Functiones et Approximatio
50.1 (2014), 151–159
doi: 10.7169/facm/2014.50.1.4

ON FUNCTIONS THAT ARE *BDS*-INTEGRABLE OVER CONVEXLY BOUNDED VECTOR MEASURES

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Dedicated to Lech Drewnowski on
the occasion of his 70th birthday

Abstract: Spaces of scalar functions that are integrable in the sense of Bartle-Dunford-Schwartz integration, with respect to a convexly bounded vector measure μ, are studied.

For instance, under the assumption that the range space X of μ is sequentially complete, the effect of the Orlicz-Pettis property (with respect to a weaker topology on X) on the size of $L^1(\mu)$ is investigated. Some completeness properties of the space $L^1_\bullet(\mu)$ of 'scalarly integrable functions' are established for general X.

Keywords: convexly bounded vector measure, Bartle-Dunford-Schwartz integration, spaces of integrable functions, σ-Lebesgue property, σ-Levi property.

1. Introduction

A *vector measure* is a countably additive set function defined on a σ-algebra, say $\mathcal{A}$, of subsets of a set T and taking values in a topological vector space, say X. We shall say that it is *bounded* (resp., *convexly bounded*) if its range (resp., the convex hull of its range) is bounded.

A recent article on the theory of *BDS*-integral of scalar functions integrated with respect to general convexly bounded measures to which we refer is [2]. The adjective 'general' means, in this case, that in [2] the measures were taking values in a topological vector space X under the only assumption that X was *sequentially complete*.

The present paper is an addendum to that article. There are at least two reasons to publish it. First, there were a few topics that, while worth studying, did not make it into [2] for reasons of space (the journal to which [2] was submitted limits the size of the papers). These topics include some consequences of the Orlicz-Pettis property as well as a formula concerning the integration against the indefinite integrals. Secondly, even though a sequentially complete Hausdorff tvs is a truly general object, there are situations where the assumption of sequential completeness is not convenient. For instance, this is the case when X is a locally

2010 Mathematics Subject Classification: primary: 28B05, 46G10; secondary: 46A40

convex space equipped with its weak topology or weak* topology. Some number of completeness type results about the 'spaces of scalarly integrable functions' $L^1_\bullet(\mu)$, results which persist without any completeness assumption on the space X, are given.

Needless to say that, in view of the remarks above, some of the results presented here will be familiar to the co-author of [2] whose jubilee is honored in the present special volume of 'Functiones et Approximatio'. This is an additional reason, besides the memory of our life-long mathematical interactions, for which the paper is dedicated to Lech with affection.

2. Vector measures and the space $L^0(\mu)$ of measurable functions

Throughout, let $X = (X, \tau)$ be a tvs (i.e., a Hausdorff topological vector space) and let

- $\mathcal{U}$ denote any base of balanced neighborhoods for 0 in X, and
- $\mathrm{Fs}(X)$ denote any base of continuous F-seminorms on X.

We recall that X is embedded as a dense subspace in its (vector topological) completion $\hat{X}$. The topology $\hat{\tau}$ of $\hat{X}$ is generated by the base $\hat{\mathcal{U}}$ whose generic member $\hat{U}$ is obtained as the closure $\overline{U}$ of the corresponding neighborhood U in $\hat{X}$. Similarly, $\mathrm{Fs}(\hat{X})$ is obtained extending the F-seminorms from $\mathrm{Fs}(X)$ by continuity onto $\hat{X}$.

Here is some terminology and notation from [2]. Let $\mathcal{A}$ be a σ-algebra of subsets of a set T, and $\mu : \mathcal{A} \to X$ be a vector measure. The triple $(T, \mathcal{A}, \mu)$ is called a *vector measure space.*

A set $A \in \mathcal{A}$ is said to be *μ-null* if $\mu(B) = 0$ for any $B \in \mathcal{A}$ contained in A. Of course, the family $\mathcal{N}(\mu)$ of all μ-null sets is a σ-ideal in $\mathcal{A}$. Given a continuous F-seminorm $|\cdot|$ on X, we denote by $|\mu|^*$ the submeasure majorant for μ with respect to $|\cdot|$; it is defined by the formula $|\mu|^*(A) = \sup\{|\mu(B)| : B \in \mathcal{A}, B \subset A\}$ for $A \in \mathcal{A}$. It is known that $|\mu|^*$ is order continuous (i.e., $|\mu|^*(A_n) \to \emptyset$ whenever $A_n \downarrow 0$), hence it is also σ-subadditive. Evidently, a set $A \in \mathcal{A}$ is μ-null iff $|\mu|^*(A) = 0$ for each $|\cdot| \in \mathrm{Fs}(X)$.

