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DIFFERENCE EQUATIONS IN WEIGHTED SPACES OF SEQUENCES

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Abstract: Let $\varphi = \{\varphi_m\}_{m=1}^{\infty}$ be a family of convex functions φ_m on $\mathbb{R}^n$ with certain growth conditions. With a help of restrictions of functions φ_m on $\mathbb{Z}^n$ a weighted space of functions on $\mathbb{Z}^n$ denoted as A_φ is defined. Linear continuous functionals on this space in terms of their Fourier-Laplace transform are described. This description and functional analysis methods allowed to study surjectivity of difference operators on A_φ and spectral synthesis problem in the kernel of such operators for a special case of a family φ.

Keywords: sequence spaces, linear difference equation, entire functions, duality.

1. Introduction

Let $\varphi = \{\varphi_m\}_{m=1}^{\infty}$ be a family of convex functions $\varphi_m : \mathbb{R}^n \to \mathbb{R}$ such that:

1) $\lim_{x\to\infty} \frac{\varphi_m(x)}{\|x\|} = +\infty$ for each $m \in \mathbb{N}$ ($\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^n$);
2) $\exists A > 0\ \forall m \in \mathbb{N}\ \exists B_m \geqslant 0$:

$$\varphi_m(x) - \varphi_{m+1}(x) \geqslant A \ln(1 + ||x||) - B_m, \qquad x \in \mathbb{R}^n.$$

For each $m \in \mathbb{N}$ let

$$A(\varphi_m) = \left\{ f : \mathbb{Z}^n \to \mathbb{C} \text{ such that } p_m(f) = \sup_{\alpha\in\mathbb{Z}^n} \frac{|f(\alpha)|}{e^{\varphi_m(\alpha)}} < \infty \right\}.$$

Obviously, for each $m \in \mathbb{N}$ $A(\varphi_{m+1}) \subset A(\varphi_m)$. Let $A_\varphi = \bigcap_{m=1}^{\infty} A(\varphi_m)$. Thus, for $n = 1$ elements of A_φ are two-sided sequences, for $n > 1$ elements of A_φ are multiple sequences. For brevity elements of A_φ will be simply called sequences. Sometimes we denote a sequence f as $(f(\alpha))_{\alpha\in\mathbb{Z}^n}$.

Under usual operations of addition and multiplication by complex numbers A_φ is a linear space. Endow A_φ with the topology of projective limit of the spaces $A(\varphi_m)$. Obviously, A_φ is a separable Fréchet space.

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In this article a description of the strong dual space of A_φ in terms of Fourier-Laplace transform of linear continuous functionals on A_φ as some space of periodic entire functions in $\mathbb{C}^n$ is obtained. Such a description allowed to study surjectivity of difference operators on A_φ and spectral synthesis problem in the kernel of such operators for a special case of a family φ.

Note that Fourier-Laplace transform of linear continuous functionals on sequence spaces was succesfully applied by many authors to study various analysis problems in these spaces. For example, L.A. Rubel and B.A. Taylor [6] considered spaces of all two-sided and one-sided sequences of complex numbers of at most exponential growth and proved some "polynomial" approximation theorems in these spaces by dualizing a gap theorem of C. Rényi [4] for periodic entire functions. This approach was also applied by A.A. Borichev [1] to describe the solutions of convolution equations in certain spaces of two-sided and one-sided sequences of exponential growth.

We shall use the following notations. For $u = (u_1, \dots, u_n) \in \mathbb{R}^n$ ($\mathbb{C}^n$), $v = (v_1, \dots, v_n) \in \mathbb{R}^n$ ($\mathbb{C}^n$) $\langle u, v\rangle = u_1v_1 + \cdots + u_nv_n$ and $\|u\|$ denotes the Euclidean norm in $\mathbb{R}^n(\mathbb{C}^n)$.

For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$, $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ $|\alpha| = \alpha_1 + \ldots + \alpha_n$, $\alpha' = (\alpha_2, \dots, \alpha_n)$, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$, $D^\alpha = \frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}}$.

For multi-indices $\alpha = (\alpha_1, \dots, \alpha_n), \beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}_+^n$ the notation $\beta \leqslant \alpha$ indicates that $\beta_j \leqslant \alpha_j$ $(j = 1, 2, \dots, n)$.

