proposition 9. *Let f, $\widehat{f}$, ξ, $\widehat{\xi}$ satisfy assumptions 1°, 2°, 3°, 5° and 6° of* Theorem 7 *and let $\nu(\widehat{\xi}) \equiv 0$ and $\widehat{\nu}(\xi) \equiv 0$. Then there exist local coordinates x, y and functions $H = \det_\theta h$, α, β, γ satisfying the system of equations*

$$\begin{aligned} \alpha &= W^2 H_y \mathrm{e}^{-2\gamma}\gamma_y + W^2 H(\mathrm{e}^{-2\gamma}\gamma_y)_y + (\mathrm{e}^{2\gamma}\gamma_x)_x \\ \beta &= -W^2(\mathrm{e}^{-2\gamma}\gamma_y)_y - \frac{1}{H}(\mathrm{e}^{2\gamma}\gamma_x)_x + \frac{H_x}{H^2}\mathrm{e}^{2\gamma}\gamma_x \\ \alpha_y &= (\alpha+\beta H)\gamma_y, \qquad \beta_x = -\frac{1}{H}(\alpha+\beta H)\gamma_x \end{aligned} \tag{74}$$

such that ϑ^i, $\widehat{\vartheta}^i$, $\omega^j_{\ k}$ and $\widehat{\omega}^j_{\ k}$ have the following form

$$\begin{aligned} &\vartheta^1 = \widehat{\vartheta}^1 = \mathrm{d}\gamma = \gamma_x\,\mathrm{d}x + \gamma_y\,\mathrm{d}y \\ &\vartheta^2 = \mathrm{e}^{-\gamma}\,\mathrm{d}x, \qquad \widehat{\vartheta}^2 = \mathrm{e}^{\gamma}\,\mathrm{d}y \\ &\omega^2_{\ 1} = -\mathrm{e}^{-\gamma}\,\mathrm{d}x, \quad \omega^3_{\ 1} = \frac{\mathrm{e}^{\gamma}}{W}\,\mathrm{d}y, \quad \omega^1_{\ 2} = -\alpha\,\mathrm{e}^{-\gamma}\,\mathrm{d}x, \quad \omega^1_{\ 3} = \frac{\beta\,\mathrm{e}^{\gamma}}{W}\,\mathrm{d}y \\ &\omega^3_{\ 2} = H\,\omega^2_{\ 3} = HW\mathrm{e}^{-2\gamma}\gamma_y\,\mathrm{d}x - \frac{\mathrm{e}^{2\gamma}}{W}\gamma_x\,\mathrm{d}y \\ &\widehat{\omega}^2_{\ 1} = \mathrm{e}^{\gamma}\,\mathrm{d}y, \quad \widehat{\omega}^3_{\ 1} = \frac{\mathrm{e}^{-\gamma}}{W}\,\mathrm{d}x, \quad \widehat{\omega}^1_{\ 2} = \frac{\beta\,\mathrm{e}^{\gamma}}{W^2}\,\mathrm{d}y, \quad \widehat{\omega}^1_{\ 3} = \alpha\,W\mathrm{e}^{-\gamma}\,\mathrm{d}x \\ &\widehat{\omega}^2_{\ 3} = HW^4\,\widehat{\omega}^3_{\ 2} = -HW^3\,\mathrm{e}^{-2\gamma}\gamma_y\,\mathrm{d}x + W\,\mathrm{e}^{2\gamma}\gamma_x\,\mathrm{d}y. \end{aligned}$$

Moreover, $\gamma_x \neq 0$, $\gamma_y \neq 0$ and $W = \det(\widehat{f} - f, \xi, \widehat{\xi})$ is a non-zero constant.

The connection ∇ is locally symmetric if and only in α is constant, and $\widehat{\nabla}$ is locally symmetric if and only if β is constant.

Note that from (74) we obtain

$$\alpha + \beta H = W^2\, H_y\, \mathrm{e}^{-2\gamma}\, \gamma_y + \frac{H_x}{H}\, \mathrm{e}^{2\gamma}\, \gamma_x. \tag{75}$$

