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DECOMPOSITION OF MEASURES AND MODULAR FUNCTIONS

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Dedicated to Lech Drewnowski on the occasion of his 70th birthday

Abstract: We deal with decomposition theorems for measures on Boolean algebras or – more generally – for modular functions on lattices. In particular, for measures on Boolean algebras with values in locally solid ℓ-groups we compare decomposition theorems obtained with the Frechét--Nikodým-approach and decomposition theorems obtained using the band decomposition theorem of Riesz.

Keywords: ℓ-group-valued measures, ℓ-group-valued modular functions, locally solid ℓ-groups.

1. Introduction

In this article we deal with decomposition theorems for measures on Boolean algebras or - more generally - for modular functions on lattices. Recall that a function μ on a lattice L is called *modular* if $\mu(x \vee y) + \mu(x \wedge y) = \mu(x) + \mu(y)$ for all $x, y \in L$.

In the ℓ-group-valued case the band decomposition theorem of Riesz is an adequate means to obtain various decomposition theorems for measures. If one studies measures with values in a Banach space - or more generally - in a topological group, then a topological approach is convenient. In this paper we present these two methods and compare them for measures with values in a locally solid ℓ-group. This is done - as far as possible - in the more general setting of modular functions on lattices.

The method that uses the Riesz band decomposition theorem to obtain decomposition theorems for measures is applied for real-valued measures e.g. in [8] and for ℓ-group-valued measures in [10]. Bauer already used the band decomposition theorem for certain decompositions of modular functions with values in a Dedekind complete Riesz space E, see [6, 7] [1]. The basic tool here is the fact that the corresponding space of modular functions is a Riesz space. Birkhoff

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[1] Bauer observed that some of his results remain true in the ℓ-group-valued case, see [6, Footnote 2a] and [7, Footnotes 3 and 7]

[9, Chapter X.6][2] proved that the space of all *real*-valued modular functions of bounded variation is a Riesz space; in particular, the decomposition $\mu = \mu^+ - \mu^-$ generalizes the classical Jordan decomposition functions of bounded variation into monotone summands and the decomposition of real-valued bounded measures into positive measures to real-valued modular functions on lattices. Bauer [6, footnote 15] observed that Birkhoff's proof also works for E-valued modular functions [3].

In Section 3 we give (in the ℓ-group-valued case) an alternative proof of Birkhoff's result mentioned above, based on the concept of a "distance function". If $\mu : L \to G$ is an ℓ-group-valued modular function, then

$$\overline{\mu}(x, y) := \mu(x \vee y) - \mu(x \wedge y) \tag{1}$$

satisfies $\overline{\mu}(x \wedge y, x) = \overline{\mu}(y, x \vee y)$ and $\overline{\mu}(x, z) = \overline{\mu}(x, y) + \overline{\mu}(y, z)$ if $x \leqslant y \leqslant z$. Therefore we study the space $d(L, G)$ of all "distance functions" $\alpha : L^2 \to G$ with the following properties:

$$x \leqslant y \leqslant z \Rightarrow \alpha(x, z) = \alpha(x, y) + \alpha(y, z), \tag{2}$$

$$\alpha(x \wedge y, x) = \alpha(y, x \vee y), \tag{3}$$

$$\alpha(x, y) = \alpha(x \wedge y, x \vee y). \tag{4}$$

The relationship between modular functions and distance functions is clarified in Theorem 2.3. In Theorem 3.5 we show that the subspace of functions of $d(L, G)$ which have finite variation is a Dedekind complete ℓ-group. This yields - combined with the band decomposition theorem of Riesz - decomposition theorems for modular functions. The advantage to consider first the space $d(L, G)$ is the fact that the order relation on $d(L, G)$ is the pointwise order [4].

Section 4 presents the known Fréchet-Nikodým-approach to decomposition theorems for measures with values in topological groups. Drewnowski was the first to develop the Fréchet-Nikodým-approach in measure theory in a systematical way, in particular with an application to decomposition theorems. In Theorem 4.1 we present Traynor's decomposition theorem [22], which was suggested by a question of Drewnowski [14, p. 47].

In Section 5 we consider modular functions or measures with values in a locally solid Dedekind complete ℓ-group G and compare decomposition theorems obtained with the Fréchet-Nikodým-approach (Section 4) and decomposition theorems obtained as band decomposition (Section 3). For *real*-valued measures on Boolean algebras both methods yield the same decomposition theorems. This is not anymore true for $\mathbb{R}^2$-valued measures: If λ is a probability measure, $\mu = (\lambda, \lambda)$, $\mu_1 = (\lambda, 0)$, $\mu_2 = (0, \lambda)$, then $\mu = \mu_1 + \mu_2$, $\mu_1 \wedge \mu_2 = 0$, but $\mu_1 \perp \mu_2$ is not true. Therefore we can only ask the converse question whether decompositions with respect to (w.r.t.) an FN-topology (Theorem 4.1) are also band decompositions.

[2] This theorem was already given in the first edition (1940) of [9] on p. 45.

[3] Bauer considered modular functions of *finite* variation to include the case of linear operators between Riesz spaces.

[4] see (8) for the order relation between modular functions

A related question is e.g.: Is it possible to decompose an order bounded measure $\mu : A \to G$ into $\mu = \mu_1 + \mu_2$ where μ_1 is a "regular" measure (according to Definition 5.5(ii), based on the topology of G) and μ_2 is "anti-regular" in the sense that the only regular measure ν with $0 \leqslant \nu \leqslant |\mu_2|$ is $\nu = 0$? The answer is yes if the topology of G has the Lebesgue property (i.e. order convergence implies topological convergence); this is a particular case of Corollary 5.17. Its proof is based on the fact that then μ and $|\mu|$ induce the same FN-topology, see Theorem 5.4. This was known under the additional assumption (M) that G has a 0-neighbourhood base consisting of sublattices. As mentioned after Corollary 5.9, Theorem 5.4 allows to avoid condition (M) in some theorems, the proof of which is based on the equality of the μ-topology and the $|\mu|$-topology.

2. Group-valued modular functions and distance functions

Throughout, let L be a lattice and G an additively written commutative group.

If a lattice has a smallest or greatest element, we denote them by 0 or 1, respectively.

Notation 2.1. By $m(L,G)$ we denote the group of all G-valued modular functions on L. For $p \in L$ let $m(L,G,p)$ be its subgroup $\{\mu \in m(L,G) : \mu(p) = 0\}$. Let $d(L,G)$ be the group of all functions $\alpha : L^2 \to G$ satisfying (2), (3), (4).

From (2) it follows that $\alpha(x,x) = 0$ for all $x \in L$ and $\alpha \in d(L,G)$.

If A is a Boolean algebra, then $m(A,G,0)$ is precisely the group of G-valued measures (=finitely additive functions) on A.

It is clear that $\overline{\mu} \in d(L,G)$ for any $\mu \in m(L,G)$. We will see in Theorem 2.3 - suggested by [9, exercise 4 on p. 241] - that (1) defines a group isomorphism from $m(L,G,p)$ onto $d(L,G)$.

Lemma 2.2. *Let $\alpha \in d(L,G)$ and $x,y,z \in L$ with $x \leqslant y$.*

(a) *Then $\alpha(t_1,t_2) = 0$ for any $t_1,t_2 \in L$ with $x \vee (y \wedge z) \leqslant t_1 \leqslant t_2 \leqslant y \wedge (x \vee z)$.*
(b) *$\alpha(x,y) = \alpha(x \wedge z, y \wedge z) + \alpha(x \vee z, y \vee z)$.*

Proof. (a) Since $\alpha(t_i \wedge z, t_i) = \alpha(z, t_i \vee z)$, $t_i \wedge z = y \wedge z$ and $t_i \vee z = x \vee z$, we have $\alpha(y \wedge z, t_i) = \alpha(z, x \vee z)$, hence $\alpha(y \wedge z, t_1) = \alpha(y \wedge z, t_2) = \alpha(y \wedge z, t_1) + \alpha(t_1,t_2)$ and therefore $\alpha(t_1,t_2) = 0$.

(b) By (a) we have in particular $0 = \alpha(x \vee (y \wedge z), y \wedge (x \vee z))$. Replacing (x,y) in (3) by $(y \wedge z, x)$ and $(y, x \vee z)$ we get $\alpha(x \wedge z, y \wedge z) = \alpha(x, x \vee (y \wedge z))$ and $\alpha(x \vee z, y \vee z) = \alpha(y \wedge (x \vee z), y)$. Adding these three equalities and using property (2) we obtain

$$\begin{aligned}\alpha(x \wedge z, y \wedge z) + \alpha(x \vee z, y \vee z) &= \alpha(x, x \vee (y \wedge z)) + \alpha(x \vee (y \wedge z), y \wedge (x \vee z)) \\ &\quad + \alpha(y \wedge (x \vee z), y) \\ &= \alpha(x,y).\end{aligned}$$

■

Theorem 2.3.

(a) $\Phi : m(L,G) \to d(L,G)$, *where* $\Phi(\mu) = \overline{\mu}$ *is defined by* (1), *is a group epimorphism. The kernel of* Φ *are the* G*-valued constant functions on* L.

(b) *For* $p \in L$, *the restriction* Φ_p *of* Φ *to* $m(L,G,p)$ *is an isomorphism onto* $d(L,G)$. *If* $\alpha \in d(L,G)$ *and* $\mu = \Phi_p^{-1}(\alpha)$, *then for any* $x \in L$

$$\mu(x) = \alpha(p, p \vee x) - \alpha(p \wedge x, p). \tag{5}$$

In particular, if $p = \min L$, *then* $\mu(x) = \alpha(p,x)$.

Proof. We already observed that Φ is well-defined. Obviously Φ is a group homomorphism.

If $\mu \in m(L,G)$ is constant, then $\Phi(\mu) = 0$ by (1); viceversa, if $\overline{\mu} := \Phi(\mu) = 0$, then for all $x, y \in L$ we have $\mu(x) = \mu(x \wedge y) + \overline{\mu}(x \wedge y, x) = \mu(x \wedge y) + \overline{\mu}(x \wedge y, y) = \mu(y)$. In particular, Φ_p is injective for any $p \in L$.

