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SOME ELEMENTARY EXPLICIT BOUNDS FOR TWO MOLLIFICATIONS OF THE MOEBIUS FUNCTION

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Abstract: We prove that the sum $\sum_{\substack{d\leqslant x\\(d,r)=1}}\mu(d)/d^{1+\varepsilon}$ is bounded by $1+\varepsilon$, uniformly in $x\geqslant 1$, r and $\varepsilon>0$. We prove a similar estimate for the quantity $\sum_{\substack{d\leqslant x\\(d,r)=1}}\mu(d)\log(x/d)/d^{1+\varepsilon}$. When $\varepsilon=0$, r varies between 1 and a hundred, and x is below a million, this sum is non-negative and this raises the question as to whether it is non-negative *for every* x.

Keywords: explicit estimates, Möbius function.

1. Introduction and results

Our first result is the following:

Theorem 1.1. *When $r\geqslant 1$ and $\varepsilon\geqslant 0$, we have*

$$\Bigg|\sum_{\substack{d\leqslant x\\(d,r)=1}}\frac{\mu(d)}{d^{1+\varepsilon}}\Bigg|\leqslant 1+\varepsilon.$$

This Lemma generalizes the estimate of [5, Lemme 10.2] which corresponds to the case $\varepsilon=0$. This generalization is *not* straightforward at all and requires a change of proof. The case $\varepsilon=0$ and $r=1$ is classical. The parameter ε that is being introduced induces some flexibility useful when applying Rankin's method (devised in [8]). As it turns out, we can do somewhat better concerning the lower bound, and we prove that

$$-\tfrac{11}{15}(1+\varepsilon)\leqslant\sum_{\substack{d\leqslant x\\(d,r)=1}}\frac{\mu(d)}{d^{1+\varepsilon}}.$$

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We ran computations covering the range $1 \leqslant x \leqslant 10^6$ and $1 \leqslant r \leqslant 100$ with $\varepsilon = 0$; we found that the lowest lower bound was met at $x = 13$ and $r = 1$. This raises the following question:

Question 1. It is true that

$$\sum_{\substack{d \leqslant x \\ (d,r)=1}} \frac{\mu(d)}{d} \geqslant -2323/30030 \quad ?$$

See section 2 for a very preliminary result in this direction.

We proceed by proving the following more involved form:

Theorem 1.2. *When $r \geqslant 1$ and $1.38 \geqslant \varepsilon \geqslant 0$, we have*

$$\left| \sum_{\substack{d \leqslant x \\ (d,r)=1}} \frac{\mu(d)}{d^{1+\varepsilon}} \log \frac{x}{d} \right| \leqslant 1.4 + 4.7\varepsilon + 3.3\varepsilon^2 + (1+\varepsilon) \frac{r^{1+\varepsilon}}{\phi_{1+\varepsilon}(r)} x^{\varepsilon}$$

where

$$\frac{r^{1+\varepsilon}}{\phi_{1+\varepsilon}(r)} = \prod_{p|r} \frac{p^{1+\varepsilon}}{p^{1+\varepsilon} - 1}. \tag{1}$$

The dependence in r is optimal as seen by taking for r the product of every primes not more than $\sqrt{x}$. The proof is again unbalanced with respect to the upper and the lower bound, and we prove a somewhat better lower bound:

$$-(1.434 + 4.992\varepsilon + 3.558\varepsilon^2) \leqslant \sum_{\substack{d \leqslant x \\ (d,r)=1}} \frac{\mu(d)}{d^{1+\varepsilon}} \log \frac{x}{d}.$$

I expect the factor x^{ε} in the upper bound to be a blemish; however, the (limited) numerical verifications we ran suggest that the factor $r^{1+\varepsilon}/\phi_{1+\varepsilon}(r)$ *cannot* be omitted even if the condition $r \leqslant x$ is added (this condition often appears in practice). It should be added that it is not difficult to prove that

$$\sum_{d \leqslant x} \frac{\mu(d)}{d} \log \frac{x}{d} \sim 1 \qquad (x \to \infty)$$

which means that one cannot expect an arbitary small constant in the right hand side of the inequality given in Theorem 1.2. We have checked that

$$0 \leqslant \sum_{\substack{d \leqslant x \\ (d,r)=1}} \frac{\mu(d)}{d} \log \frac{x}{d} \leqslant \frac{r}{\phi(r)} + 0.007 \qquad (x \leqslant 10^6,\ 1 \leqslant r \leqslant 100)$$