For multi-indices $\alpha = (\alpha_1, \dots, \alpha_n), \beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}_+^n$ such that $\beta \leqslant \alpha$ let $C_\alpha^\beta = \prod_{j=1}^n C_{\alpha_j}^{\beta_j}$ where $C_{\alpha_j}^{\beta_j}$ are the combinatorial numbers.

For $r > 0$ and $z \in \mathbb{C}^n$ let $B(z, r) = \{\zeta \in \mathbb{C}^n : \|\zeta - z\| < r\}$.

For a locally convex space X let X' be the space of linear continuous functionals on X and let X^* be the strong dual space.

For a function $\Phi \in C(\mathbb{R}^n)$ such that $\lim_{x\to\infty} \frac{\Phi(x)}{\|x\|} = +\infty$ let

$$\Phi^\star(x) := \sup_{\alpha\in\mathbb{Z}^n} (\langle x, \alpha\rangle - \Phi(\alpha)), \qquad x \in \mathbb{R}^n;$$

$$\Phi^*(x) := \sup_{y\in\mathbb{R}^n} (\langle x, y\rangle - \Phi(y)), \qquad x \in \mathbb{R}^n.$$

Recall that Φ^* is called the Young conjugate of the function Φ. It is well known that if Φ is convex on $\mathbb{R}^n$ then $(\Phi^*)^* = \Phi$.

Obviously, for each $z \in \mathbb{C}^n$ the sequence $f_z : \alpha \in \mathbb{Z}^n \to e^{-i\langle z,\alpha\rangle}$ belongs to A_φ since for each $m \in \mathbb{N}$

$$p_m(f_z) = \sup_{\alpha\in\mathbb{Z}^n} \frac{|e^{-i\langle z,\alpha\rangle}|}{e^{\varphi_m(\alpha)}} = \exp\left(\sup_{\alpha\in\mathbb{Z}^n} (\langle Imz, \alpha\rangle - \varphi_m(\alpha))\right) = e^{\varphi_m^\star(Imz)} < \infty. \tag{1}$$

Thus, for each linear continuous functional S on A_φ the function $\hat{S}(z) = S(f_z)$ is correctly defined on $\mathbb{C}^n$. It is called the Fourier-Laplace transform of S. The mapping $\mathcal{F} : S \in A_\varphi^* \to \hat{S}$ is called the Fourier-Laplace transformation.

For each $m \in \mathbb{N}$ let

$$P(\varphi_m^\star) = \Big\{ F \in H(\mathbb{C}^n) : F(z + 2\pi l) = F(z) \text{ for all } z \in \mathbb{C}^n, \; l \in \mathbb{Z}^n \\ \text{and such that } \|F\|_m = \sup_{z \in \mathbb{C}^n} \frac{|F(z)|}{e^{\varphi_m^\star(Imz)}} < \infty \Big\}.$$

Let $\varphi^\star = \{\varphi_m^\star\}_{m=1}^\infty$ and $P_{\varphi^\star} = \bigcup_{m=1}^\infty P(\varphi_m^\star)$. Under usual operations of addition and multiplication by complex numbers $P_{\varphi^\star}$ is a linear space. Endow $P_{\varphi^\star}$ with the topology of inductive limit of the spaces $P(\varphi_m^\star)$.

The main results of the paper are the following.

Theorem 1.1. *The mapping $\mathcal{F} : S \in A_\varphi^* \to \hat{S}$ establishes an isomorphism between the spaces A_φ^* and $P_{\varphi^\star}$.*

Theorem 1.1 is proved in the second section. In the third section we apply Theorem 1.1 to study difference operators in A_φ. For $f \in A_\varphi$ and $h \in \mathbb{Z}^n$ define a sequence f_h by the rule: $f_h(\alpha) = f(\alpha + h), \alpha \in \mathbb{Z}^n$. Let H be a finite subset of $\mathbb{Z}^n$ and for $h \in H$ let γ_h be a complex number.