Proof: If we insert $A = 0$ and $\widehat{A} = 0$ into (22) – (25) and (49) – (59), then we obtain

$$\vartheta^2 + \omega^2_{\ 1} = 0, \qquad \widehat{\vartheta}^2 = W\,\omega^3_{\ 1} = \widehat{\omega}^2_{\ 1}, \qquad \vartheta^2 = W\,\widehat{\omega}^3_{\ 1}, \qquad W^4\, H\, \widehat{H} = 1$$

$$\omega^1_{\ 2} = \alpha\,\omega^2_{\ 1}, \quad \omega^1_{\ 3} = \beta\,\omega^3_{\ 1}, \quad \widehat{\omega}^1_{\ 2} = \frac{1}{W}\,\omega^1_{\ 3}, \quad \widehat{\omega}^1_{\ 3} = -\,W\,\omega^1_{\ 2}$$

$$\omega^3_{\ 2} = u\,\omega^2_{\ 1} + v\,\omega^3_{\ 1}, \;\; \widehat{\omega}^2_{\ 3} = -\,W^2\,\omega^3_{\ 2}, \;\; \widehat{\omega}^3_{\ 2} = \frac{-\,1}{HW^2}\,\omega^3_{\ 2}, \;\; \omega^2_{\ 3} = \frac{1}{H}\,\omega^3_{\ 2}$$

$$\widehat{\vartheta}^1 = \vartheta^1 = v\,\omega^2_{\ 1} - \frac{u}{H}\,\omega^3_{\ 1}, \qquad \omega^1_{\ 1} = \omega^2_{\ 2} = \widehat{\omega}^1_{\ 1} = \widehat{\omega}^2_{\ 2} = 0, \qquad \mathrm{d}W = 0.$$

From structural equations with $\omega^2_{\ 1} = -\,\vartheta^2$ and $\omega^1_{\ 2} = \alpha\,\omega^2_{\ 1}$ it follows that $\mathrm{d}\vartheta^1 = 0$. Hence locally there exists function γ such that $\vartheta^1 = \mathrm{d}\gamma$. It is easy to check that $\mathrm{d}(\mathrm{e}^{\gamma}\,\vartheta^2) = 0$ and $\mathrm{d}(\mathrm{e}^{-\,\gamma}\,\widehat{\vartheta}^2) = 0$. Moreover $(\mathrm{e}^{\gamma}\,\vartheta^2) \wedge (\mathrm{e}^{-\,\gamma}\,\widehat{\vartheta}^2) \neq 0$. Therefore there exist local coordinates x, y such that $\mathrm{e}^{\gamma}\,\vartheta^2 = \mathrm{d}x$ and $\mathrm{e}^{-\,\gamma}\,\widehat{\vartheta}^2 = \mathrm{d}y$. Next we find the basic one-forms $\omega^2_{\ 1} = -\,\vartheta^2 = -\,\mathrm{e}^{-\,\gamma}\,\mathrm{d}x$ and $\omega^3_{\ 1} = \frac{1}{W}\,\widehat{\vartheta}^2 = \frac{\mathrm{e}^{\gamma}}{W}\,\mathrm{d}y$. Looking at ϑ^1 we may find u and v, and the rest of one-forms is easy to obtain. The system of differential equations for α, β, γ and H we get from the fundamental equations. Since $\vartheta^1 \wedge \vartheta^2 \neq 0$ and $\widehat{\vartheta}^1 \wedge \widehat{\vartheta}^2 \neq 0$, we have $\gamma_x \neq 0$ and $\gamma_y \neq 0$.

We have also

$$R(X_1, X_2)X_1 = X_2, \qquad R(X_1, X_2)X_2 = -\,\beta\,H\,X_1$$

$$(\nabla_Y R)(X_1, X_2)X_1 = -\,(\alpha + \beta\,H)\,\vartheta^2(Y)\,X_1$$

$$(\nabla_Y R)(X_1, X_2)X_2 = -\,Y(\beta\,H)\,X_1 + (\alpha + \beta\,H)\,\vartheta^2(Y)\,X_2$$

and

$$\widehat{R}(\widehat{X}_1, \widehat{X}_2)\widehat{X}_1 = \widehat{X}_2, \qquad \widehat{R}(\widehat{X}_1, \widehat{X}_2)\widehat{X}_2 = \frac{-\,\alpha}{W^2\,H}\,\widehat{X}_1$$

$$(\widehat{\nabla}_Y \widehat{R})(\widehat{X}_1, \widehat{X}_2)\widehat{X}_1 = \frac{\alpha + \beta\,H}{W^2\,H}\,\widehat{\vartheta}^2(Y)\,\widehat{X}_1$$

$$(\widehat{\nabla}_Y \widehat{R})(\widehat{X}_1, \widehat{X}_2)\widehat{X}_2 = Y\left(\frac{\alpha}{W^2\,H}\right)\,\widehat{X}_1 - \frac{\alpha + \beta\,H}{W^2\,H}\,\widehat{\vartheta}^2(Y)\,\widehat{X}_2.$$

If $\nabla R = 0$ then $\alpha + \beta\,H = 0$ and $\beta\,H = \text{const}$, hence $\alpha = -\,\beta\,H$ is also constant. Conversely, if $\alpha = \text{const}$, then $\alpha_y = 0$ and from the system of differential equations we obtain $\alpha + \beta\,H = 0$, next $\beta\,H = -\,\alpha = \text{const}$ and $\nabla R = 0$.