For the real-valued and, more generally, extended real-valued functions on T, we use the following terminology. Two such functions, say f and g, are said to be *μ-equivalent* (or equal μ-a.e.) if there is a μ-null set N such that $f = g$ on $T \setminus N$. A function $f : T \to \overline{\mathbb{R}}$ is called *μ-simple* (resp., *μ-measurable*) if it is μ-equivalent to an $\mathcal{A}$-simple (resp., $\mathcal{A}$-measurable) real valued function. The class of all μ-measurable functions is stable under the μ-a.e. convergence (i.e., the pointwise convergence outside a μ-null set) of sequences.

We denote by $L^0(\mu)$ the vector space of all μ-measurable real functions on T, by $L^\infty(\mu)$ its subspace of μ-essentially bounded functions, and by $S(\mu)$ the subspace of the latter consisting of μ-simple functions. Actually, we usually identify functions that are equal μ-a.e., and consider these spaces as consisting of μ-equivalence classes of functions. But, in practice, we work with the most convenient representatives of such classes. In particular, when we consider an $f \in L^0(\mu)$, we always assume that it is (represented by) a finite $\mathcal{A}$-measurable function.

Treated this way, $L^0(\mu)$ is a Dedekind σ-complete and laterally σ-complete vector lattice, $L^\infty(\mu)$ is a solid subspace of $L^0(\mu)$ and a Banach space under the norm $\|\cdot\|_\infty$ of μ-essential supremum, and $S(\mu)$ is a dense subspace of $L^\infty(\mu)$.

In what follows,

$$B_\infty(\mu) = \text{ the closed unit ball of } L^\infty(\mu) \qquad \text{and} \qquad S_\infty(\mu) = B_\infty(\mu) \cap S(\mu).$$

A natural topology in $L^0(\mu)$ is the Hausdorff vector topology $\tau^0 = \tau^0(\mu)$ of convergence in measure μ. It has a base for the neighborhoods of zero consisting of the sets

$$\{f \in L^0(\mu) : |\mu|^*(\{t \in T : |f(t)| > \varepsilon\}) \leqslant \varepsilon\},$$

where $|\cdot| \in \mathrm{Fs}(X)$ and $\varepsilon > 0$. Alternatively, τ^0 is defined by the F-seminorms $|\cdot|^0$, where $|f|^0 = \inf\{\varepsilon > 0 : |\mu|^*(\{t \in T : |f(t)| > \varepsilon\}) \leqslant \varepsilon\}$ and $|\cdot| \in \mathrm{Fs}(X)$. Unless stated otherwise, the space $L^0(\mu)$ is always considered with the topology τ^0.

Evidently, if (f_n) is a sequence in $L^0(\mu)$, then $f_n \to 0$ (τ^0), written also as $f_n \to 0$ (μ), iff $\lim_n |\mu|^*(\{t \in T : |f_n(t)| > \varepsilon\}) = 0$ for all $\varepsilon > 0$ and $|\cdot| \in \mathrm{Fs}(X)$. Furthermore, if $f_n \to 0$ μ-a.e., then $f_n \to 0$ (μ).

A vector measure space $(T, \mathcal{A}, \mu)$, or the vector measure μ itself, *is of type* (C) (resp. (SC)) if the corresponding space $L^0(\mu)$ is complete (resp. sequentially complete).

Theorem 2.1. *For a vector measure space $(T, \mathcal{A}, \mu)$, the following are equivalent.*

(a) *μ is of type* (C).
(b) *$L^0(\mu)$ is a universally complete Lebesgue Levi topological vector lattice.*

Moreover, if this is the case, $L^0(\mu)$ has the countable sup property iff it is metrizable.