Let now $\alpha \in d(L,G)$ and μ be defined by (5). Then obviously $\mu(p) = 0$. We show that μ is modular and thus $\mu \in m(L,G,p)$. Let $x, y \in L$ and $a := p \wedge x \wedge y$. Then

$$\begin{aligned}\alpha(a, x \wedge y) + \alpha(a, x \vee y) &= \alpha(a, x \wedge y) + \alpha(a, y) + \alpha(y, x \vee y) \\ &= \alpha(a, x \wedge y) + \alpha(a, y) + \alpha(x \wedge y, x) = \alpha(a, x) + \alpha(a, y).\end{aligned}$$

Subtracting from this equality $2\alpha(a,p)$ one obtains the modularity law for μ observing that

$$\begin{aligned}\alpha(a,t) - \alpha(a,p) &= \alpha(a,t) - \alpha(a, p \wedge t) - \alpha(p \wedge t, p) = \alpha(p \wedge t, t) - \alpha(p \wedge t, p) \\ &= \alpha(p, p \vee t) - \alpha(p \wedge t, p) = \mu(t)\end{aligned}$$

for any $t \geqslant a$.

It remains to show that $\Phi(\mu) = \alpha$. Using property (2) and Lemma 2.2(b) we obtain for $x \leqslant y$

$\mu(y) - \mu(x) = \alpha(p, p \vee y) - \alpha(p \wedge y, p) - (\alpha(p, p \vee x) - \alpha(p \wedge x, p)) = \alpha(p \vee x, p \vee y) + \alpha(p \wedge x, p \wedge y) = \alpha(x,y)$. ■

The next lemma collects some rules which can be extracted from [19, 16, 17].

Lemma 2.4. *Let* $\alpha \in d(L,G)$ *and* $\alpha^*(x,y) := \{\alpha(u,v) : x \wedge y \leqslant u \leqslant v \leqslant x \vee y\}$. *Then for all* $x, y, z \in L$:

(a) $\alpha^*(x \wedge y, x) = \alpha^*(y, x \vee y)$;

(b) $\alpha^*(x \vee z, y \vee z) \subseteq \alpha^*(x,y)$ *and dually* $\alpha^*(x \wedge z, y \wedge z) \subseteq \alpha^*(x,y)$;

(c) $\alpha^*(x,y) \subseteq \alpha^*(x,z) + \alpha^*(z,y)$.

(d) *If* $\mu \in m(L,G)$ *and* $\alpha = \Phi(\mu)$, *then* $\alpha^*(x,y) = \{\mu(v) - \mu(u) : x \wedge y \leqslant u \leqslant v \leqslant x \vee y\} \subseteq \{\mu(v) - \mu(u) : u, v \in [x \wedge y, x \vee y]\} \subseteq \alpha^*(x,y) - \alpha^*(x,y)$.

Proof. (a) $\subseteq$: Let $x\wedge y \leqslant u \leqslant v \leqslant x$. Then $y \leqslant u\vee y \leqslant v\vee y \leqslant x\vee y$ and therefore $\alpha(u,v) = \alpha(u\vee y, v\vee y) \in \alpha^*(y, x\vee y)$ by Lemma 2.2(b). The other inclusion $\supseteq$ holds by duality.

(b) We prove the first inclusion. Using $(x\vee z)\wedge(y\vee z) \geqslant [(x\wedge y)\vee z]$, (a) with $[(x\wedge y)\vee z], x\vee y$ instead of x, y and $[(x\wedge y)\vee z]\wedge(x\vee y) \geqslant x\wedge y$ we get

$$\begin{aligned}\alpha^*(x\vee z, y\vee z) &= \alpha^*((x\vee z)\wedge(y\vee z), (x\vee z)\vee(y\vee z)) \\ &\subseteq \alpha^*([(x\wedge y)\vee z], [(x\wedge y)\vee z]\vee(x\vee y)) \\ &= \alpha^*([(x\wedge y)\vee z]\wedge(x\vee y), x\vee y) \subseteq \alpha^*(x\wedge y, x\vee y) = \alpha^*(x,y).\end{aligned}$$

(c) We essentially follow the calculation[5] of [17, page 290/291]. Let $x\wedge y \leqslant u \leqslant v \leqslant x\vee y$ and set $r = x\wedge z$, $s = x\vee z$. Using Lemma 2.2(b) we have

$$\begin{aligned}\alpha(u,v) &= \alpha(u\vee r, v\vee r) + \alpha(u\wedge r, v\wedge r) \\ &= \alpha((u\vee r)\wedge s, (v\vee r)\wedge s) + \alpha((u\vee r)\vee s, (v\vee r)\vee s) + \alpha(u\wedge r, v\wedge r).\end{aligned}$$

The first term belongs to $\alpha^*(x,z)$. It remains to show that

$$\alpha(u\vee s, v\vee s) + \alpha(u\wedge r, v\wedge r) \in \alpha^*(z,y).$$

Since $y' := y\vee(u\wedge r) \geqslant y$, we have $\alpha(u\vee s\vee y', v\vee s\vee y') = \alpha(s\vee y', s\vee y') = 0$ and therefore by Lemma 2.2(b)

$$\alpha(u\vee s, v\vee s) = \alpha((u\vee s)\wedge y', (v\vee s)\wedge y'). \tag{6}$$

Since $y'' := y\wedge(v\vee s) \leqslant y$ we similarly get

$$\alpha(u\wedge r, v\wedge r) = \alpha((u\wedge r)\vee y'', (v\wedge r)\vee y''). \tag{7}$$

Applying Lemma 2.2(a) we obtain

$$\alpha((v\wedge r)\vee y'', ((v\wedge r)\vee y)\wedge(v\vee s)) = 0 \text{ and } \alpha((u\wedge r)\vee y'', ((u\wedge r)\vee y)\wedge(v\vee s)) = 0,$$

hence with (7)

$$\begin{aligned}\alpha(u\wedge r, v\wedge r) &= \alpha((u\wedge r)\vee y'', ((v\wedge r)\vee y)\wedge(v\vee s)) \\ &= \alpha(((u\wedge r)\vee y)\wedge(v\vee s), ((v\wedge r)\vee y)\wedge(v\vee s))) \\ &= \alpha((v\vee s)\wedge y', ((v\wedge r)\vee y)\wedge(v\vee s)).\end{aligned}$$

Adding this equality to (6) we have

$$\begin{aligned}\alpha(u\vee s, v\vee s) + \alpha(u\wedge r, v\wedge r) &= \alpha((u\vee s)\wedge y', (v\vee s)\wedge y') \\ &\quad + \alpha((v\vee s)\wedge y', ((v\wedge r)\vee y)\wedge(v\vee s))) \\ &= \alpha((u\vee s)\wedge y', (v\wedge r)\vee y)\wedge(v\vee s))) \in \alpha^*(z,y).\end{aligned}$$

(d) Observe that for $u, v \in [x\wedge y, x\vee y]$ one has

$$\mu(v) - \mu(u) = (\mu(v) - \mu(u\wedge v)) - (\mu(v) - \mu(u\wedge v)) \in \alpha^*(x,y) - \alpha^*(x,y). \quad \blacksquare$$

[5]Much simpler is the proof of $\alpha^*(x,y) \subseteq \alpha^*(x,z) + \alpha^*(y,z) + \alpha^*(y,z)$ which is sufficient to deduce 2.5.

An immediate consequence of Lemma 2.2(a) and 2.4 is

Corollary 2.5 ([16, 17]). *Let $\alpha \in d(L,G)$.*

(a) *Then $N(\alpha) := \{(x,y) : \alpha^*(x,y) = \{0\}\}$ is a lattice congruence, and the quotient $L/N(\alpha)$ is a modular lattice.*
(b) *If G is a topological group, then the sets $\{(x,y) \in L^2 : \alpha^*(x,y) \subseteq U\}$ where U is a 0-neighbourhood in G, form a base for a lattice uniformity*[6] *$u(\alpha)$ on L.*

It is clear that the uniformity $u(\alpha)$ defined in the preceding lemma is the weakest lattice uniformity on L making α uniformly continuous. If $\mu \in m(L,G)$ and $\alpha = \Phi(\mu)$, then $u(\alpha)$ is also the weakest lattice uniformity on L making μ uniformly continuous. Therefore we write $u(\mu) := u(\alpha)$ and this uniformity is called the μ-uniformity or the α-uniformity. The induced topology is denoted by $\tau(\mu)$ or by $\tau(\alpha)$ and is called the μ-topology or the α-topology. By [26, Proposition 3.2] this is the weakest locally convex lattice topology making μ (or equivalently, making α) continuous. Recall that a lattice topology is a topology making the lattice operations $\vee$ and $\wedge$ continuous; it is called locally convex if every point has a neighbourhood base consisting of sets U such that $a, b \in U$ and $a \leqslant b$ implies $[a,b] \subseteq U$. If ρ is a locally convex lattice topology on L and μ (or α) is continuous w.r.t. ρ, we write $\mu \ll \rho$ (or $\alpha \ll \rho$). Therefore $\mu \ll \rho$ iff $\tau(\mu) \subseteq \rho$.

The congruence defined in Corollary 2.5 (a) can also be described with the aid of μ:

$$N(\mu) := \{(x,y) : \mu \text{ is constant on } [x \wedge y, x \vee y]\} = N(\alpha).$$

3. ℓ-Group-valued modular functions and distance functions

In this section let G be a Dedekind complete ℓ-group.

We extend $+$ and $\leqslant$ in the usual way onto $\overline{G} := G \cup \{+\infty\}$. The pointwise order on spaces of $\overline{G}$ -valued functions we denote by $\leqslant$, too. Let $p \in L$; with the aid of the isomorphism Φ_p one can transfer the order relation $\leqslant$ of $d(L,G)$ to $m(L,G,p)$ such that Φ_p becomes an order isomorphism (cf. Theorem 2.3). Thus, for $\mu, \nu \in m(L,G,p)$, we set $\nu \preceq \mu$ iff $\Phi_p(\nu) \leqslant \Phi_p(\mu)$, i.e.

$$\nu \preceq \mu \qquad \text{iff} \qquad \mu - \nu \ \text{ is increasing.} \tag{8}$$

In particular, $\mu \succeq 0$ means that μ is increasing. This order relation $\preceq$ was used by Birkhoff [9, p. 240] and later by Bauer [6, 7]. If L is a Boolean algebra and $\mu, \nu \in m(L,G,0)$, then $\nu \preceq \mu$ iff $\nu \leqslant \mu$.