(where x is a real number and not especially an integer) and all these maxima were in fact very close to $r/\phi(r)$. These computations raise two questions:

Question 2. Is it true that

$$\sum_{\substack{d\leqslant x\\(d,r)=1}}\frac{\mu(d)}{d}\log\frac{x}{d}\geqslant 0,\qquad (x\geqslant 1,\ r\geqslant 1)\quad ?$$

Question 3. Is it true that

$$\sum_{\substack{d\leqslant x\\(d,r)=1}}\frac{\mu(d)}{d}\log\frac{x}{d}\leqslant\frac{r}{\phi(r)}+1,\qquad (x\geqslant 1,\ r\geqslant 1)\quad ?$$

In both these questions, x is only assumed to be a positive real number. On recalling what happens in the case of Turán's conjecture on the summatory function of the Liouville function divided by its argument, see [2], we believe that the answer to the first question is no. The sum is however less likely to be very erratical because of the smoothing factor, a factor that is absent in Turán's problem. In direction of these conjecture, we note the following formula

$$\int_1^\infty\sum_{\substack{d\leqslant x\\(d,r)=1}}\frac{\mu(d)}{d}\log\frac{x}{d}\frac{dx}{x^{s+1}}=\frac{r^{1+s}}{\phi_{1+s}(r)}\frac{1}{s^2\zeta(1+s)}$$

from which we easily deduce (on taking $s=\varepsilon>0$ and letting ε go to infinity) that

$$\limsup_x\sum_{\substack{d\leqslant x\\(d,r)=1}}\frac{\mu(d)}{d}\log\frac{x}{d}\geqslant\frac{r}{\phi(r)}.$$

We discuss some related points in the last section.

Notation

We use here the notation $h=\mathcal{O}^*(k)$ to mean that $|h|\leqslant k$. We denote by $\tau(m)$ the number of (positive) divisors of m, and by (a,b) the gcd of a and b. For $\varepsilon\geqslant 0$ and $r\geqslant 1$ any natural squarefree number, we define two functions. The first one is alternatively defined by

$$f_{r,\varepsilon}(n)=\sum_{\substack{\ell|n\\(\ell,r)=1}}\frac{\mu(\ell)}{\ell^\varepsilon}\tau(n/\ell)\tag{2}$$

or, in multiplicative form, by:

$$f_{r,\varepsilon}(n)=\prod_{\substack{p^\nu\|n\\p\nmid r}}\Bigl(\nu+1-\frac{\nu}{p^\varepsilon}\Bigr)\prod_{\substack{p^\nu\|n\\p|r}}(\nu+1).\tag{3}$$

We easily determine its Dirichlet series: $\sum_{n\geqslant 1} f_{r,\varepsilon}(n)/n^s = \zeta(s)^2/\zeta(s+\varepsilon)$. We shall further write

$$f_{r,\varepsilon}(n) = \mathbb{1} \star g_{r,\varepsilon}(n) \tag{4}$$

where the function $g_{r,\varepsilon}$ has the essential property of being non-negative and is being defined by:

$$g_{r,\varepsilon}(n) = \sum_{\substack{\ell|n \\ (\ell,r)=1}} \frac{\mu(\ell)}{\ell^\varepsilon} \geqslant 0. \tag{5}$$

Thanks. Sincere thanks are due to the careful referee who has checked our computations and indeed has rooted out several mistakes.

2. Verifying Theorem 1.1 for small values

We study what happens for small values here. The proof is pedestrian and painful, but I have not seen any way to avoid it, or to present it in a more general frame.