A vector measure μ is said to satisfy the *countable chain condition*, (ccc), if every family of disjoint sets from $\mathcal{A} \setminus \mathcal{N}_\mu$ is at most countable.

Theorem 2.2. *The following conditions are equivalent.*

(a) *The measure μ satisfies (ccc).*
(b) *There exists an F-normed space $Y = (Y, |\cdot|)$, and a continuous linear map $h : X \to Y$ such that $\mathcal{N}_\mu = \mathcal{N}_{h\circ\mu}$.*
(c) *$L^0(\mu)$ is an F-lattice.*

Let τ' be another Hausdorff vector topology on X. We say that τ-subseries convergence *is contained* in τ'-subseries convergence, if every series that is subseries convergent in (X, τ) is also subseries convergent in (X, τ'). If μ satisfies (ccc) when treated as a vector measure into (X, τ'), then it satisfies (ccc). We quote the following result.

Corollary 2.3. *Suppose X admits a metrizable topology τ' such that τ-subseries convergence is contained in τ'-subseries convergence. Then $L^0(\mu)$ is a Dedekind complete F-lattice with the countable sup property.*

The reference for all undefined terms, notions and results given without proofs is [2].

3. *BDS*-integrable functions

Let $\mathcal{A}$ be a σ-algebra of subsets of a set T and let $\mu : \mathcal{A} \to X$ be a vector measure space. A function $f \in L^0(\mu)$ is said to be *BDS-integrable* (or *μ-integrable*) if there exists a sequence (f_n) of simple functions such that $f_n \to f$ μ-a.e. and, for each $A \in \mathcal{A}$, $\lim_n \int_A f_n \, d\mu$ exists in X. Then, by definition,

$$\int_A f \, d\mu = \lim_{n\to\infty} \int_A f_n \, d\mu.$$

As already mentioned, one of the global assumptions in [2] was that the space X was sequentially complete. It is clear, however, that the above definition does not require any completeness condition on X. Moreover, a quick look at the arguments used in [2] reveals that the correctness of the definition is not affected, and that the indefinite integral μ_f is a vector measure on $\mathcal{A}$.

3.A. Special case: X is sequentially complete

In this section, as in [2], $X = (X, \tau)$ is a Hausdorff topological vector space which is sequentially complete, and $\mu : \mathcal{A} \to X$ is convexly bounded.

Let ρ, $\rho \subset \tau$, be another Hausdorff vector topology on X. Denote by ρ-$L^1(\mu)$ the space of functions that are (BDS) integrable with respect to $\mu : \mathcal{A} \to (X, \rho)$.

Theorem 3.1. *Suppose τ has the OP-property with respect to ρ, i.e., ρ-subseries convergent series are τ-convergent. Then ρ-$L^1(\mu) = L^1(\mu)$.*

Proof. It is clear that $L^1(\mu)$ is contained in ρ-$L^1(\mu)$. Let $f \in L^0(\mu)$ be BDS-integrable with respect to $\mu : \mathcal{A} \to (X, \rho)$. Then μ_f is a ρ-measure by the Nikodym Theorem, and therefore also a τ-measure by the assumed OP-property. Let (E_n) be a disjoint partition of T such that $f|E_n$ is bounded for each $n \in \mathbb{N}$. Then the integrals τ-$\int_{E_n} f \, d\mu$ exist and coincide with ρ-$\int_{E_n} f \, d\mu$. Further, the series $\sum_n \int_A f \chi_{E_n} \, d\mu$ τ-converges for each $A \in \mathcal{A}$. Hence $f \in L^1(\mu)$ by Remark 3.3(3) in [2]. ∎

Assume additionally that X possesses a vector space X' of continuous linear functionals on X that separate the points of X. We will say that $f \in L^0(\mu)$ is *X'-weakly integrable* if for each $A \in \mathcal{A}$ there exists $x_A \in X$ such that

$$\langle x_A, x' \rangle = \int_A f \, d\langle \mu, x' \rangle$$

for all $x' \in X'$.