Proposition 3.1. *The semivariation $\|\alpha\| : L^2 \to \overline{G}$ of $\alpha \in d(L,G)$ defined by $\|\alpha\|(x,y) := \sup\{|g| : g \in \alpha^*(x,y)\}$ has the following properties:*

(a) $0 = \|\alpha\|(x,x) \leqslant \|\alpha\|(x,y) = \|\alpha\|(x \wedge y, x \vee y)$,
(b) $\|\alpha\|(x \wedge z, y \wedge z) \leqslant \|\alpha\|(x,y)$, $\|\alpha\|(x \vee z, y \vee z) \leqslant \|\alpha\|(x,y)$,
(c) $\|\alpha\|(x \wedge y, y) = \|\alpha\|(x, x \vee y)$,
(d) $\|\alpha\|(x,y) \leqslant \|\alpha\|(x,z) + \|\alpha\|(z,y)$.

[6] that is a uniformity making the lattice operations $\vee$ and $\wedge$ uniformly continuous

This can easily be verified with the aid of Lemma 2.4.

If $0 \leqslant \alpha \in d(L,G)$, then $\|\alpha\| = \alpha$ and therefore by the last inequality we have for any $x, y, z \in L$

$$\alpha(x,y) \leqslant \alpha(x,z) + \alpha(z,y).$$

In general $\|\alpha\|$ doesn't belong to the space $d(L,\overline{G})$ of $\overline{G}$-valued functions on L satisfying (2), (3), (4). To find for $\alpha \in d(L,G)$ a positive majorant in $d(L,\overline{G})$ we use the following

Proposition 3.2. *Let $\varphi : \{(x,y) \in L^2 : x \leqslant y\} \to \overline{G}$ be a function satisfying*

$$\varphi(x,x) = 0, \qquad \varphi(x \wedge y, y) = \varphi(x, x \vee y), \tag{9}$$

$$\varphi(x,y) \leqslant \varphi(x \wedge z, y \wedge z) + \varphi(x \vee z, y \vee z) \qquad \text{if } \ x \leqslant y. \tag{10}$$

Set $\psi(x,y) := \sup\{\sum_{i=1}^n \varphi(x_{i-1}, x_i) : n \in \mathbb{N},\ x \wedge y = x_0 \leqslant x_1 \leqslant \ldots \leqslant x_n = x \vee y\}$.

Then $\psi \in d(L,\overline{G})$. ψ is the smallest function of $d(L,\overline{G})$ such that $\varphi(x,y) \leqslant \psi(x,y)$ for all $x, y \in L$ with $x \leqslant y$.

Proof. We first show $\psi(x \wedge y, y) = \psi(x, x \vee y)$. Let $x \wedge y = x_0 \leqslant \ldots \leqslant x_n = y$. Then $x = x_0 \vee x \leqslant \ldots \leqslant x_n \vee x = x \vee y$ and $\varphi(x_{i-1} \wedge x, x_i \wedge x) = \varphi(x \wedge y, x \wedge y) = 0$, hence $\sum_{i=1}^n \varphi(x_{i-1}, x_i) \leqslant \sum_{i=1}^n \varphi(x_{i-1} \vee x, x_i \vee x) \leqslant \psi(x, x \vee y)$. It follows $\psi(x \wedge y, y) \leqslant \psi(x, x \vee y)$. The other inequality $\psi(x \wedge y, y) \geqslant \psi(x, x \vee y)$ can be seen analogously.

Let now $x \leqslant y \leqslant z$. Then obviously $\psi(x,z) \geqslant \psi(x,y) + \psi(y,z)$. To prove $\leqslant$ let $x = x_0 \leqslant \ldots \leqslant x_n = z$. Then $x = x_0 \wedge y \leqslant \ldots \leqslant x_n \wedge y = y$ and $y = x_0 \vee y \leqslant \ldots \leqslant x_n \vee y = z$, hence $\sum_{i=1}^n \varphi(x_{i-1}, x_i) \leqslant \sum_{i=1}^n \varphi(x_{i-1} \wedge y, x_i \wedge y) + \sum_{i=1}^n \varphi(x_{i-1} \vee y, x_i \vee y) \leqslant \psi(x,y) + \psi(y,z)$. It follows $\psi(x,z) \leqslant \psi(x,y) + \psi(y,z)$.

We have proved that $\psi \in d(L,\overline{G})$. Obviously $\varphi \leqslant \chi \in d(L,\overline{G})$ implies $\psi \leqslant \chi$. ■

Remark 3.3. If L is modular, then condition (10) in Proposition 3.2 can be replaced by the simpler condition

$$\varphi(x,z) \leqslant \varphi(x,y) + \varphi(y,z) \qquad \text{if } \ x \leqslant y \leqslant z. \tag{11}$$

This cannot be done if L is not modular.

Proof. Let $x \leqslant y$. Then we have for any $z \in L$ by (11), modularity and (9)

$$\begin{aligned}
\varphi(x,y) &\leqslant \varphi(x, x \vee (y \wedge z)) + \varphi(x \vee (y \wedge z), y) \\
&= \varphi(x, x \vee (y \wedge z)) + \varphi(y \wedge (x \vee z), y) \\
&= \varphi(x \wedge z, y \wedge z) + \varphi(x \vee z, y \vee z).
\end{aligned}$$

For the second statement take the non-modular lattice consisting of five elements and, if $x < y$, let $\varphi(x,y) = 1$. ■

For $\alpha \in d(L,G)$ the function $\varphi(x,y) = |\alpha(x,y)|$ satisfies the assumption of Proposition 3.2 by Lemma 2.2(b). Therefore the *variation* $var(\alpha)$ defined by

$$(var(\alpha))(x,y) = \sup\left\{\sum_{i=1}^{n} |\alpha(x_{i-1},x_i)| : n \in \mathbb{N},\ x \wedge y = x_0 \leqslant x_1 \leqslant \ldots \leqslant x_n = x \vee y\right\}$$

belongs to $d(L,\overline{G})$.

Definition 3.4. Let $\mu \in m(L,G)$ and $\overline{\mu} = \Phi(\mu)$. We say that μ and $\overline{\mu}$ have *finite variation* if $var(\overline{\mu})$ is G-valued. We denote by $fvd(L,G), fvm(L,G), fvm(L,G,p)$ the spaces of functions of finite variation belonging to $d(L,G), m(L,G), m(L,G,p)$, respectively.

Theorem 3.5. *$fvd(L,G)$ is a Dedekind complete ℓ-group. For $\alpha,\beta \in fvd(L,G)$ we have*

$$(\alpha \vee \beta)(x,y) = \sup\left\{\sum_{i=1}^{n} \alpha(x_{i-1},x_i) \vee \beta(x_{i-1},x_i) : n \in \mathbb{N},\ x \wedge y = x_0 \leqslant \ldots \leqslant x_n = x \vee y\right\} \tag{12}$$

$$(\alpha \wedge \beta)(x,y) = \inf\left\{\sum_{i=1}^{n} \alpha(x_{i-1},x_i) \wedge \beta(x_{i-1},x_i) : n \in \mathbb{N},\ x \wedge y = x_0 \leqslant \ldots \leqslant x_n = x \vee y\right\} \tag{13}$$

$$|\alpha| := \alpha \vee (-\alpha) = var(\alpha). \tag{14}$$

If $(\alpha_\gamma)_{\gamma\in\Gamma}$ is an increasing bounded net in $fvd(L,G)$, then its pointwise supremum is its supremum in $fvd(L,G)$.

Proof. Since $var(-\alpha) = var(\alpha)$ and $var(\alpha+\beta) \leqslant var(\alpha) + var(\beta)$, $fvd(L,G)$ is a subgroup of $d(L,G)$. To prove (12) we apply Proposition 3.2 with $\varphi(x,y) := \alpha(x,y) \vee \beta(x,y)$. Observe that by Lemma 2.2(b) φ satisfies the assumption of Proposition 3.2 and that the right-hand side of (12) is $\leqslant var(\alpha) + var(\beta)$, hence finite. Thus we have proved that $fvd(L,G)$ is an ℓ-group. (14) is a special case of (12) (for $\beta = -\alpha$) and (13) follows from (12) using $\alpha \wedge \beta = -(-\alpha) \vee (-\beta)$. The last statement obviously holds. ■

It follows that for $p \in L$ also $(\Phi_p^{-1}(fvd(L,G)), +, \preceq)$ is a Dedekind complete ℓ-group.

Always we have $\|\alpha\| \leqslant var(\alpha)$; but α can be bounded without having finite variation: Take for L the real unit interval $[0,1]$ and $\alpha = \Phi(\mu)$ for some continuous function $\mu : [0,1] \to \mathbb{R}$ which is not of finite total variation.

We now give a condition under which α has finite variation iff it has finite semivariation.

Proposition 3.6. *Let $\alpha \in d(L,G)$ such that $\|\alpha\|$ is G-valued. Suppose that for any chain $x_0 \leqslant x_1 \leqslant \ldots \leqslant x_n$ in L and $I \subseteq \{0,\ldots,n\}$ there are $x,y \in L$ such that $x_0 \leqslant x \leqslant y \leqslant x_n$ and $\sum_{i\in I}\alpha(x_{i-1},x_i) = \alpha(x,y)$. Then $\alpha \in fvd(L,G)$ and*

$$\alpha^+(x,y) = \sup\{\alpha(a,b) : x\wedge y \leqslant a \leqslant b \leqslant x \vee y\} \tag{15}$$

$$\alpha^-(x,y) = -\inf\{\alpha(a,b) : x\wedge y \leqslant a \leqslant b \leqslant x \vee y\} \tag{16}$$

$$|\alpha|(x,y) = \sup\{\alpha(a,b)-\alpha(c,d) : x\wedge y \leqslant a \leqslant b \leqslant x\vee y, x\wedge y \leqslant c \leqslant d \leqslant x\vee y\}. \tag{17}$$

In particular, $\|\alpha\| \leqslant |\alpha| \leqslant 2\|\alpha\|$.