Theorem 1.2. *Let φ satisfies the following additional conditions:*

i$_1$) *for each $m \in \mathbb{N}$ there exist numbers $a_m > 0$, $b_m > 0$ and $\mu_m > 1$ such that*

$$\varphi_m(x) \geqslant a_m \|x\|^{\mu_m} - b_m, \qquad x \in \mathbb{R}^n;$$

i$_2$) *for each $m \in \mathbb{N}$ there exists $d_m > 0$ such that for all $x \in \mathbb{R}^n$ and $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ with $|\xi_j| \leqslant 1$ ($j = 1, \ldots, n$)*

$$\varphi_{m+1}(x + \xi) \leqslant \varphi_m(x) + d_m.$$

Then the equation $\sum_{h \in H} \gamma_h f_h = g$ is solvable in A_φ for each $g \in A_\varphi$.

For $\mu \in \mathbb{Z}_+^n$ and $\zeta \in \mathbb{C}^n$ define a sequence $E_{\mu,\zeta}$ by the rule: $\alpha \in \mathbb{Z}^n \to \alpha^\mu e^{-i\langle \alpha, \zeta \rangle}$. Note that in view of the second condition on φ $E_{\mu,\zeta}$ is in A_φ.

Denote the zero element of A_φ by $\mathbf{0}$. Let W be the set of all solutions $f \in A_\varphi$ of the equation $\sum_{h \in H} \gamma_h f_h = \mathbf{0}$ and $\mathcal{E}$ be the set of all solutions of the form $E_{\mu,\zeta}$.

Theorem 1.3. *Let the family φ satisfies the conditions of Theorem 1.2 and $\mathcal{E}$ is not empty. Then a closure of a linear envelope of $\mathcal{E}$ in A_φ is W.*

2. Space A_φ and its dual

To note some special properties of the spaces A_φ and $P_{\varphi^\star}$ we need to remember definitions of (M^*)-space and (LN^*)-space from [7], [8].

(M^*)-space is a locally convex space F which is the projective limit of a sequence of normed spaces F_k with linear continuous mappings $g_{mk} : F_k \to F_m$, $m < k$, such that $g_{k,k+1}$ is compact for each $k \in \mathbb{N}$.

(LN^*)-space is a locally convex space E which is the inductive limit of an increasing sequence of normed spaces E_k such that the unit ball of E_k is relatively compact in E_{k+1} for each $k \in \mathbb{N}$, i.e. such that the inclusion map from E_k into E_{k+1} is compact.

It is known that if E (the inductive limit of an increasing sequence of normed spaces E_k) is an (LN^*)-space then a set B is bounded in E iff for some $m \in \mathbb{N}$ it is contained in E_m and bounded there ([7], Theorem 1).

It is easy to show that the inclusions $I_m : A(\varphi_{m+1}) \to A(\varphi_m)$ are compact for each $m \in \mathbb{N}$. So A_φ is an (M^*)-space. Therefore, A_φ is a reflexive space ([7], Proposition 7).

From conditions on φ it follows that $\lim_{x\to\infty}(\varphi^\star_{m+1}(x) - \varphi^\star_m(x)) = +\infty$. Using this fact and Montel's theorem it can be shown that the mappings $J_m : P(\varphi^\star_m) \to P(\varphi^\star_{m+1})$ are compact for each $m \in \mathbb{N}$. Thus, the space $P_{\varphi^\star}$ is an (LN^*)-space.

Lemma 2.1. *For each $S \in A'_\varphi$ we have $\hat{S} \in P_{\varphi^\star}$.*

Proof. First show that for $S \in A'_\varphi$ $\hat{S}$ is an entire function. For $f \in A_\varphi$ and $N \in \mathbb{N}$ define the mapping $f_N : \mathbb{Z}^n \to \mathbb{C}$ by the rule: $f_N(\alpha) = f(\alpha)$ for $|\alpha| \leqslant N$, $f_N(\alpha) = 0$ for $|\alpha| > N$. Then for each $s \in \mathbb{N}$

$$p_s(f - f_N) = \sup_{|\alpha|>N} \frac{|f(\alpha)|}{e^{\varphi_s(\alpha)}} \leqslant p_{s+1}(f) \exp\left(\sup_{|\alpha|>N} (\varphi_{s+1}(\alpha) - \varphi_s(\alpha))\right).$$