Here $\langle \mu, x' \rangle = \mu \circ x' : \mathcal{A} \to X$. If X' is understood, we write $x_A = weak$-$\int_A f \, d\mu$; $x_T = weak$-$\int f \, d\langle \mu, x' \rangle$, and the space of all such functions is denoted by $weak$-$L^1(\mu)$.

Proposition 3.2. *Suppose τ has the OP-property with respect to $\sigma = \sigma(X, X')$. Then $L^1(\mu) = \sigma$-$L^1(\mu) = weak$-$L^1(\mu)$.*

Proof. In view of the previous Proposition, it is sufficient to show that the first and the third L^1 are equal. But, by its very definition, the weak integral of f, as a set function on $\mathcal{A}$, is a $\sigma(X, X')$-countably additive measure into X. Hence $f \in L^1(\mu)$ by the same argument as in the previous proposition. ■

Let us recall how the space $L^1_\circ(\mu)$, the largest vector subspace of $L^1(\mu)$ which is a solid subset of $L^0(\mu)$, was introduced. For each $f \in L^0(\mu)$, we put

$$N(f) = \{g \in L^0(\mu) : |g| \leqslant |f| \ \mu\text{-a.e.}\} = \{fh : h \in B_\infty(\mu)\},$$

and $L^1_\circ(\mu)$ was the set of all $f \in L^1(\mu)$ such that $N(f) \subset L^1(\mu)$.

The first theorem characterizing $L^1_\circ(\mu)$ was the following

Theorem 3.3. *If $f \in L^1_\circ(\mu)$, then*

(a) *the indefinite integrals $\int g \, d\mu$ are equi-exhaustive for $g \in N(f)$;*
(b) *for $g \in N(f)$, the operators $h \to \int_T gh \, d\mu$ from $L^\infty(\mu)$ to X, and the operators $h \to \int gh \, d\mu$ from $L^\infty(\mu)$ to $cbca(\mathcal{A}, X)$, are equi-continuous;*
(c) *the indefinite integral μ_f of f is convexly bounded and*

$$\int_T fh \, d\mu = \int_T h \, d\mu_f \qquad \text{for all } \ h \in L^\infty(\mu). \tag{+}$$

Conversely, if $f \in L^1(\mu)$ and μ_f is convexly bounded, then $f \in L^1_\circ(\mu)$.

We are interested in an extension of the integration formula (+) of the above Theorem 3.3 (c). It will give us the formula for the integration against the indefinite integrals.

Since this has some delicate points, we precede it with a preliminary discussion. Recall that $\mathcal{N}(\mu) \subset \mathcal{N}(\mu_f)$ for each $f \in L^1(\mu)$.

Lemma 3.4. *Let $f \in L^1_\circ(\mu)$, and let g be a μ_f-measurable function on T. Then fg is a μ-measurable function on T. If, moreover, $g = 0$ μ_f-a.e., then $fg = 0$ μ-a.e.*

Proof. By assumption, there is a μ_f-null set N_1 such that $g|(T \smallsetminus N_1)$ is $\mathcal{A}$-measurable. Then, by Proposition 3.5 of [2], $f = 0$ μ-a.e. on N_1 so that there is a μ-null set $N_2 \subset N_1$ such that $f = 0$ on $N_1 \smallsetminus N_2$. Since f is μ-measurable, there is a μ-null set N_3 containing N_2 such that $f|(T \smallsetminus N_3)$ is $\mathcal{A}$-measurable. Then fg is obviously $\mathcal{A}$-measurable on $T \smallsetminus (N_1 \cup N_3)$ as well as on $N_1 \smallsetminus N_2$ because on that set $fg = 0$. Thus, denoting by E the union of these two sets, we see that $E \in \mathcal{A}$ and that fg is $\mathcal{A}$-measurable on E. Since $T \smallsetminus E = N_2 \cup (N_3 \smallsetminus N_1) \in \mathcal{N}(\mu)$, fg is μ-measurable.