Proof. (i) In our calculation we use the following fact: If $+,\cdot$ are two binary commutative and associative operations on a set X satisfying the distributive law, then $\prod_{i=1}^n(x_{i,0}+x_{i,1}) = \sum_{(\varepsilon_i)\in\{0,1\}^n}\prod_{i=1}^n x_{i,\varepsilon_i}$. In particular, for $(X,+,\cdot) = (G,\vee,+)$ we have $\sum_{i=1}^n(x_{i,0}\vee x_{i,1}) = \sup_{(\varepsilon_i)\in\{0,1\}^n}\sum_{i=1}^n x_{i,\varepsilon_i}$.

(ii) Let now $x\wedge y = x_0 \leqslant \ldots \leqslant x_n = x\vee y$ be a chain in L. Then, using (i), we have

$$\begin{aligned}\sum_{i=1}^n \alpha(x_{i-1},x_i)\vee 0 &= \sup\{\sum_{i\in I}\alpha(x_{i-1},x_i) : I \subseteq \{1,\ldots,n\}\}\\ &\leqslant \sup\{\alpha(a,b) : x\wedge y \leqslant a \leqslant b \leqslant x\vee y\}\\ &\leqslant \|\alpha\|(x,y)\end{aligned} \tag{18}$$

and as well

$$\sum_{i=1}^n(-\alpha(x_{i-1},x_i))\vee 0 \leqslant \|\alpha\|(x,y).$$

Adding these two inequalities we get $\sum_{i=1}^n|\alpha(x_{i-1},x_i))| \leqslant 2\|\alpha\|(x,y)$.

Consequently $var(\alpha) \leqslant 2\|\alpha\|$.

(18) and (12) yield $\alpha^+(x,y) \leqslant \sup\{\alpha(a,b) : x\wedge y \leqslant a \leqslant b \leqslant x\vee y\}$ whereas $\geqslant$ is obviously true.

(16) and (17) now follow from (15) observing $\alpha^- = (-\alpha)^+$ and $|\alpha| = \alpha^- + (-\alpha)^+$. ∎

Remark 3.7. Let $\alpha \in d(L,G)$ be of finite semivariation. Then the assumption of Proposition 3.6 is satisfied if L is complemented or sectionally complemented or relatively complemented, or if L is a commutative ℓ-group and $\alpha = \Phi(\mu)$ for some homomorphism $\mu : L \to G$, see [26, Proposition 2.8]

In particular, it follows the known fact that if A is a Boolean algebra the space $fvm(A,0,G)$ coincides with the space $b(A,G)$ of bounded G-valued measures on A.

Theorem 3.5 allows to use the band decomposition theorem of Riesz to obtain decomposition theorems for functions of $fvd(L,G)$ or equivalently for G-valued modular functions on L of finite variation.

We need a preparatory lemma. If a net (x_γ) order converges to x, we write $x_\gamma \xrightarrow{o} x$.

Lemma 3.8. *(x_γ) be a bounded net in L and $x \in L$.*

(a) *Let $\alpha \in d(L,G)$. Then $\|\alpha\|(x_\gamma, x) \xrightarrow{o} 0$ iff $\|\alpha\|(x_\gamma \wedge x, x_\gamma \wedge a) \xrightarrow{o} 0$ for any $a \geqslant x$ and $\|\alpha\|(x_\gamma \vee x, x_\gamma \vee a) \xrightarrow{o} 0$ for any $a \leqslant x$.*

(b) *If $\alpha \in fvd(L,G)$, then the following conditions are equivalent:*

(i) *$\alpha(x_\gamma \wedge x, x_\gamma \wedge a) \xrightarrow{o} 0$ for any $a \geqslant x$ and $\alpha(x_\gamma \vee x, x_\gamma \vee a) \xrightarrow{o} 0$ for any $a \leqslant x$;*

(ii) *$\|\alpha\|(x_\gamma, x) \xrightarrow{o} 0$;*

(iii) *$|\alpha|(x_\gamma, x) \xrightarrow{o} 0$.*

Proof. Since (x_γ) is bounded, we may assume that L is bounded.

(a) $\Rightarrow$: Let $a \geqslant x$. Using the rules of Proposition 3.1 we get

$$\begin{aligned}\|\alpha\|(x_\gamma \wedge x, x_\gamma \wedge a) &\leqslant \|\alpha\|(x_\gamma \wedge x, x) + \|\alpha\|(x, x_\gamma \wedge a)\\ &= \|\alpha\|(x_\gamma \wedge x, x \wedge x) + \|\alpha\|(x_\gamma \wedge a, x \wedge a)\\ &\leqslant 2\|\alpha\|(x_\gamma, x) \xrightarrow{o} 0,\end{aligned}$$

hence $\|\alpha\|(x_\gamma \wedge x, x_\gamma \wedge a) \xrightarrow{o} 0$. The second condition holds by duality.

$\Leftarrow$: With $a = 1$ we have $\|\alpha\|(x, x_\gamma \vee x) = \|\alpha\|(x_\gamma \wedge x, x_\gamma) = \|\alpha\|(x_\gamma \wedge x, x_\gamma \wedge 1) \xrightarrow{o} 0$ and by duality $\|\alpha\|(x_\gamma \wedge x, x) \xrightarrow{o} 0$, hence $\|\alpha\|(x_\gamma, x) \leqslant \|\alpha\|(x_\gamma \wedge x, x) + \|\alpha\|(x, x_\gamma \vee x) \xrightarrow{o} 0$.

(b) (iii) $\Rightarrow$ (ii) follows from $\|\alpha\| \leqslant |\alpha|$ and (ii) $\Rightarrow$ (i) from (a).

(i) $\Rightarrow$ (iii): Let $y_\gamma = x_\gamma \vee x$. Then by (2), Lemma 2.2(a) and (3) we have for $x \leqslant a \leqslant b$

$$\begin{aligned}\alpha(x, y_\gamma \wedge a) &= \alpha(x, (x_\gamma \wedge a) \vee x) + \alpha((x_\gamma \wedge a) \vee x, (x_\gamma \vee x) \wedge a)\\ &= \alpha(x, (x_\gamma \wedge a) \vee x) = \alpha(x_\gamma \wedge x, x_\gamma \wedge a) \xrightarrow{o} 0\end{aligned}$$

and as well $\alpha(x, y_\gamma \wedge b) \xrightarrow{o} 0$, hence

$$\alpha(y_\gamma \wedge a, y_\gamma \wedge b) = \alpha(x, y_\gamma \wedge b) - \alpha(x, y_\gamma \wedge a) \xrightarrow{o} 0. \tag{19}$$

Let $x = z_0 \leqslant \ldots \leqslant z_n = 1$ be a chain in L. Then

$$\sum_{i=1}^n |\alpha(z_{i-1}, z_i)| \leqslant \sum_{i=1}^n |\alpha(z_{i-1} \wedge y_\gamma, z_i \wedge y_\gamma)| + \sum_{i=1}^n |\alpha(z_{i-1} \vee y_\gamma, z_i \vee y_\gamma)|,$$

moreover

$$\sum_{i=1}^n |\alpha(z_{i-1} \vee y_\gamma, z_i \vee y_\gamma)| \leqslant |\alpha|(y_\gamma, 1) = |\alpha|(x,1) - |\alpha|(x, y_\gamma),$$

hence

$$|\alpha|(x, y_\gamma) + \sum_{i=1}^n |\alpha(z_{i-1}, z_i)| \leqslant \sum_{i=1}^n |\alpha(z_{i-1} \wedge y_\gamma, z_i \wedge y_\gamma)| + |\alpha|(x,1),$$

therefore using (19)

$$\limsup |\alpha|(x, y_\gamma) + \sum_{i=1}^{n} |\alpha(z_{i-1}, z_i)| \leqslant \limsup \sum_{i=1}^{n} |\alpha(z_{i-1} \wedge y_\gamma, z_i \wedge y_\gamma)| + |\alpha|(x, 1)$$
$$= |\alpha|(x, 1).$$

It follows

$$\limsup |\alpha|(x, y_\gamma) + |\alpha|(x, 1) \leqslant |\alpha|(x, 1)$$

and thus $|\alpha|(x, x_\gamma \vee x) \xrightarrow{o} 0$.

Dually we have $|\alpha|(x_\gamma \wedge x, x) \xrightarrow{o} 0$ and finally $|\alpha|(x, x_\gamma) = |\alpha|(x_\gamma \wedge x, x) + |\alpha|(x, x_\gamma \vee x) \xrightarrow{o} 0$. ■

Proposition 3.9. *Let $\mathcal{C}$ be a class of bounded nets in L and $\Lambda : \mathcal{C} \to L$ a "limit operator". Then $B := \{\alpha \in fvd(L, G) : |\alpha|(x_\gamma, x) \xrightarrow{o} 0$ whenever $(x_\gamma)_{\gamma\in\Gamma} \in \mathcal{C}$ and $x = \Lambda((x_\gamma)_{\gamma\in\Gamma})\}$ is a band in $fvd(L, G)$. Moreover,*

$$B = \{\alpha \in fvd(L, G) : \|\alpha\|(x_\gamma, x) \xrightarrow{o} 0 \textit{ whenever } (x_\gamma)_{\gamma\in\Gamma} \in \mathcal{C} \textit{ and } x = \Lambda((x_\gamma)_{\gamma\in\Gamma})\}.$$

Proof. Since $|\alpha-\beta| \leqslant |\alpha|+|\beta|$ for $\alpha, \beta \in fvd(L, G)$, B is a subgroup of $fvd(L, G)$. Obviously $0 \leqslant \alpha \leqslant \beta \in B$ implies $\alpha \in B$.

Let now (α_ι) be an increasing net in B and $0 \leqslant \alpha_\iota \uparrow \alpha \in fvd(L, G)$. To show that $\alpha \in B$ let $(x_\gamma)_{\gamma\in\Gamma}$ be a net in $\mathcal{C}$, $x = \Lambda((x_\gamma))$ and $a, b \in L$ such that $a \leqslant x_\gamma \leqslant b$ for all $\gamma \in \Gamma$. Then

$$\alpha(x_\gamma, x) = (\alpha - \alpha_\iota)(x_\gamma, x) + \alpha_\iota(x_\gamma, x) \leqslant (\alpha - \alpha_\iota)(a, b) + \alpha_\iota(x_\gamma, x)\,,$$

hence

$$\limsup_\gamma \alpha(x_\gamma, x) \leqslant (\alpha - \alpha_\iota)(a, b) + \limsup_\gamma \alpha_\iota(x_\gamma, x) = (\alpha - \alpha_\iota)(a, b)\,,$$

therefore

$$0 \leqslant \limsup_\gamma \alpha(x_\gamma, x) \leqslant \inf_\iota (\alpha - \alpha_\iota)(a, b) = 0\,,$$

i.e. $|\alpha|(x_\gamma, x) = \alpha(x_\gamma, x) \xrightarrow{o} 0$.