We study the following quantity:

$$m_0(r,x) = \sum_{\substack{d\leqslant x \\ (d,r)=1}} \frac{\mu(d)}{d^{1+\varepsilon}}. \tag{6}$$

Lemma 2.1. *When $x < 10$ and $\varepsilon \geqslant 0$, we have $-1/30 \leqslant m_0(r,x) \leqslant 1$.*

Proof. The sum we consider reads

$$1 - \frac{h(2)}{2^{1+\varepsilon}} - \frac{h(3)}{3^{1+\varepsilon}} - \frac{h(5)}{5^{1+\varepsilon}} + \frac{h(6)}{6^{1+\varepsilon}} - \frac{h(7)}{7^{1+\varepsilon}}$$

where h is the characteristic function of the integers $\leqslant x$ that are coprime with r. The minimum is clearly

$$1 - \frac{1}{2^{1+\varepsilon}} - \frac{1}{3^{1+\varepsilon}} - \frac{1}{5^{1+\varepsilon}}$$

which is minimal when $\varepsilon = 0$. This is the $-1/30$. The maximum contains the summand 1. If the summand $1/6^{1+\varepsilon}$ is present, then so is the summand $-1/2^{1+\varepsilon}$. This concludes the proof. ■

3. Auxiliaries

Lemma 3.1. *When $\varepsilon \geqslant 0$, we have*

$$\sum_{h\leqslant H} h^\varepsilon = \frac{H^{1+\varepsilon}}{1+\varepsilon} + \mathcal{O}^*\big(H^\varepsilon\big).$$

This is also $\leqslant H^{1+\varepsilon}$. When H is an integer, we have $\sum_{h\leqslant H} h^\varepsilon \geqslant \frac{H^{1+\varepsilon}}{1+\varepsilon}$.

Proof. Indeed, when $\varepsilon > 0$, a summation by parts gives us directly

$$\begin{aligned}\sum_{h\leqslant H} h^\varepsilon &= \sum_{h\leqslant H} \varepsilon \int_0^h dt/t^{1-\varepsilon} = \varepsilon \int_0^H \sum_{t<h\leqslant H} 1\, dt/t^{1-\varepsilon} \\ &= \varepsilon \int_0^H (H-t)\, dt/t^{1-\varepsilon} + \mathcal{O}^*(H^\varepsilon).\end{aligned}$$

We proceed by continuity to cover the case $\varepsilon = 0$. When H is an integer, a comparison to an integral gives the result. ■

Lemma 3.2. *For $L > 1$, we have*

$$\sum_{n\leqslant L} f_{r,\varepsilon}(n) \leqslant L \sum_{\ell\leqslant L} g_{r,\varepsilon}(\ell)/\ell. \tag{7}$$

Proof. We recall (4) and write, since $g_{r,\varepsilon} \geqslant 0$

$$\sum_{n\leqslant L} f_{r,\varepsilon}(n) = \sum_{km\leqslant L} g_{r,\varepsilon}(m) \leqslant L \sum_{m\leqslant L} g_{r,\varepsilon}(m)/m.$$

The Lemma follows readily. ■

Lemma 3.3. *For every integer n and any $\varepsilon \geqslant 0$, we have*

$$g_{1,\varepsilon}(\ell) \leqslant \sum_{mn=\ell} g_{1,\varepsilon/2}(m) g_{1,\varepsilon/2}(n).$$

Proof. We check that, when $\alpha \geqslant 1$ is an integer and p a prime number,

$$\begin{aligned} g_{1,\varepsilon}\big(p^\alpha\big) &= 1 - \frac{1}{p^\varepsilon} = 1 - \frac{1}{p^{\varepsilon/2}} + \frac{1}{p^{\varepsilon/2}}\Big(1 - \frac{1}{p^{\varepsilon/2}}\Big) \\ &\leqslant g_{1,\varepsilon/2}\big(p^\alpha\big) g_{1,\varepsilon/2}(1) + g_{1,\varepsilon/2}(1) g_{1,\varepsilon/2}\big(p^\alpha\big) \\ &\leqslant \sum_{0\leqslant\beta\leqslant\alpha} g_{1,\varepsilon/2}\big(p^{\alpha-\beta}\big) g_{1,\varepsilon/2}\big(p^\beta\big). \end{aligned}$$

We conclude by invoking the multiplicativity of $g_{1,\varepsilon/2}$. ■

Lemma 3.4. *We have when $L \geqslant 7.2$,*

$$\sum_{p\leqslant L} \frac{\log p}{p-1} \leqslant \log L.$$

Proof. We cite [9, (2.8)]:

$$\sum_{p\leqslant L} \frac{\log p}{p} \leqslant \log L - \gamma - \sum_{p\geqslant 2} \frac{\log p}{p(p-1)} + \frac{1}{2\log L}, \qquad (L \geqslant 319)$$

from which we deduce, for $L \geqslant 319$,

$$\sum_{p \leqslant L} \frac{\log p}{p-1} \leqslant \log L - \gamma + \frac{1}{2\log L}.$$