To verify the other assertion simply note that if $g = 0$ μ_f-a.e., then in the argument above one can choose the set $N_1 \in \mathcal{N}(\mu_f)$ so that $g = 0$ on $T \smallsetminus N_1$. In consequence, $fg = 0$ on E. ■

Theorem 3.5. *Let $f \in L^1_\circ(\mu)$, and let g be a μ_f-measurable function on T. Then fg is μ-integrable iff g is μ_f-integrable, and in this case*

$$\int_A fg\,d\mu = \int_A g\,d\mu_f \qquad \text{for all} \quad A \in \mathcal{A}.$$

Moreover, if $g \in L^1_\circ(\mu_f)$, then $fg \in L^1_\circ(\mu)$.

Proof. There is a μ_f-null set N such that $g|(T \setminus N)$ is $\mathcal{A}$-measurable. Note that replacing g with $g\chi_{T\setminus N}$ will make no effect on the existence and equality of the integrals involved. Therefore, we may assume that g is $\mathcal{A}$-measurable. Fix an increasing sequence (E_n) with union T such that g is bounded on each of the sets E_n. Then, by Theorem 3.3 (c), for each $A \in \mathcal{A}$ one has

$$\int_A fg\chi_{E_n}\,d\mu = \int_A g\chi_{E_n}\,d\mu_f.$$

Since $E_n \uparrow T$, the μ-integrability of fg or the μ_f-integrability of g imply that both the integrals in the last displayed formula converge in X for each $A \in \mathcal{A}$ as $n \to \infty$. To finish, apply Theorem 3.7 of [2].

The 'moreover' part is an obvious consequence of the first part. ■

Corollary 3.6. *Let $f \in L^1_\circ(\mu)$. Then the map $g \to fg$ is a well-defined continuous injective linear operator from $L^1(\mu_f)$ into $L^1(\mu)$ and from $L^1_\circ(\mu_f)$ into $L^1_\circ(\mu)$. In fact, for each $|\cdot| \in \mathrm{Fs}(X)$ one has $|fg|^1 = |g|^1$ and $|fg|^1_\circ = |g|^1_\circ$, respectively.*

Question 3.7. If $g \in L^1_\circ(\mu)$ is such that $fg \in L^1_\circ(\mu)$ for all $f \in L^1_\circ(\mu)$, does then $g \in L^\infty(\mu)$? This is not always true, but maybe it is true under the assumption that X contains no copy of the space $\omega = \mathbb{R}^{\mathbb{N}}$?

3.B. General case: X is arbitrary

The setting in this section does not change except for the fact that *we do not assume* anymore that X is sequentially τ-complete.

For every $f \in L^0(\mu)$, let us set

$$N_{\circ\circ}(f) = \{g \in S(\mu) : |g| \leqslant |f|\} \qquad \text{and} \qquad M_{\circ\circ}(f) = \Big\{ \int_T g\,d\mu : g \in N_{\circ\circ}(f)\Big\}.$$

We denote by $\tau^\bullet$ the translation invariant topology on $L^0(\mu)$ for which a neighborhood base at zero is formed by the sets

$$\{f \in L^0(\mu) : M_{\circ\circ}(f) \subset U\} \qquad (U \in \mathcal{U})$$

where $\mathcal{U}$ is a neighborhood base at zero for τ. It is determined by the FG-seminorms $\mu_i^\bullet$ associated with the F-seminorms $\{|\,.\,|_i : i \in I\} = \mathrm{Fs}(X)$ via the formula

$$\mu_i^\bullet(f) = \sup\Big\{ \Big| \int_T g\,d\mu \Big|_i : g \in N_{\circ\circ}(f)\Big\}.$$

Consider $L^0(\mathcal{A},\mu,X)$ corresponding to $\mu:\mathcal{A}\to X$ and, then, consider μ into the completion $\hat{X}$ together with its $L^0(\mathcal{A},\mu,\hat{X})$. In view of our definitions, these two spaces are clearly identical, so we will use the previously used symbol $L^0(\mu)$. Let $|\cdot|$ be a continuous F-seminorm on X, and $|\cdot|\hat{}$ be its extension on $\hat{X}$. Working for a moment with $\hat{X}$, let us note that it was already observed in Section 5 of [2] (in which $L^1_\bullet(\mu)$ was introduced) that we have the identity $\mu^\bullet = |\,.\,|^0_\bullet$ when these two FG-seminorms are defined using $|\cdot|\hat{}$. But on the integrals of simple functions the F-seminorm $|\,.\,|\hat{}$ reduces to $|\cdot|$. It follows that we have