The second statement immediately follows from Proposition 3.8. ■

Remark 3.10. If in Proposition 3.9 a net $(x_\gamma)_{\gamma\in\Gamma} \in \mathcal{C}$ is increasing or decreasing, then it is enough to require that (x_γ) is bounded from above or bounded from below, respectively (replacing $(x_\gamma)_{\gamma\in\Gamma} \in \mathcal{C}$ by $(x_\gamma)_{\gamma\geqslant\gamma_0}$ for some $\gamma_0 \in \Gamma$).

Easy examples show that in general the boundedness assumption of Proposition 3.9 is not superfluous. (Take e.g. for L the ring $\mathcal{R}$ of all finite subsets of $\mathbb{N}$, let $\mathcal{C}$ be the class of all sequences (A_n) in $\mathcal{R}$ such that the characteristic functions (χ_{A_n}) converge pointwise to 0 and $\Lambda((A_n)) = 0$ for $(A_n) \in \mathcal{C}$.)

Definition 3.11. Let $p \in L$, $K \subseteq L$, H an upwards directed subset of L, $\beta \in d(L,G)$ and let $\aleph$ be a cardinal number. We call $\alpha \in fvd(L,G)$

(i) *K-smooth at p* if $\|\alpha\|(x_\gamma, x) \xrightarrow{o} 0$ for any monotone net (x_γ) in K with $x_\gamma \downarrow p$, and *K-smooth* if α is K-smooth at q for every $q \in L$,

(ii) *H-inner-regular* if $\inf\{\|\alpha\|(h,x) = 0 : x \geqslant h \in H\}$ for any $x \in L$,

(iii) *β-continuous* if $\alpha(x_\gamma, x) \xrightarrow{o} 0$ for any bounded net (x_γ) in L and $x \in L$ with $\|\beta\|(x_\gamma, x) \xrightarrow{o} 0$,

(iv) *$\aleph$-order continuous* ($\aleph$-o.c.) if $\alpha(x_\gamma, x) \xrightarrow{o} 0$ whenever $(x_\gamma)_{\gamma\in\Gamma}$ is a monotone net in L with $x_\gamma \uparrow x$ or $x_\gamma \downarrow x$ and the cardinality of Γ is $\leqslant \aleph$. If $\aleph = \aleph_0$ is countable, we say σ-o.c. instead of $\aleph_0$-o.c..

Theorem 3.12. *Let p, K, H, β as in Definition 3.11. Then the sets*

$$B_1 = \{\alpha \in fvd(L,G) : \alpha \textit{ is } K\textit{-smooth (at } p\textit{)}\},$$
$$B_2 = \{\alpha \in fvd(L,G) : \alpha \textit{ is } H\textit{-inner-regular}\},$$
$$B_3 = \{\alpha \in fvd(L,G) : \alpha \textit{ is } \beta\textit{-continuous}\},$$
$$B_4 = \{\alpha \in fvd(L,G) : \alpha \textit{ is } \aleph\textit{-o.c.}\}$$

are bands in $fvd(L,G)$.

Proof. For B_1 and B_2 this immediately follows from Proposition 3.9 and Remark 3.10. To prove that B_3 is a band, it is by Proposition 3.9 enough to show that $\alpha \in B_3$ iff $|\alpha|(x_\gamma, x) \xrightarrow{o} 0$ for any bounded net (x_γ) in L and $x \in L$ with $\|\beta\|(x_\gamma, x) \xrightarrow{o} 0$. We prove the non-obvious implication ($\Rightarrow$) using Lemma 3.8. Let $\alpha \in B_3$, (x_γ) be a bounded net in L and $x \in L$ with $\|\beta\|(x_\gamma, x) \xrightarrow{o} 0$. Then, for any $a \geqslant x$, we have $\|\beta\|(x, (x_\gamma \wedge a) \vee x) = \|\beta\|((x \wedge a) \vee x, (x_\gamma \wedge a) \vee x) \leqslant \|\beta\|(x, x_\gamma) \xrightarrow{o} 0$ Therefore $\alpha(x_\gamma \wedge x, x_\gamma \wedge a) = \alpha(x, (x_\gamma \wedge a) \vee x) \xrightarrow{o} 0$ since $\alpha \in B_3$. Analogously $\alpha(x_\gamma \vee x, x_\gamma \vee a) \xrightarrow{o} 0$ if $a \leqslant x$. It follows by Lemma 3.8 that $|\alpha|(x_\gamma, x) \xrightarrow{o} 0$.

Similarly one proceeds to show that B_4 is a band: Let $\alpha \in B_4$, moreover $(x_\gamma)_{\gamma\in\Gamma}$ a decreasing net in L , $x_\gamma \downarrow x$ and $|\Gamma| \leqslant \aleph$. Then, for $a \geqslant x$, we have $x_\gamma \wedge a \downarrow x$, hence $\alpha(x_\gamma \wedge a, x_\gamma \wedge x) = \alpha(x_\gamma \wedge a, x) \xrightarrow{o} 0$; moreover $\alpha(x_\gamma \vee a, x_\gamma \vee x) = \alpha(x_\gamma, x_\gamma) = 0$ for $a \leqslant x$. Thus $|\alpha|(x_\gamma, x) \xrightarrow{o} 0$ by Lemma 3.8. Similarly $|\alpha|(x_\gamma, x) \xrightarrow{o} 0$ if $x_\gamma \uparrow x$. Now Proposition 3.9 yields that B_4 is a band. ■

Let p, K, H, β, $\aleph$ as in Definition 3.11 and $\mu \in fvm(L,G)$. We say that μ is *K-smooth* (at p), *H-inner-regular*, *β-continuous*, *$\aleph$-o.c.*, *σ-o.c.* if $\overline{\mu} := \Phi(\mu)$ is so. Obviously, μ is σ-o.c. iff $x_n \uparrow x$ or $x_n \downarrow x$ implies $\mu(x_n) \xrightarrow{o} \mu(x)$ for any sequence $(x_n)_{n\in\mathbb{N}}$ in L.

Let us verify that μ is β-continuous iff $\mu(x_\gamma) \xrightarrow{o} \mu(x)$ whenever (x_γ) is a bounded net $\|\beta\|(x_\gamma, x) \xrightarrow{o} 0$: ($\Leftarrow$) If $\|\beta\|(x_\gamma, x) \xrightarrow{o} 0$, then $\|\beta\|(x \vee x_\gamma, x) \xrightarrow{o} 0$, hence $\mu(x \vee x_\gamma) \xrightarrow{o} \mu(x)$; similarly $\mu(x \wedge x_\gamma) \xrightarrow{o} \mu(x)$. Thus $\overline{\mu}(x_\gamma, x) = (\mu(x \vee x_\gamma) - \mu(x)) - (\mu(x \wedge x_\gamma) - \mu(x)) \xrightarrow{o} 0$. ($\Rightarrow$) Use $|\mu(x) - \mu(x_\gamma)| \leqslant |\mu(x) - \mu(x \vee x_\gamma)| + |\mu(x \vee x_\gamma) - \mu(x_\gamma)| \leqslant 2|\overline{\mu}|(x, x_\gamma)$ and 3.12.

To obtain decomposition theorems for modular functions we will apply the following version of the

Band Decomposition Theorem of Riesz. *Let B be a band in a Dedekind complete ℓ-group E. Then $B^{\perp} = \{x \in E : y \in B$ and $0 \leqslant y \leqslant |x|$ imply $y = 0\}$ is a band in E, and every $x \in E$ has a unique decomposition $x = x_1 + x_2$ with $x_1 \in B$ and $x_2 \in B^{\perp}$.*

Moreover, if B_0 is another band in E and $x \in B_0$, then the components x_1 and x_2 also belong to B_0.

Combining the band decomposition theorem with the preceding results we get the following decomposition theorems for modular functions.

Theorem 3.13 (Alexandroff decomposition theorem). *Let $K \subseteq L$, $p \in L$ and $\mu \in fvm(L,G,p)$. Then there are unique $\mu_1, \mu_2 \in fvm(L,G,p)$ with the following properties:*

(i) $\mu = \mu_1 + \mu_2$,
(ii) μ_1 *is K-smooth (at p),*
(iii) *If $\lambda : L \to G$ is a modular function K-smooth (at p) with $0 \preceq \lambda \preceq |\mu_2|$ and $\lambda(p) = 0$, then $\lambda = 0$.*

Moreover, if H is an upwards directed subset of L and μ is H-inner-regular, then μ_1 and μ_2 are H-inner-regular, too.

Theorem 3.14 (Hewitt-Yosida decomposition theorem). *Let $\aleph$ be a cardinal number, $p \in L$ and $\mu \in fvm(L,G,p)$. Then there are unique $\mu_1, \mu_2 \in fvm(L,G,p)$ with the following properties:*

(i) $\mu = \mu_1 + \mu_2$,
(ii) μ_1 *is $\aleph$-o.c.,*
(iii) *If $\lambda : L \to G$ is an $\aleph$-o.c. modular function with $0 \preceq \lambda \preceq |\mu_2|$ and $\lambda(p) = 0$, then $\lambda = 0$.*

Moreover, if H is an upwards directed subset of L and μ is H-inner-regular, then μ_1 and μ_2 are H-inner-regular, too.