A simple GP script shows that

$$\sum_{p \leqslant L} \frac{\log p}{p-1} \leqslant \log L$$

when $1000 \geqslant L \geqslant 7.2$, and the reader will conclude readily. ■

Lemma 3.5. *We have, when $L \geqslant 1$ and $\varepsilon \geqslant 0$,*

$$\sum_{\ell \leqslant L} g_{1,\varepsilon}(\ell)/\ell \leqslant L^{\varepsilon}. \tag{8}$$

Proof. Verifying the stated inequality for $1 \leqslant L < 8$ is (tedious but) easy, hence we can now assume that $L \geqslant 8$. We readily find that the sum in question is not more than

$$T = \prod_{p \leqslant L} \frac{1 - p^{-1-\varepsilon}}{1 - p^{-1}} = \exp \sum_{p \leqslant L} \log\left(1 + \frac{1 - p^{-\varepsilon}}{p-1}\right).$$

We apply $\log(1+x) \leqslant x$ for non-negative x and $1 - p^{-\varepsilon} \leqslant \varepsilon \log p$ to get, when $L \geqslant 8$,

$$T \leqslant \exp \varepsilon \sum_{p \leqslant L} \frac{\log p}{p-1} \leqslant L^{\varepsilon}$$

by invoking Lemma 3.4. ■

Lemma 3.6. *We have, when $L \geqslant 1$, $r \geqslant 1$ and $\varepsilon \geqslant 0$,*

$$\sum_{\ell \leqslant L} g_{r,\varepsilon}(\ell)/\ell \leqslant \frac{r^{1+\varepsilon}}{\phi_{1+\varepsilon}(r)} L^{\varepsilon}. \tag{9}$$

Proof. We use the notation $d|r^{\infty}$ to say that each prime factor of d divides r. We write

$$\begin{aligned}
\sum_{\ell \leqslant L} \frac{g_{r,\varepsilon}(\ell)}{\ell} &= \sum_{\substack{d|r^{\infty} \\ d \leqslant L}} \sum_{\substack{\ell \leqslant L/d \\ (\ell, r)=1}} \frac{g_{r,\varepsilon}(\ell)}{\ell d} \\
&\leqslant L^{\varepsilon} \sum_{d|r^{\infty}} \frac{1}{d^{1+\varepsilon}} = L^{\varepsilon} \frac{r^{1+\varepsilon}}{\phi_{1+\varepsilon}(r)}
\end{aligned}$$

by Lemma 3.5. The Lemma follows readily. ■

Lemma 3.7.

$$\sum_{m\leqslant M} m^\varepsilon \tau(m) = \frac{M^{1+\varepsilon}}{1+\varepsilon}\left(\log M + 2\gamma - \frac{1}{1+\varepsilon}\right) + \mathcal{O}^*\left(0.961(1+2\varepsilon)M^{\frac12+\varepsilon}\right)$$

Proof. We recall part of [1, Theorem 1.1]:

$$\sum_{m\leqslant t} \tau(m) = t\log t + (2\gamma-1)t + \mathcal{O}^*(0.961\sqrt{t}), \quad (t\geqslant 1).$$

Since $(t\log t + (2\gamma-1)t)/\sqrt{t}$ is seen to vary between -0.681 and 0.155 when t varies between 0 and 1, this estimate is also valid for $t>0$. We use summation by parts and find that

$$\begin{aligned}
\sum_{m\leqslant M} m^\varepsilon\tau(m) &= M^\varepsilon \sum_{m\leqslant M}\tau(m) - \varepsilon\int_0^M \sum_{m\leqslant t}\tau(m) dt/t^{1-\varepsilon}\\
&= M^{1+\varepsilon}(\log M + 2\gamma - 1) + \mathcal{O}^*\left(0.961 M^{\frac12+\varepsilon}\right)\\
&\quad - \varepsilon\int_0^M (\log t + 2\gamma-1)t^\varepsilon dt + \mathcal{O}^*\left(0.961\varepsilon\int_0^M t^{\varepsilon-1/2}dt\right)\\
&= \frac{M^{1+\varepsilon}}{1+\varepsilon}\left(\log M + 2\gamma - \frac{1}{1+\varepsilon}\right) + \mathcal{O}^*\left(0.961(1+2\varepsilon)M^{\frac12+\varepsilon}\right).
\end{aligned}$$