Taking into account condition 2) on φ we conclude that $p_s(f - f_N) \to 0$ as $N \to \infty$. This means that $f_N \to f$ in A_φ as $N \to \infty$. Hence, $S(f_N) \to S(f)$ as $N \to \infty$. For each $\alpha \in \mathbb{Z}^n$ let e_α be the mapping $e_\alpha : \mathbb{Z}^n \to \mathbb{C}$ acting by the rule: $e_\alpha(\beta) = 1$ if $\alpha = \beta$, $e_\alpha(\beta) = 0$ if $\alpha \neq \beta$. Then $f_N = \sum_{|\alpha|\leqslant N} f(\alpha)e_\alpha$. Consequently, $S(f) = \lim_{n\to\infty} S(f_N) = \sum_{\alpha\in\mathbb{Z}^n} \gamma(\alpha)f(\alpha)$, where $\gamma(\alpha) = S(e_\alpha)$. In particular,

$$\hat{S}(z) = \sum_{\alpha\in\mathbb{Z}^n} \gamma(\alpha)e^{-i\langle z,\alpha\rangle}, \qquad z \in \mathbb{C}^n. \tag{2}$$

Obviously, $\hat{S}(z + 2\pi l) = \hat{S}(z)$ for all $z \in \mathbb{C}^n$ and $l \in \mathbb{Z}^n$. Since S is a linear continuous functional on A_φ then for some $m \in \mathbb{N}$ and $c > 0$

$$|S(f)| \leqslant cp_m(f), \qquad f \in A_\varphi.$$

From this we have for each $\alpha \in \mathbb{Z}^n$

$$|\gamma(\alpha)| = |S(e_\alpha)| \leqslant cp_m(e_\alpha) = ce^{-\varphi_m(\alpha)}. \tag{3}$$

Using the second condition on φ we can choose $k \in \mathbb{N}$ so that the inequality $\sum_{\alpha\in\mathbb{Z}^n} e^{\varphi_{m+k}(\alpha)-\varphi_m(\alpha)} < \infty$ holds. Now using (3) we have for each $z \in \mathbb{C}^n$

$$\begin{aligned} |\hat{S}(z)| &= \left|\sum_{\alpha\in\mathbb{Z}^n} \gamma(\alpha)e^{-i\langle z,\alpha\rangle}\right| \leqslant c\sum_{\alpha\in\mathbb{Z}^n} e^{-\varphi_m(\alpha)+\langle \alpha, Imz\rangle} \\ &\leqslant ce^{\varphi^\star_{m+k}(Imz)} \sum_{\alpha\in\mathbb{Z}^n} e^{\varphi_{m+k}(\alpha)-\varphi_m(\alpha)}. \end{aligned}$$

From this it follows that the series $\sum_{\alpha\in\mathbb{Z}^n}\gamma(\alpha)e^{-i\langle z,\alpha\rangle}$ converges uniformly on compact subsets of $\mathbb{C}^n$. Hence, $\hat{S}$ is an entire function and $\hat{S}$ is in $P_{\varphi^\star}$. ■

Remark 2.1. Using the representation (2), inequality (3) and the second condition on φ it is easy to see that for each $S \in A'_\varphi$ we have

$$(D^\nu \hat{S})(z) = S((-i\alpha)^\nu e^{-i\langle z,\alpha\rangle})_{\alpha\in\mathbb{Z}^n}), \qquad \nu \in \mathbb{Z}^n_+,\ z \in \mathbb{C}^n.$$

Proof of Theorem 1.1. By Lemma 2.1 $\mathcal{F}(S) \in P_{\varphi^\star}$ for each $S \in A^*_\varphi$.

Let us show now that the linear mapping $\mathcal{F}$ is continuous. But first note that the topology of A^*_φ can be described as follows. For each $k \in \mathbb{N}$ let $W_k = \{f \in A_\varphi :\ p_k(f) \leqslant 1\}$ and $W_k^0 = \{S \in A'_\varphi : |S(f)| \leqslant 1,\ \forall f \in W_k\}$ be a polar of W_k in A'_φ. Let $T_k = \bigcup_{\alpha>0}(\alpha W_k^0)$ be a vector subspace in A'_φ generated by W_k^0 ($k \in \mathbb{N}$). Define a topology in T_k with a help of the norm