Theorem 3.15 (Lebesgue decomposition theorem). *Let $p \in L$, $\mu \in fvm(L,G,p)$, $\nu \in m(L,G)$ and $\overline{\nu} = \Phi(\nu)$. Then there are unique $\mu_1, \mu_2 \in fvm(L,G,p)$ with the following properties:*

(i) $\mu = \mu_1 + \mu_2$,
(ii) μ_1 *is $\overline{\nu}$-continuous,*
(iii) *If $\lambda : L \to G$ is a $\overline{\nu}$-continuous modular function with $0 \preceq \lambda \preceq |\mu_2|$ and $\lambda(p) = 0$, then $\lambda = 0$.*

The relation $0 \preceq \lambda \preceq |\mu_2|$ appearing in the last theorems means precisely that, with $\overline{\mu_2} = \Phi(\mu_2)$, for any $x, y \in L$ with $x \leqslant y$

$$\begin{aligned} 0 \leqslant \lambda(y) - \lambda(x) &\leqslant |\overline{\mu_2}|(x,y) \\ &= \sup\left\{ \sum_{i=1}^{n} |\mu_2(x_i) - \mu_2(x_{i-1})| : x = x_0 \leqslant \ldots \leqslant x_n = y \right\}. \end{aligned}$$

Bauer [6, 2.3.6] also proved a version of the Hewitt-Yosida decomposition theorem. Bauer *defines* μ to be σ-o.c. if $|\mu|$ is σ-o.c., whereas we *prove* that μ is σ-o.c. iff $|\mu|$ is so.

ℓ-group-valued modular measures on D-lattices are also studied in [4]. It follows from [1, Corollary 2.4] and Proposition 3.6 that the space of all G-valued bounded modular measures on a D-lattice D is a regular embedded ℓ-subgroup of the Dedekind complete ℓ-group of all G-valued modular functions of bounded variation defined on D.

4. Measures with values in topological groups

In this section let $G = (G, +, \tau)$ be a Hausdorff topological group and A a Boolean algebra.

The following decomposition theorem, which answers a question of Drewnowski [14, p.47], was first proved by Traynor [22]. Another proof was given in [23]. We here use as in [23] the singularity condition proposed by Drewnowski [14, p.47]: If $\mu : A \to G$ is a measure and ρ an FN-topology on A, then $\mu \perp \rho$ means that the infimum of ρ and the μ-topology taken in the lattice of all FN-topologies on A is the trivial topology. Recall that μ is *exhaustive* if $\mu(a_n) \to 0$ for every disjoint sequence (a_n) in A.

Theorem 4.1. *Let G be complete, $\mu : A \to G$ an exhaustive measure and ρ an FN-topology on A. Then μ has a unique decomposition*

$$\mu = \mu_1 + \mu_2 \qquad \text{where } \mu_1, \mu_2 : A \to G \text{ are measures with } \mu_1 \ll \rho \text{ and } \mu_2 \perp \rho.$$

Moreover $\mu_1(A), \mu_2(A) \subseteq \overline{\mu(A)}$. The μ-topology is the supremum of the μ_1-topology and the μ_2-topology.

It is clear (also considering the completion of G) that the assumption of G being complete can be replaced by the assumption of the range of μ being contained in a complete subset of G. This will be used in the proof of Corollary 5.17.

Different choices of ρ in Theorem 4.1 yield different decomposition theorems, see [22], [23]. One obtains e.g. the Hewitt-Yosida decomposition taking for ρ the finest $\aleph$-o.c. FN-topology on A.

An essential tool in the proof of Theorem 4.1 given in [23] is the fact that an (as lattice) complete Boolean algebra admits at most one o.c. FN-topology. A common generalization of this fact and a theorem about Lebesgue topologies on Riesz spaces of Amemiya-Mori (see [3, Note on p. 90]) is

Theorem 4.2 ([24, Corollary 5.11]). *If u and v are o.c.*[7] *Hausdorff lattice uniformities on a complete lattice, then u and v induce the same topology.*

Following the method of [23], Theorem 4.1 was generalized for additive functions on orthomodular lattices [25], on complemented lattices [28], on MV-algebras [5] and for modular measures on D-lattices [2]. An essential tool in the proofs is Theorem 4.2; it is also used in the proof of Theorem 5.4.

[7] i.e. order convergence implies topological convergence

5. Modular functions and distance functions with values in locally solid ℓ-groups

In this section let $G = (G, +, \leqslant, \tau)$ be a Hausdorff locally solid Dedekind complete ℓ-group, where "locally solid" means that there is a 0-neighbourhood base consisting of solid subsets[8]. The uniformity of G is then a lattice uniformity, in particular τ is a locally convex lattice topology.

Proposition 5.1. *Let $\alpha_1, \alpha_2, \alpha, \beta \in d(L, G)$.*

(a) *If $0 \leqslant \alpha \leqslant \beta$, then $u(\alpha) \leqslant u(\beta)$.*
(b) $u(\alpha_1 + \alpha_2) \leqslant u(\alpha_1) \vee u(\alpha_2)$
(c) *If $\alpha_1 \geqslant 0$ and $\alpha_2 \geqslant 0$, then $u(\alpha_1 + \alpha_2) = u(\alpha_1) \vee u(\alpha_2)$.*
(d) $u(\alpha) \leqslant u(|\alpha|) = u(\alpha^+) \vee u(\alpha^-)$.

Proof. For (a) it is enough to observe that $\beta^*(x, y) \subseteq U$ implies $\alpha^*(x, y) \subseteq U$ for any solid subset U of G and $x, y \in G$ with $x \leqslant y$ (notation as in Lemma 2.4). For (b) observe that $\alpha_1 + \alpha_2$ is uniformly continuous w.r.t. $u(\alpha_1) \vee u(\alpha_2)$. (c) follows from (a) and (b). (d) follows from (b), (c) and the formulas $\alpha = \alpha^+ - \alpha^-$ and $|\alpha| = \alpha^+ + \alpha^-$. ■

The following example[9] shows that $\tau(\alpha)$ and $\tau(|\alpha|)$, hence $u(\alpha)$ and $u(|\alpha|)$, can be different.

Example 5.2. Let $\mathcal{A}$ be the σ-algebra of Borel subsets of $\mathbb{R}$, $E_n := (\mathbb{R}^n, \|\cdot\|_2)$ and $E := \{(x_n) \in \prod_{n=1}^{\infty} E_n : \|(x_n)\|_\infty := \sup \|x_n\|_2 < +\infty\}$. E is then a Dedekind complete Banach lattice. Let $r_1, r_2, r_3, \ldots$ be the Rademacher functions on $[0, 1]$ and $\nu_k(B) = \int_B r_k(x)dx$ for a Borel set $B \subseteq [0, 1]$. For any $A \in \mathcal{A}$ and $n \in \mathbb{N}$ let $\mu_n(A) = \frac{1}{\sqrt{n}}(\nu_k([0, 1] \cap (A - n))_{k=1}^n$. Then $\mu = (\mu_n) : \mathcal{A} \to E$ is a σ-additive, hence exhaustive, measure of finite variation (according to Definition 3.4) , but $|\mu|$ is not exhaustive.

Proof. If B is a Borel set in $[0, 1]$, then $\nu_k(B)$, $k \in \mathbb{N}$, are Fourier coefficients of the characteristic function χ_B. Therefore $\nu(B) := (\nu_k(B)) \in \ell_2$ and $\|\nu(B)\|_2 \leqslant \lambda(B)$ where λ denotes the Lebesgue measure. Consequently $\|\mu_n(A)\|_2 \leqslant \frac{1}{\sqrt{n}}\|\nu([0, 1] \cap (A - n))\|_2 \leqslant \frac{1}{\sqrt{n}}$, i.e. $(\|\mu_n(A)\|_2)_{n\in\mathbb{N}}$ is a null sequence. Using this fact it is easy to see that μ is σ-additive.

For a Borel set $B \subseteq [0, 1]$ we have $|\nu_k|(B) = \lambda(B)$. Hence, for $A \in \mathcal{A}$, $|\mu_n|(A) = \frac{\lambda([0,1]\cap(A-n))}{\sqrt{n}} e_n$ where $e_n = (1, \ldots, 1) \in E_n$ and $\||\mu_n|(A)\|_2 = \lambda([0, 1] \cap (A - n)$. It follows $|\mu|(A) = (\frac{\lambda([0,1]\cap(A-n))}{\sqrt{n}} e_n)_{n\in\mathbb{N}}$ and $\||\mu|(]n, n + 1[)\|_\infty = 1$. Thus $|\mu|$ is not exhaustive. ■

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[8] U is solid if $|x| \leqslant |y|$ and $y \in U$ implies $x \in U$.
[9] A similar example is given in [15, 7.10].

In view of Theorem 5.4 recall that a lattice uniformity is *(locally) exhaustive* if every (bounded) monotone sequence is Cauchy. If the uniformity of $(G,+,\tau)$ is locally exhaustive, τ is also called a *pre-Lebesgue topology.* Let $\mu \in m(L,G)$ and $\overline{\mu} = \Phi(\mu)$. Then μ and $\overline{\mu}$ are called (locally) exhaustive if $u(\mu) = u(\overline{\mu})$ is (locally) exhaustive, or equivalently if $\overline{\mu}(x_n, x_{n+1}) \to 0$ w.r.t. τ for any (bounded) monotone sequence in L (see [26, p.43]).

[21, Example 3] shows that a σ-additive[10] (hence exhaustive) measure on a σ-algebra need not be order bounded. On the other hand we have

Proposition 5.3. *The following conditions are equivalent:*

(i) *For any bounded lattice L' every $\alpha \in fvd(L',G)$ is exhaustive.*
(ii) *Every G-valued bounded*[11] *monotone modular function on a lattice is exhaustive.*
(iii) *Every G-valued monotone modular function on a lattice is locally exhaustive.*
(iv) *Any positive measure $\mu : \mathbb{P}(\mathbb{N}) \to G$ defined on the power set of $\mathbb{N}$ is exhaustive.*
(v) *τ is a pre-Lebesgue topology.*

Proof. (i)⇒(iv) is obvious.

(iv)⇒(v): Let (a_n) be sequence of positive elements of G such that $\{\sum_{i=1}^n a_i : n \in \mathbb{N}\}$ is bounded. Then $\mu(A) := \sum_{n\in A} a_n := \sup_{n\in\mathbb{N}} \sum_{n\geqslant i\in A} a_i$ defines a measure on $\mathbb{P}(\mathbb{N})$ (which is even σ-o.c. in the sense of Definition 3.11). Then $a_n = \mu(\{n\}) \to 0$ w.r.t. τ. As known (see e.g. [5, Proposition 4.1.9]) and easy to see, this implies that τ is a pre-Lebesgue topology.