■

Lemma 3.8. *We have, when* $n\geqslant 2$,

$$g_{r,\varepsilon}(n) \leqslant 1 - \frac{\mathbb{1}_{(n,r)=1}\mu^2(n)}{n^\varepsilon}.$$

Proof. Indeed, we verify that $(1-a)(1-b)\leqslant(1-ab)$ when $0\leqslant a,b\leqslant 1$. The Lemma readily follows by recursion on the number of prime factors of n. ■

4. Some lemmas on squarefree numbers

Here is a Lemma from [4]:

Lemma 4.1. *We have, for* $D\geqslant 1\,664$

$$\sum_{d\leqslant D}\mu^2(d) = \frac{6D}{\pi^2} + \mathcal{O}^*\left(0.1333\sqrt{D}\right).$$

In particular, this is not more than $0.62D$ *when* $D\geqslant 1700$.

Lemma 4.2. *We have*

$$\sum_{d\leqslant x}\mu^2(d)/\sqrt{d} \leqslant 1.33\sqrt{x}, \qquad (x\geqslant 1).$$

If we are ready to assume larger, we would not save much since the best constant one can get is $12/\pi^2 = 1.215 + \mathcal{O}^*(0.001)$.

Proof. We use PARI/GP see [7] and the following script:

```
{check(borne) =
   my(res = 0.0, coef = 0);
   for(d = 1, borne,
       res += moebius(d)^2/sqrt(d);
       coef = max(coef, res/sqrt(d)));
   return(coef)}
```

It is then almost immediate to check our result when $x \leqslant 10^7$, despite the lack of refinement of the script proposed. For larger values, we use a summation by parts together with Lemma 4.1. ■

Lemma 4.3. *We have*

$$\sum_{d\leqslant x} \mu^2(d) \leqslant \tfrac{11}{15}\, x, \qquad (x \geqslant 9).$$

We note that $11/15 = 0.7333\ldots$ while the asymptotically best constant is rather lower, namely $6/\pi^2 = 0.607927\ldots$. Reaching $73/115 = 0.63478\ldots$ already requires to take $x \geqslant 75$, and this means we would have to handle the possible divisibility by 21 primes in section 2. This is out of reach of the simple-minded method we have at our disposal.

Proof. We use PARI/GP see [7] and the following script:

```
{check(borneinf, bornesup) =
   my(res = 0.0, coef = 0);
   res = sum(d = 1, borneinf-1, moebius(d)^2);
   for(d = borneinf, bornesup,
       res += moebius(d)^2;
       coef = max(coef, res/d));
   return(coef)}
```

It is then almost immediate to check our result when $x \leqslant 10^7$, despite the lack of refinement of the script proposed. For larger values, the result is an immediate consequence of Lemma 4.1. ■

5. Proof of Theorem 1.1

Lemma 2.1 establishes Theorem 1.1 when $x < 10$, so we may assume $x \geqslant 10$. We further may restrict our attention to integer values of x. We start with

$$S_0 = \sum_{n\leqslant x} n^\varepsilon g_{r,\varepsilon}(n) = \sum_{n\leqslant x} \sum_{\substack{d|n\\(d,r)=1}} \mu(d)(n/d)^\varepsilon.$$

Using the first expression yields $0 \leqslant S_0$ as well as

$$S_0/x^\varepsilon \leqslant 1 + \sum_{2\leqslant n\leqslant x}\Bigl(g_{r,\varepsilon}(n) + \frac{\mathbb{1}_{(n,r)=1}\mu^2(n)}{n^\varepsilon}\Bigr) - \sum_{\substack{2\leqslant n\leqslant x\\(n,r)=1}}\frac{\mu^2(n)}{n^\varepsilon}$$

Each summand in the second sum is bounded above by 1 by Lemma 3.8. We get

$$0 \leqslant S_0/x^\varepsilon \leqslant x - \sum_{\substack{2\leqslant n\leqslant x\\(n,r)=1}}\frac{\mu^2(n)}{n^\varepsilon}.$$

Let us write the second expression for S_0:

$$S_0 = \sum_{\substack{d\leqslant x\\(d,r)=1}}\mu(d)\sum_{m\leqslant x/d} m^\varepsilon.$$