$$N_k(S) = \sup_{f\in W_k} |S(f)|, \qquad S \in T_k.$$

Obviously, $A'_\varphi = \bigcup_{k=1}^\infty T_k$. Define in A'_φ the topology λ of an inductive limit of spaces T_k. Since A_φ is a reflexive space then the strong topology in A'_φ coincides with the topology λ ([2], chapter 8). Now let $S \in T_k,\ k \in \mathbb{N}$. Then $|S(f)| \leqslant N_k(S),\ f \in W_k$. Hence, $|S(f)| \leqslant N_k(S)p_k(f),\ f \in A_\varphi$. Putting here $f = f_z$ with $z \in \mathbb{C}^n$ and using (1) we obtain that

$$|\hat{S}(z)| \leqslant N_k(S)e^{\varphi_k^\star(Imz)}.$$

From this it follows that $\|\hat{S}\|_k \leqslant N_k(S),\ S \in T_k\ (k = 1, 2, \ldots)$. Thus, $\mathcal{F}$ is continuous.

Let us prove that L is injective. Let $S \in A^*_\varphi$ and $\hat{S}(z) = 0$ for each $z \in \mathbb{C}^n$. For some $m \in \mathbb{N}$ and $c > 0$ we have $|S(f)| \leqslant cp_m(f),\ f \in A_\varphi$. As it was shown in the proof of Lemma 2.1 the functional S admits the representation

$$S(f) = \sum_{\alpha\in\mathbb{Z}^n} \gamma(\alpha)f(\alpha), \qquad f \in A_\varphi,$$

where complex numbers $\gamma(\alpha)$ satisfy the inequality

$$|\gamma(\alpha)| \leqslant ce^{-\varphi_m(\alpha)}, \qquad \alpha \in \mathbb{Z}^n.$$

From this representation we have for each $x \in \mathbb{R}^n$

$$\hat{S}(x) = \sum_{\alpha\in\mathbb{Z}^n} \gamma(\alpha)e^{-i\langle x,\alpha\rangle} = 0$$

Therefore, coefficients $\gamma(\alpha) = 0$ for all $\alpha \in \mathbb{Z}^n$ and S is a zero functional. Thus, $\mathcal{F}$ is injective.

Now we prove that $\mathcal{F}$ is surjective. Let $F \in P_{\varphi^\star}$. Then $F \in P(\varphi_m^\star)$ for some $m \in \mathbb{N}$. Represent $F(x)$ by the Fourier series

$$F(x) = \sum_{\alpha \in \mathbb{Z}^n} c_\alpha e^{-i\langle x, \alpha \rangle}, \qquad x \in \mathbb{R}^n.$$

For each $\alpha \in \mathbb{Z}^n$ we have

$$c_\alpha = \frac{1}{(2\pi)^n} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} F(x) e^{i\langle x, \alpha \rangle}\, dx.$$

In view of periodicity of F for each $\alpha \in \mathbb{Z}^n$ and $y \in \mathbb{R}^n$

$$c_\alpha = \frac{1}{(2\pi)^n} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} F(x+iy) e^{i\langle x+iy, \alpha \rangle}\, dx.$$

From this we get

$$|c_\alpha| \leqslant \frac{\|F\|_m}{(2\pi)^n} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} e^{\varphi_m^\star(y) - \langle y, \alpha \rangle}\, dx, \qquad \alpha \in \mathbb{Z}^n,\ y \in \mathbb{R}^n.$$

Since $\varphi_m^\star(y) \leqslant \varphi_m^*(y)$ for every $y \in \mathbb{R}^n$ then for each $\alpha \in \mathbb{Z}^n$ we have

$$|c_\alpha| \leqslant \|F\|_m \exp\left(\inf_{y \in \mathbb{R}^n} ((\varphi_m^*(y) - \langle y, \alpha \rangle) \right) = \|F\|_m e^{-(\varphi_m^*)^*(\alpha)} = \|F\|_m e^{-\varphi_m(\alpha)}. \tag{4}$$

Define a functional S on A_φ by the formula $S(f) = \sum_{\alpha \in \mathbb{Z}^n} c_\alpha f(\alpha),\ f \in A_\varphi$. Using the estimate (4) and the second condition on φ it is easy to see that the linear functional S is continuous. Obviously, $\hat{S}(z) = F(z),\ z \in \mathbb{C}^n$. Thus, $\mathcal{F}$ is surjective.