Proposition 3.8. $(L^0(\mu),\mu^\bullet)=(L^0(\mu),|\,.\,|^0_\bullet)$, *where the first FG-seminorm was calculated for* $\mu:\mathcal{A}\to X$ *and the second for* $\mu:\mathcal{A}\to\hat{X}$. *In particular,* $(L^0(\mu),\tau^\bullet)=(L^0(\mu),\tau^0_\bullet)$.

As a consequence, the results about spaces of BDS-integrable functions obtained in Section 5 of [2] apply in the present setting although no completeness assumption was imposed on X. Let us spell out a few that may be of interest in the present context.

Proposition 3.9. *The topology* $\tau^\bullet$ *is stronger than* τ^0.

Proof. Indeed, this is true by Prop 5.4 of [2] on the right of our equation in 3.8. ■

Let us denote by $L^1_\bullet(\mu)$ the largest vector subspace (in fact, an ideal) in $L^0(\mu)$ on which $\tau^\bullet$ is a vector topology (in fact, a vector lattice topology). It consists of precisely those $f\in L^0(\mu)$ for which the set $M_{\circ\circ}(f)$ is bounded in X. Alternatively,

$$L^1_\bullet(\mu)=\left\{f\in L^0(\mu):\lim_n \mu_i^\bullet\left(\frac{1}{n}f\right)=0\right\}$$

for each basic F-seminorm. We may use the same notation as in [2], because in the situation discussed there $L^1_\bullet(\mu)$ is the same here and there.

In view of Prop. 5.5 in [2], we have

Proposition 3.10. $(L^0(\mu),\tau^\bullet)$ *has the* σ*-Fatou property, that is, for every continuous* F*-seminorm* $|\cdot|$ *on* X, *if* $0\leqslant f_n\uparrow f$ *in* $L^0(\mu)$, *then* $\mu^\bullet(f_n)\to\mu^\bullet(f)$. *Moreover, if* $f_n\to f$ μ*-a.e. in* $L^0(\mu)$, *then* $\mu^\bullet(f)\leqslant\liminf_n\mu^\bullet(f_n)$. *The same holds when* $f_n\to f$ *in measure* μ.

The following corresponds to Theorem 5.8 of [2].

Theorem 3.11. $L^1_\bullet(\mu)$ *has the* σ*-Levi property.*

Theorem 3.12. *If* μ *is of type* (SC), *then* $(L^0(\mu),\tau^\bullet)$ *and* $L^1_\bullet(\mu)$ *are sequentially complete. If* μ *satisfies (ccc), then* $L^1_\bullet(\mu)$ *is Fatou Levi and complete. In particular, if* X *is an* F*-space, then* $(L^0(\mu),|\cdot|^0_\bullet)$ *is a complete metric vector lattice-group and, consequently,* $L^1_\bullet(\mu)$ *is an* F*-lattice.*

Let us provide a proof, although it is very similar to the one of Prop. 5.6 in [2].

Proof. By Proposition 3.10, $\tau^\bullet$ has a base of sequentially τ^0-closed neighborhoods of zero. Since $\tau^\bullet \geqslant \tau^0$ (Proposition 3.9), the sequential $\tau^\bullet$-completeness of $L^0(\mu)$ follows. To see the same for $L^1_\bullet(\mu)$, observe that $L^1_\bullet(\mu)$, as the largest vector subspace on which $\tau^\bullet$ is a vector topology, is $\tau^\bullet$-closed.