(v)⇒(ii): If $0 \preceq \mu \in m(L,G)$ is bounded and (x_n) is a monotone sequence in L, then $\mu(x_n)$ is a bounded monotone sequence in G, hence Cauchy. Therefore $\overline{\mu}(x_n, x_{n+1}) \to 0$ w.r.t. τ.

(ii)⇒(iii): Apply (ii) to the restriction $\mu \mid [a,b]$ where $a,b \in L$ with $a \leqslant b$.

(iii)⇒(i): Since $\alpha = \alpha^+ - \alpha^-$, we may assume that $\alpha \geqslant 0$. Then $\alpha = \Phi(\mu)$ for some monotone modular function. Since L' is bounded, μ is exhaustive by (iii). Hence α is exhaustive. ∎

Theorem 5.4. *Let $\alpha \in fvd(L,G)$ such that $|\alpha|$ is exhaustive. Then the α-topology coincides with the $|\alpha|$-topology.*

Proof. Let $\beta := |\alpha|$ and $w := u(\beta)$. Passing to the quotient $L/N(\beta)$ we may assume that w is Hausdorff. Let $(\widetilde{L}, \widetilde{w})$ be the completion of (L,w) and $\widetilde{G}$ the (uniform) completion of G. Then $\widetilde{L}$ is a complete lattice, $\widetilde{w}$ is o.c. (see e.g. [24, Corollary 6.15]) and $\widetilde{G}$ is a complete locally solid ℓ-group (not necessarily Dedekind complete). Let $\widetilde{\alpha}, \widetilde{\beta} : \widetilde{L} \to \widetilde{G}$ be the continuous extensions of α, β. Then $\widetilde{\alpha}, \widetilde{\beta} \in d(\widetilde{L}, \widetilde{G})$ and $|\widetilde{\alpha}(x,y)| \leqslant \widetilde{\beta}(x,y)$ for all $x,y \in \widetilde{L}$.

[10] using τ-convergence, not order convergence in G
[11] "bounded" here always means "order bounded"

We claim that $u(\widetilde{\alpha})$ is Hausdorff. Let $a, b \in N(\widetilde{\alpha})$ with $a \leqslant b$. Therefore by Lemma 2.2 (b) and Corollary 2.5 (a) we have for any $x, y \in \widetilde{L}$ with $x \leqslant y$

$$\widetilde{\alpha}(x,y) = \widetilde{\alpha}(x \wedge a, y \wedge a) + \widetilde{\alpha}(x \vee a, y \vee a) = \widetilde{\alpha}(x \wedge a, y \wedge a) + \widetilde{\alpha}(x \vee b, y \vee b),$$

hence

$$|\widetilde{\alpha}(x,y)| \leqslant \widetilde{\beta}(x \wedge a, y \wedge a) + \widetilde{\beta}(x \vee b, y \vee b) =: \widetilde{\varphi}(x,y).$$

Therefore for any chain $x_0 \leqslant \ldots \leqslant x_n$ in $\widetilde{L}$

$$\sum_{i=1}^{n} |\widetilde{\alpha}(x_{i-1}, x_i)| \leqslant \sum_{i=1}^{n} \widetilde{\beta}(x_{i-1} \wedge a, x_i \wedge a) + \sum_{i=1}^{n} \widetilde{\beta}(x_{i-1} \vee b, x_i \vee b) = \widetilde{\varphi}(x_0, x_n).$$

In particular, for $x, y \in L$ with $x \leqslant y$

$$\begin{aligned} \beta(x,y) &= |\alpha|(x,y) \\ &= \sup\left\{ \sum_{i=1}^{n} |\alpha(x_{i-1}, x_i)| : x_i \in L, x = x_0 \leqslant \ldots \leqslant x_n = y \right\} \leqslant \widetilde{\varphi}(x,y). \end{aligned}$$

By continuity we have $\widetilde{\beta}(x,y) \leqslant \widetilde{\varphi}(x,y)$ for any $x, y \in \widetilde{L}$ with $x \leqslant y$. In particular

$$0 \leqslant \widetilde{\beta}(a,b) \leqslant \widetilde{\varphi}(a,b) = \widetilde{\beta}(a \wedge a, b \wedge a) + \widetilde{\beta}(a \vee b, b \vee b) = 0,$$

i.e. $\widetilde{\beta}(a,b) = 0$. Hence $a = b$ since $\widetilde{w} = u(\widetilde{\beta})$ is Hausdorff.

It now follows from Theorem 4.2 that $\widetilde{\alpha}$ and $\widetilde{\beta}$ induce the same topology on $\widetilde{L}$. Hence α and β induce the same topology on L, i.e. $\tau(\alpha) = \tau(|\alpha|)$. ■

With the aid of Theorem 5.4 we are able to compare the concepts of order continuity, smoothness and regularity introduced in Definition 3.11, which use order convergence in G, with the corresponding concepts using convergence w.r.t. τ, instead.

Definition 5.5. Let p, K, H, $\aleph$, α be as in Definition 3.11. We call $\alpha \in fvd(L,G)$

(i) *K-smooth at p* w.r.t. τ if $x_\gamma \to x$ w.r.t. $\tau(\alpha)$ for any monotone net (x_γ) in K with $x_\gamma \downarrow p$, and *K-smooth* w.r.t. τ if α is K-smooth at q w.r.t. τ for every $q \in L$,

(ii) *H-inner-regular* w.r.t. τ if for any 0-neighbourhood U in G and $x \in L$ there exists $h \in H$ with $x \geqslant h$ and $\alpha^*(h,x) \subseteq U$,

(iii) *$\aleph$-o.c.* w.r.t. τ if $\alpha(x_\gamma, x) \to 0$ w.r.t. τ whenever $(x_\gamma)_{\gamma\in\Gamma}$ is a monotone net in L with $x_\gamma \uparrow x$ or $x_\gamma \downarrow x$ and the cardinality of Γ is $\leqslant \aleph$.

If $\mu \in m(L,G)$ such that $\alpha = \overline{\mu} = \Phi(\mu)$, we call μ *K-smooth* (at p), *H-inner-regular*, *$\aleph$-o.c.* w.r.t. τ if $\overline{\mu}$ is so.

Lemma 5.6. *Let $0 \leqslant \alpha \in d(L,G)$, $x \in L$ and (x_γ) be a net in L. Then $x_\gamma \to x$ w.r.t. $\tau(\alpha)$ iff $\alpha(x_\gamma, x) \to 0$ w.r.t. τ.*

Proof. Observe that for any solid subset U of G and $x, y \in L$ we have $\alpha(x,y) \in U$ iff $\alpha^*(x,y) \subseteq U$ since $\alpha \geqslant 0$. ■

Lemma 5.7. *Let $\alpha \in fvd(L,G)$, $x \in L$ and (x_γ) be a monotone net in L. Suppose that τ has the Lebesgue property. Then $x_\gamma \to x$ w.r.t. $\tau(\alpha)$ iff $|\alpha|(x_\gamma, x) \xrightarrow{o} 0$.*

Proof. Let $x_\gamma \downarrow$. We may assume that (x_γ) is bounded and therefore that L is bounded. Since τ has the pre-Lebesgue property, it follows by Proposition 5.3 that $|\tau|$ is exhaustive. Therefore $\tau(\alpha) = \tau(|\alpha|)$ by Theorem 5.4 and thus, replacing α by $|\alpha|$, we may assume that $\alpha \geqslant 0$. Moreover, replacing x_γ by $x_\gamma \vee x$ we may assume that $x_\gamma \geqslant x$. In view of Lemma 5.6 it remains to verify that $\alpha(x_\gamma, x) \to 0$ w.r.t. τ iff $\alpha(x_\gamma, x) \xrightarrow{o} 0$. The implication $\Rightarrow$ holds since $0 \leqslant \alpha(x_\gamma, x) \downarrow$ and τ is Hausdorff. $\Leftarrow$ holds since τ is o.c.. ■

Corollary 5.8. *Suppose that τ has the Lebesgue property. Let p, K, H, $\aleph$ be as in Definitions 3.11/5.5. Then $\alpha \in fvd(L,G)$ is K-smooth at p (K-smooth, H-inner-regular, $\aleph$-o.c.) according to Definition 3.11 iff α is K-smooth at p (K-smooth, H-inner-regular, $\aleph$-o.c.) w.r.t. τ according to Definition 5.5.*

Proof. First observe that α is $\aleph$-o.c. iff $\alpha(x_\gamma, x) \to 0$ w.r.t. τ whenever $(x_\gamma)_{\gamma \in \Gamma}$ is a monotone net in L with order limit x and the cardinality of Γ is $\leqslant \aleph$. Now the assertion immediately follows from Lemma 5.7 and Theorem 3.12. ■

Corollary 5.9. *If G is endowed with a Hausdorff locally solid Lebesgue topology τ, then the conditions "K-smooth (at p)", "H-inner-regular", "$\aleph$-o.c." can be replaced in Theorems 3.13 and 3.14 by the corresponding conditions w.r.t. τ according to Definition 5.5.*

This generalizes the decomposition theorems [11, pp. 119, 123] in two directions: First, here we consider modular functions on lattices instead of measures on Boolean algebras. Second, in the decomposition theorems [11, pp. 119, 123] it is assumed that G satisfies - besides the assumption of Corollary 5.9 - additionally condition (M), i.e. that G has a 0-neighbourhood base consisting of sublattices. With the aid of Theorem 5.4 one also sees that in the decomposition theorems [12, Theorem 4.4] and [20, Theorem 5.1] it is enough to assume that G satisfies only the assumption of Corollary 5.9 (without condition (M)). Dedekind complete ℓ-groups endowed with a Hausdorff locally solid Lebesgue topology satisfying condition (M) are characterized in [29].

Notation 5.10. For a locally convex lattice topology ρ on L and $p \in L$ let

$$fvm(L,G,p,\rho) := \{\mu \in fvm(L,G,p) : \mu \ll \rho\}$$

and

$$fvd(L,G,\rho) := \{\alpha \in fvd(L,G) : \alpha \ll \rho\}.$$

If ρ is the topology induced by a lattice uniformity w on L, we write also $fvm(L,G,p,w)$ and $fvd(L,G,w)$ instead of $fvm(L,G,p,\rho)$ and $fvd(L,G,\rho)$.