If $\widehat{\nabla}\,\widehat{R} = 0$, then $\alpha + \beta\,H = 0$ and $\frac{\alpha}{H}$ is constant, and now $\beta = \frac{-\,\alpha}{H}$. Conversely, if β is constant, then from $\beta_x = 0$ we obtain $\alpha + \beta\,H = 0$, hence $\frac{\alpha}{H} = -\,\beta$ is constant and $\widehat{\nabla}\,\widehat{R} = 0$.

6. The Particular Case when Connections are Induced by Affine Normal Vector Fields

Theorem 10. *Let $f : M \to \mathbb{R}^3$ and $\widehat{f} : M \to \mathbb{R}^3$ be non-degenerate immersions of a two-dimensional real manifold M into affine space $\mathbb{R}^3$.*

We denote by ξ and $\widehat{\xi}$ the affine normal vector field for f and $\widehat{f}$ respectively, by h and $\widehat{h}$ the corresponding affine fundamental forms, and by ν and $\widehat{\nu}$ the conormal maps. Let $\varepsilon = \operatorname{sign}\det h_{ij}$ and $\widehat{\varepsilon} = \operatorname{sign}\det \widehat{h}_{ij}$.

Let f and $\widehat{f}$ satisfy the following conditions

i) *for every $p \in M$ $f(p) \neq \widehat{f}(p)$, moreover the vector $\widehat{f}(p) - f(p)$ is tangent to $f(M)$ at $f(p)$ and is tangent to $\widehat{f}(M)$ at $\widehat{f}(p)$*

ii) *the spherical representation of $\widehat{f} - f$, $M \ni p \mapsto \pi(\widehat{f}(p) - f(p)) \in \mathbb{P}^2(\mathbb{R})$, has rank 2 at every point of M*

iii) *the functions $\nu(\widehat{\xi})$ and $\widehat{\nu}(\xi)$ are constant*

iv) $\det(\widehat{f} - f, \xi, \widehat{\xi})$ *is a non-zero constant*

v) $\left|\det(\widehat{f} - f, \xi, \widehat{\xi})\right| = \left|1 - \nu(\widehat{\xi})\,\widehat{\nu}(\xi)\right|$

vi) $\varepsilon = \widehat{\varepsilon}$.

Then affine fundamental forms h and $\widehat{h}$ are conformal to each other.

If moreover

vii) $\widehat{\nu}(\xi) + \varepsilon\,\nu(\widehat{\xi}) = 0$

then the Blaschke connections ∇ and $\widehat{\nabla}$, of f and $\widehat{f}$ respectively, are locally symmetric.

Proof: Without loss of generality we may assume that $\det(\widehat{f} - f, \xi, \widehat{\xi}) = 1 - \nu(\widehat{\xi})\,\widehat{\nu}(\xi)$, because affine normal vector field $\widehat{\xi}$ may be replaced by $-\widehat{\xi}$. We retain our previous notation, so we have now $W = 1 - A\widehat{A}$. The case $A = \widehat{A} = 0$ is described in Theorem 1.5 of [7]. We may also use (75) with constant H and next use Proposition 9.

If $A \neq 0$ or $\widehat{A} \neq 0$, then (f, ξ) and $(\widehat{f}, \widehat{\xi})$ satisfy the assumptions 1° – 5° and 7° of Theorem 7. It suffices to check whether they satisfy 6°.