We employ Lemma 3.1; we treat the case $d=1$ separately for the lower bound to reach

$$\begin{aligned}\frac{x^{1+\varepsilon}}{1+\varepsilon}\sum_{\substack{d\leqslant x\\(d,r)=1}}\frac{\mu(d)}{d^{1+\varepsilon}} - x^\varepsilon\sum_{\substack{2\leqslant d\leqslant x\\(d,r)=1}}\mu^2(d)d^{^{-\varepsilon}} &\leqslant S_0\\ &\leqslant \frac{x^{1+\varepsilon}}{1+\varepsilon}\sum_{\substack{d\leqslant x\\(d,r)=1}}\frac{\mu(d)}{d^{1+\varepsilon}} + x^\varepsilon\sum_{\substack{d\leqslant x\\(d,r)=1}}\mu^2(d)d^{^{-\varepsilon}}.\end{aligned}$$

The lower bound requires x to be an integer, but not the upper bound. We rewite the above as

$$S_0 - x^\varepsilon\sum_{\substack{d\leqslant x\\(d,r)=1}}\mu^2(d)d^{^{-\varepsilon}} \leqslant \frac{x^{1+\varepsilon}}{1+\varepsilon}\sum_{\substack{d\leqslant x\\(d,r)=1}}\frac{\mu(d)}{d^{1+\varepsilon}} \leqslant S_0 + x^\varepsilon\sum_{\substack{2\leqslant d\leqslant x\\(d,r)=1}}\mu^2(d)d^{^{-\varepsilon}}.$$

By conjugating both estimates, we get,

$$-x^\varepsilon\sum_{\substack{d\leqslant x\\(d,r)=1}}\mu^2(d)d^{^{-\varepsilon}} \leqslant \frac{x^{1+\varepsilon}}{1+\varepsilon}\sum_{\substack{d\leqslant x\\(d,r)=1}}\frac{\mu(d)}{d^{1+\varepsilon}} \leqslant x^{1+\varepsilon}.$$

The right hand side is easily handled. We use Lemma 4.3 for the left hand side via, when $x \geqslant 9$:

$$\sum_{\substack{d\leqslant x\\(d,r)=1}}\mu^2(d)d^{^{-\varepsilon}} \leqslant \sum_{d\leqslant x}\mu^2(d) \leqslant \tfrac{11}{15}\,x.$$

By conjugating both estimates, we get

$$-\tfrac{11}{15}(1+\varepsilon) \leqslant \sum_{\substack{d\leqslant x\\(d,r)=1}}\frac{\mu(d)}{d^{1+\varepsilon}} \leqslant 1+\varepsilon, \qquad (x\geqslant 9). \tag{10}$$

Theorem 1.1 is proved.

6. Proof of Theorem 1.2

The proof relies on two ways of writing the sum

$$S_1 = \sum_{n\leqslant x} n^\varepsilon f_{r,\varepsilon}(n) = \sum_{n\leqslant x} \sum_{\substack{d|n\\(d,r)=1}} \mu(d)(n/d)^\varepsilon \tau(n/d).$$

The first form shows that $0 \leqslant S_1 \leqslant x^{1+2\varepsilon} r^{1+\varepsilon}/\phi_{1+\varepsilon}(r)$ by combining Lemma 3.2 together with Lemma 3.6. Let us write this sum differently:

$$S_1 = \sum_{\substack{d\leqslant x\\(d,r)=1}} \mu(d) \sum_{m\leqslant x/d} m^\varepsilon \tau(m)$$

and we use Lemma 3.7 to reach

$$S_1 = \frac{x^{1+\varepsilon}}{1+\varepsilon} \sum_{\substack{d\leqslant x\\(d,r)=1}} \frac{\mu(d)}{d^{1+\varepsilon}}\Bigl(\log\frac{x}{d} + 2\gamma - \frac{1}{1+\varepsilon}\Bigr) + \mathcal{O}^*\Bigl(0.961\times 1.33\,(1+2\varepsilon)x^{1+\varepsilon}\Bigr)$$

since $\sum_{d\leqslant x}\mu^2(d)/\sqrt{d} \leqslant 1.33\sqrt{x}$ by Lemma 4.2. We set