Proposition 5.11. *Suppose that τ has the Lebesgue property and L is bounded. Let ρ be a locally convex lattice topology on L. Then $fvd(L,G,\rho)$ is a band in $fvd(L,G)$.*

Proof. If $\alpha_1, \alpha_2 \in fvd(L,G)$ are continuous w.r.t. ρ, then $\alpha_1 - \alpha_2$ is so; i.e. $\alpha_1 - \alpha_2 \ll \rho$.

If $\alpha, \beta \in fvd(L,G)$ with $|\alpha| \leqslant |\beta|$ and $\beta \ll \rho$, then $\tau(\alpha) \subseteq \tau(|\beta|) = \tau(\beta) \subseteq \rho$ by Proposition 5.1 and Theorem 5.4. Hence $\alpha \ll \rho$.

Let $\alpha_\gamma \in fvd(L,G,\rho)$ such that $0 \leqslant \alpha_\gamma \uparrow \alpha \in fvd(L,G)$. Then $0 \leqslant \alpha(x,y) - \alpha_\gamma(x,y) \leqslant \alpha(0,1) - \alpha_\gamma(0,1) \downarrow 0$. Therefore $\alpha_\gamma(x,y) \to \alpha(x,y)$ w.r.t. τ uniformly on $L \times L$. Hence α is continuous w.r.t. ρ, too. ■

The following example shows that in Proposition 5.11 it is not enough to suppose that τ is a pre-Lebesgue topology.

Example 5.12. Let $\nu : \mathbb{P}(\mathbb{N}) \to \{0,1\}$ be an ultrafilter measure which is not σ-additive, τ_0 the topology on ℓ_∞ induced by the seminorm $f \mapsto \int |f| d\nu$, τ_p the product topology on ℓ_∞ and τ be the supremum of τ_0 and τ_p; τ is then a locally solid pre-Lebesgue topology on ℓ_∞. The restrictions ρ_0 and ρ_p on $\{0,1\}^{\mathbb{N}} = \mathbb{P}(\mathbb{N})$ of τ_0 and τ_p, respectively, are exhaustive FN-topologies. Define positive measures $\mu, \mu_n : \mathbb{P}(\mathbb{N}) \to (\ell_\infty, \tau)$ by $\mu(A) = \chi_A$, $\mu_n = \chi_{A \cap \{1,\dots,n\}}$. Then $\mu_n \uparrow \mu$, $\mu_n \ll \rho_p$, $\mu_n \perp \rho_0$, but neither $\mu \ll \rho_p$ nor $\mu \perp \rho_0$ holds true since the μ-topology is the supremum of ρ_0 and ρ_p. Therefore neither $\{\lambda \in b(\mathbb{P}(\mathbb{N}), \ell_\infty) : \lambda \ll \rho_p\}$ nor $\{\lambda \in b(\mathbb{P}(\mathbb{N}), \ell_\infty) : \lambda \perp \rho_0\}$ are bands in $b(\mathbb{P}(\mathbb{N}), \ell_\infty)$. Moreover, $\mu \in \{\lambda \in b(\mathbb{P}(\mathbb{N}), \ell_\infty) : \lambda \ll \rho_0\}^\perp$.

Theorem 5.13. *Suppose that τ has the Lebesgue property and L is bounded. Let ρ be a locally convex lattice topology on L and $\mu \in fvm(L,G,0)$. Then there are unique $\mu_1, \mu_2 \in fvm(L,G,0)$ with the following properties:*

(i) $\mu = \mu_1 + \mu_2$,
(ii) $\mu_1 \ll \rho$,
(iii) *if $\lambda : L \to G$ is a modular function such that $\lambda \ll \rho$, $0 \preceq \lambda \preceq |\mu_2|$ and $\lambda(p) = 0$, then $\lambda = 0$.*

Proof. This immediately follows from Riesz decomposition theorem, Proposition 5.11 and the isomorphism between $fvd(L,G)$ and $fvm(L,G,0)$, see Theorem 2.3. ■

We now ask when the singularity condition (iii) of the preceding theorem based on the order in G can be replaced by a topological singularity condition generalizing the one used in Theorem 4.1.

If v, w are uniformities on a set X, we write $v \perp w$ if for every $V \in v$ and $W \in w$ we have $V \circ W = X^2$, i.e. for every $a, b \in X$ there exits $x \in X$ such that $(a,x) \in V$ and $(x,b) \in W$.[12] If L is a bounded lattice and v, w are lattice uniformities on L, then $v \perp w$ iff for every $V \in v$ and $W \in w$ there exits $x \in L$ such that $(0,x) \in V$ and $(x,1) \in W$.

[12] Such uniformities are called in [18] orthogonal, and in [27] and the papers cited in [27, Definition 4.1] independent. Conditions equivalent to $v \perp w$ are given in [27, Proposition 4.3].

If w is a lattice uniformity on L, $\mu \in m(L,G)$ and $\overline{\mu} = \Phi(\mu)$, we write (compatible with the notation of the last section) $\mu \perp w$ and $\overline{\mu} \perp w$ if $u(\mu) \perp w$.

Lemma 5.14. *Let v, w be uniformities on a set X such that $v \perp w$. Then the trivial uniformity is the only uniformity on X such that its induced topology is weaker than the v-topology (=topology induced by v) and the w-topology.*

Proof. Let u be a uniformity on X such that the u-topology is weaker than the v-topology and the w-topology, moreover $a, b \in X$ and $U \in u$. We show that $(a,b) \in U$. Choose symmetric sets $U_0 \in u$, $V \in v$, $W \in w$ with $U_0 \circ U_0 \subseteq U$, $V(a) \subseteq U_0(a)$ and $W(b) \subseteq U_0(b)$. Since $v \perp w$, there is an $x \in X$ such that $(a,x) \in V$ and $(x,b) \in W$. Therefore $x \in V(a) \subseteq U_0(a)$ and $x \in W(b) \subseteq U_0(b)$, i.e. $(a,x) \in U_0$ and $(b,x) \in U_0$, and finally $(a,b) \in U_0 \circ U_0 \subseteq U$. ■

Corollary 5.15. *Let L be a bounded lattice, w a lattice uniformity on L and τ a pre-Lebesgue topology. If $\alpha \in fvd(L,G)$ and $\alpha \perp w$, then $\alpha \in fvd(L,G,w)^{\perp}$.*

Proof. Let $\beta \in fvd(L,G,w)$ and $\gamma = |\alpha| \wedge |\beta|$. By Proposition 5.3 and Theorem 5.4 we have $\tau(|\alpha|) = \tau(\alpha)$ and $\tau(|\beta|) = \tau(\beta)$. Therefore $\tau(\gamma) \subseteq \tau(\alpha)$ and $\tau(\gamma) \subseteq \tau(\beta)$, thus $\tau(\gamma)$ is weaker than the w-topology. It follows with Lemma 5.14 that $u(\gamma)$ is the trivial uniformity, i.e. $\gamma = 0$. ■

Example 5.12 shows that in general $\alpha \in fvd(L,G,w)^{\perp}$ doesn't imply $\alpha \perp w$.

We now give an answer to the question when a " topological decomposition" for measures is also a band decomposition. If w is a lattice uniformity and $\mu \in m(L,G)$, we write $\mu \ll w$ if μ is continuous w.r.t. the w-uniformity.

Theorem 5.16. *Let τ be a pre-Lebesgue topology, L a bounded lattice and w a lattice uniformity on L. Suppose that for any $\mu \in fvm(L,G,0)$ there are $\mu_1, \mu_2 \in fvm(L,G,0)$ such that*

$$\mu = \mu_1 + \mu_2 \qquad \text{with } \mu_1 \ll w \text{ and } \mu_2 \perp w\,. \tag{20}$$

Then $fvm(L,G,0,w) = \{\mu \in fvm(L,G,0) : \mu \perp w\}^{\perp}$ and $\{\mu \in fvm(L,G,0) : \mu \perp w\} = fvm(L,G,0,w)^{\perp}$.

In particular, the decomposition (20) is a band decomposition w.r.t. the band $fvm(L,G,0,w)$.

Proof. Use the isomorphism between $fvm(L,G,0)$ and $fvd(L,G)$, and apply with $H = fvd(L,G)$, $A = fvd(L,G,w)$, $B = \{\alpha \in fvd(L,G) : \alpha \perp w\}$ the following fact: Let H be an ℓ-group, $H = A + B$ and $B \subseteq A^{\perp}$, then $A = B^{\perp}$ and $B = A^{\perp}$, in particular, A and B are bands in H.

(20) implies that $H = A + B$, and $B \subseteq A^{\perp}$ follows from Corollary 5.15. ■

Corollary 5.17. *Let τ be a Lebesgue topology, A a Boolean algebra and ρ an FN-topology on A. Then $\{\mu \in b(A,G) : \mu \ll \rho\}$ and $\{\mu \in b(A,G) : \mu \perp \rho\}$ are bands in $b(A,G)$ and $b(A,G)$ is the direct sum of these bands.*

Proof. We verify the assumption of Theorem 5.16. Let $\mu \in b(A,G)$, then μ is exhaustive by Proposition 5.3. Since by [24, Corollary 4.7] the order intervals $[a,b]$ of G are τ-complete, $\mu(A)$ is contained in a τ-complete subset of G. Therefore, by Theorem 4.1 and the succeeding remark, μ has a decomposition $\mu = \mu_1 + \mu_2$ where $\mu_i : A \to G$ are measures with $\mu_1 \ll \rho$, $\mu_2 \perp \rho$ and $\mu_i(A) \subseteq \overline{\mu(A)}$. In particular, the measures μ_i are bounded. Now 5.17 immediately follows from Theorem 5.16. ■

In recent years the authors of [2] and [1] have studied modular measures on D-lattices in several articles. One easily sees that a theorem analogous to 5.17 also holds true in this setting - replacing the Boolean algebra by a D-lattice, measures by modular measures and FN-topologies by D-uniformities. In the proof one has to replace Theorem 4.1 by the decomposition theorem [2, Theorem 3.5] for modular measures on D-lattices.

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