We will show that the assumption $\widehat{A}+\varepsilon\, A=0$ implies $\omega^1{}_1=0$, which is equivalent to 6°. We proceed as in the first part of the proof of Theorem 7 and obtain the formulae corresponding to (47), (25), (49), (48) and (23), when $W=1-A\widehat{A}=$ constant

$$\begin{aligned}
&\widehat{\vartheta}^1=\vartheta^1+\omega^1{}_1, \qquad \widehat{\vartheta}^2=\omega^3{}_1, \qquad \widehat{\omega}^3{}_1=\vartheta^2\\
&\widehat{\omega}^3{}_2=\frac{-1}{W}\,\omega^2{}_3, \quad -\,\omega^1{}_1=\widehat{A}\,\omega^3{}_2+\frac{A}{W}\,\omega^2{}_3, \qquad \vartheta^2=-\,\omega^2{}_1+\widehat{A}\,\omega^3{}_1.
\end{aligned} \tag{76}$$

If we bring together $\widehat{\omega}^3{}_2$ and $-\,\omega^1{}_1$, then we obtain

$$\widehat{\omega}^3{}_2=\frac{\widehat{A}}{A}\,\omega^3{}_2+\frac{1}{A}\,\omega^1{}_1. \tag{77}$$

Note, that if $\widehat{A}+\varepsilon\, A=0$ and $(A,\widehat{A})\neq(0,0)$, then $A\neq 0$.

Substituting (76) and (77) into

$$\widehat{\omega}^3{}_1\wedge\widehat{\vartheta}^1+\widehat{\omega}^3{}_2\wedge\widehat{\vartheta}^2=0, \qquad \widehat{\omega}^3{}_1\wedge\widehat{\omega}^3{}_2=\widehat{\varepsilon}\,\widehat{\vartheta}^1\wedge\widehat{\vartheta}^2=\varepsilon\,\widehat{\vartheta}^1\wedge\widehat{\vartheta}^2$$

we obtain

$$\begin{aligned}
\omega^1{}_1\wedge\left(-\,\vartheta^2+\frac{1}{A}\,\omega^3{}_1\right)-\left(\vartheta^1\wedge\vartheta^2+\frac{\widehat{A}}{A}\,\omega^3{}_1\wedge\omega^3{}_2\right)&=0\\
\omega^1{}_1\wedge\left(-\,\frac{1}{A}\,\vartheta^2-\varepsilon\,\omega^3{}_1\right)+\left(\varepsilon\,\omega^3{}_1\wedge\vartheta^1-\frac{\widehat{A}}{A}\,\omega^3{}_2\wedge\vartheta^2\right)&=0.
\end{aligned}$$

But $\dfrac{\widehat{A}}{A}=-\,\varepsilon$, therefore

$$\begin{aligned}
\vartheta^1\wedge\vartheta^2+\frac{\widehat{A}}{A}\,\omega^3{}_1\wedge\omega^3{}_2&=\varepsilon\left(\varepsilon\,\vartheta^1\wedge\vartheta^2-\omega^3{}_1\wedge\omega^3{}_2\right)=0\\
\varepsilon\,\omega^3{}_1\wedge\vartheta^1-\frac{\widehat{A}}{A}\,\omega^3{}_2\wedge\vartheta^2&=\varepsilon\,\left(\omega^3{}_1\wedge\vartheta^1+\omega^3{}_2\wedge\vartheta^2\right)=0
\end{aligned}$$

and consequently

$$\omega^1{}_1\wedge\left(-\,\vartheta^2+\frac{1}{A}\,\omega^3{}_1\right)=0, \qquad \omega^1{}_1\wedge\left(-\,\frac{1}{A}\,\vartheta^2-\varepsilon\,\omega^3{}_1\right)=0.$$

The one-forms

$$\begin{aligned}
-\,\vartheta^2+\frac{1}{A}\,\omega^3{}_1&=\omega^2{}_1+\frac{1-A\widehat{A}}{A}\,\omega^3{}_1\\
-\,\frac{1}{A}\,\vartheta^2-\varepsilon\,\omega^3{}_1&=\frac{1}{A}\,\omega^2{}_1-\frac{\widehat{A}+\varepsilon\, A}{A}\,\omega^3{}_1=\frac{1}{A}\,\omega^2{}_1
\end{aligned}$$

are linearly independent, hence $\omega^1{}_1=0$ and we may apply Theorem 7.

7. Conclusion

It is possible to formulate the Bäcklund theorem in $\mathbb{R}^3$ without using notions depending on the metric. The conditions of constant length and constant angle may be replaced by some conditions involving the volume form and the conormal mapping. If we impose such requirements on the affine normal vector fields, then both induced connections are locally symmetric (Theorem 10). It seems to be a common generalization of Euclidean and Minkowski space Bäcklund theorems. One may also consider a pair of surfaces endowed with arbitrary equiaffine transversal vector fields. Some sufficient conditions under which the induced connections are locally symmetric are given in Theorem 7.

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