If μ satisfies (ccc), $L^0(\mu)$ has the countable sup property and μ is of type (C) (see Theorem 2.1 and Theorem 2.2). Hence, $L^1_\bullet(\mu)$ has the countable sup property and so it is Fatou. It also is Levi. To see it, let (f_α) be an increasing positive bounded net in $L^1_\bullet(\mu)$. Considering this net in $L^0(\mu)$ and taking advantage of the fact that the latter space has the Levi property by Theorem 2.1, we find its supremum f in $L^0(\mu)$. By the countable sup property, we can find a subsequence (f_{α_n}) such that $f_{\alpha_n} \uparrow f$. The σ-Levi property of $L^1_\bullet(\mu)$, Theorem 3.11, locates f in that space. Hence $L^1_\bullet(\mu)$ is indeed Levi. The completeness of $L^1_\bullet(\mu)$ now follows from the Nakano Theorem [1, Theorem 4.37]. ■

As above, let X' be a separating space of linear continuous functionals on X. We say that $f \in L^0(\mu)$ is X'-scalarly integrable (resp. scalarly integrable when X' is understood) if $\int f\, d < \mu, x' >$ exists for each $x' \in X'$.

Remark 3.13. Note that for the scalar measure $\mu \circ x' : \mathcal{A} \to \mathbb{R}$ its semivariation equals its total variation and if $| < \mu, x' > |$ denotes the total variation, then $\int f\, d < \mu, x' >$ exists iff $\int |f|\, d| < \mu, x' > | < \infty$.

Proposition 3.14. *Let X be locally convex and X' its dual. Then $L^1_\bullet(\mu)$ is the space of all scalarly integrable functions.*

Proof. Clearly, if f is scalarly integrable, then $M_{\circ\circ}(f) = \left\{ \int_T g\, d\mu : g \in N_{\circ\circ}(f) \right\}$ is weakly bounded in X. But then, it is also τ-bounded by the Mackey theorem. Hence $f \in L^1_\bullet(\mu)$. ■

Here is an interesting corollary. We stress that no completeness assumption is imposed on X.

Corollary 3.15. *Let X be a metrizable locally convex space or a locally convex space that admits a weaker F-norm. Then $L^1_\bullet(\mu)$, i.e., the space of all scalarly μ-integrable functions is Fatou, Levi and complete.*

Another interesting case occurs with $L^1_\circ(\mu)$. It is defined in [2] in such a way that the sequential completeness of X seems to be really needed. However, by Corollary 4.11 and Proposition 4.10 in [2] , $L^1_\circ(\mu) = \overline{S(\mu)}$ and, moreover, it is the largest σ-Lebesgue subspace of $(L^0(\mu), \tau^0_\bullet)$. Thus, one can *define* the space of integrable functions as $\overline{S(\mu)}$ in $(L^0(\mu), \tau^\bullet)$. This actually is how L^1 is defined by Turpin in [8] and by Thomas in [6], [5]. Incidentally, we see that their L^1 corresponds to our $L^1_\circ(\mu)$.

Besides the work of Thomas and of Turpin on their TT-integral discussed in [2], there is also an early Polish contribution. Namely, in his book [3], S. Rolewicz proposes the following definition of integrability of a scalar function f with respect to a vector measure μ.

A positive μ-measurable function f is integrable if, for each increasing sequence of positive simple functions (f_n) converging μ-a.e to f, the corresponding sequence of integrals $(\int f_n \, d\mu)$ is convergent and $\int f \, d\mu = \lim_n \int f_n \, d\mu$. Then the integral for an arbitrary f is defined via its positive and negative part.

To be precise, there is a misprint in [3] which makes the definition unintelligible. The definition is given according to Turpin [7, 2.16], who attributes it to Rolewicz. In the book, Rolewicz does not go beyond proposing a definition and, in fact, it is not obvious at all how to check the unicity of his integral. Yet, the attempt is historically interesting because Turpin showed (loc.cit.) that the definition of Rolewicz, for convexly bounded measures, produces an integral which is equivalent with the TT-integral. In the approach of [2] the latter fact can be dealt with in the same way as in the proof of Proposition 3.9 there.

In the second edition of the book [4], Rolewicz does not return to his own definition. He gives a sketch of the TT-integral following the approach presented in Turpin's thesis [8].

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Received: 7 February 2013; **revised:** 3 July 2013