$$\alpha = 2\gamma - \frac{1}{1+\varepsilon} \in [0,1]. \tag{11}$$

All of that amounts to:

$$\begin{aligned} S_1 &= \frac{x^{1+\varepsilon}}{1+\varepsilon} \sum_{\substack{d\leqslant x\\(d,r)=1}} \frac{\mu(d)}{d^{1+\varepsilon}}\Bigl(\log\frac{x}{d} + \alpha\Bigr) + \mathcal{O}^*\bigl(1.279(1+2\varepsilon)x^{1+\varepsilon}\bigr)\\ &= S_1^* + \alpha S_0 + \mathcal{O}^*\bigl(1.279(1+2\varepsilon)x^{1+\varepsilon}\bigr)\end{aligned}$$

say. We thus have

$$-1.279(1+2\varepsilon)x^{1+\varepsilon} \leqslant S_1^* + \alpha S_0 \leqslant 1.279(1+2\varepsilon)x^{1+\varepsilon} + x^{1+2\varepsilon}\frac{r^{1+\varepsilon}}{\phi_{1+\varepsilon}(r)}.$$

We use (10) and Lemma 2.1, and reach

$$-1.279(1+2\varepsilon) - \alpha \leqslant x^{-1-\varepsilon}S_1^* \leqslant 1.279(1+2\varepsilon) + \tfrac{11}{15}\alpha + x^\varepsilon\frac{r^{1+\varepsilon}}{\phi_{1+\varepsilon}(r)}.$$

We use $\alpha \leqslant 2\gamma - 1 + \varepsilon$. This gives

$$\begin{aligned} -1.434 - 4.992\varepsilon - 3.558\varepsilon^2 \leqslant \sum_{\substack{d\leqslant x\\(d,r)=1}} \frac{\mu(d)}{d^{1+\varepsilon}}\log\frac{x}{d}&\\ \leqslant 1.393 + 4.684\varepsilon + 3.292\varepsilon^2 + (1+\varepsilon)\frac{x^\varepsilon r^{1+\varepsilon}}{\phi_{1+\varepsilon}(r)}&.\end{aligned}$$

Since $x^\varepsilon r^{1+\varepsilon}/\phi_{1+\varepsilon}(r) \geqslant 1$, we check that the right hand side is larger than minus times the left hand side. Theorem 1.2 follows.

7. A generalization and a remark

It is not difficult to get along these lines the following Lemma:

Lemma 7.1. *When $r \geqslant 1$ and $k \geqslant 1$, we have*

$$\sum_{\substack{d \leqslant x \\ (d,r)=1}} \frac{\mu(d)}{d^{1+\varepsilon}} \log^k \frac{x}{d} \ll_k \left(\frac{r}{\phi(r)}\right)^k (\log x)^{k-1}.$$

Such quantities appear for instance in [10] where cases $k = 0$ and $k = 1$ are used, while case $k = 2$ is evaluated (there is a main term), but all with no coprimality conditions (i.e. $r = 1$) and no ε. The reader will find in [3, Chapter 1] the evaluation of case $k = 3$, $r = 1$ and $\varepsilon = 0$. [6] also pertains to these quantities.

Proof. Indeed, we first prove that

$$\sum_{n \leqslant x} \sum_{\substack{d \mid n \\ (d,r)=1}} \mu(d)(n/d)^{\varepsilon} \tau_{k+1}(n/d) \ll \left(\frac{r}{\phi(r)}\right)^k x(\log x)^{k-1}.$$

We then continue as in section 6. ∎

Here is a surprising elementary consequence.

Lemma 7.2. *For any $c > 0$, we have*

$$\sum_{\substack{d \leqslant x \\ (d,r)=1}} \frac{\mu(d)}{d} - x^{\varepsilon} \sum_{\substack{d \leqslant x \\ (d,r)=1}} \frac{\mu(d)}{d^{1+\varepsilon}} \ll_c \varepsilon \frac{r}{\phi(r)}$$

provided that $0 \leqslant \varepsilon \leqslant c(\log x)^{-1}$.

Proof. It is enough to consider

$$\int_0^{\varepsilon} \sum_{\substack{d \leqslant x \\ (d,r)=1}} \frac{\mu(d)x^{\eta}}{d^{1+\eta}} \log(x/d) d\eta \ll \varepsilon \frac{r}{\phi(r)}.$$

∎

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