By the open mapping theorem [2], [5] $\mathcal{F}^{-1}$ is continuous. Thus, $\mathcal{F}$ is a topological isomorphism and the proof of theorem is complete. ■

3. A difference operator on A_φ

3.1. In the proof of Theorems 1.2 and 1.3 we will use Lemma 3.2. But first let us prove the following auxiliary result.

Lemma 3.1. *Let a function $h : \mathbb{Z}^n \to \mathbb{R}$ be such that for some constants $\nu > 1$, $C > 0$ and $D > 0$*

$$h(\alpha) \geqslant C\|\alpha\|^\nu - D, \qquad \alpha \in \mathbb{Z}^n. \tag{5}$$

Then there exists a constant $M_h > 0$ such that

$$|h^\star(\xi) - h^\star(x)| \leqslant M_h \tag{6}$$

for all $x, \xi \in \mathbb{R}^n$ satisfying the condition $\|\xi - x\| \leqslant (1 + \|x\|)^{-\frac{1}{\nu - 1}}$.

Proof. For each $x \in \mathbb{R}^n$ let $\alpha(x) \in \mathbb{Z}^n$ be a point where the supremum of the function $u_x : \alpha \in \mathbb{Z}^n \to \langle x, \alpha\rangle - h(\alpha)$ over $\mathbb{Z}^n$ is attained.

First prove that there exists a constant $m_h > 0$ not depending on x such that

$$\|\alpha(x)\| \leqslant m_h \cdot (1 + \|x\|^{\frac{1}{\nu-1}}).$$

Using (5) we have for each $\alpha \in \mathbb{Z}^n$

$$u_x(\alpha) \leqslant \|\alpha\| \cdot \|x\| - C\|\alpha\|^\nu + D.$$

Since $h^\star(x) = \sup_{\alpha\in\mathbb{Z}^n} u_x(\alpha) \geqslant -h(0)$ (here 0 is the zero element of $\mathbb{Z}^n$) then the supremum of u_x over $\mathbb{Z}^n$ is attained on the set

$$G_x = \{\alpha \in \mathbb{Z}^n : \|\alpha\| \cdot \|x\| \geqslant C\|\alpha\|^\nu - D - h(0)\}.$$

Put $L_h = D + h(0)$. By the condition on h we have $L_h \geqslant 0$. For each $\lambda \geqslant 0$ denote by T_λ the set of solutions of the inequality

$$\lambda t \geqslant Ct^\nu - L_h,$$

belonging to $\mathbb{R}_+$. This set is a segment of a form $[0, t_\lambda]$, where $t_\lambda < \infty$. Let us estimate t_λ from above. We have $\lambda t_\lambda = Ct_\lambda^\nu - L_h$. If $t_\lambda \geqslant 1$ then

$$\lambda = Ct_\lambda^{\nu-1} - \frac{L_h}{t_\lambda} \geqslant Ct_\lambda^{\nu-1} - L_h.$$

From this $t_\lambda \leqslant \left(\frac{\lambda+L_h}{C}\right)^{\frac{1}{\nu-1}}$. Taking into account a case $t_\lambda \in [0,1)$ we have

$$t_\lambda \leqslant \left(\frac{\lambda + L_h}{C}\right)^{\frac{1}{\nu-1}} + 1.$$

From this if $0 \leqslant \lambda \leqslant 1$ then $t_\lambda \leqslant \left(\frac{1+L_h}{C}\right)^{\frac{1}{\nu-1}} + 1$. Moreover if $\lambda > 1$ then $t_\lambda \leqslant \lambda^{\frac{1}{\nu-1}} \left(\frac{1+L_h}{C}\right)^{\frac{1}{\nu-1}} + 1$. Let $m_h := \left(\frac{1+L_h}{C}\right)^{\frac{1}{\nu-1}} + 1$. Then

$$t_\lambda \leqslant m_h(1 + \lambda^{\frac{1}{\nu-1}}).$$

Let $d_\lambda := m_h(1 + \lambda^{\frac{1}{\nu-1}})$. Then $T_\lambda \subseteq [0, d_\lambda]$. Since $\alpha \in G_x \Leftrightarrow \|\alpha\| \in T_{\|x\|}$, then for all $\alpha \in G_x$ we have

$$\|\alpha\| \leqslant m_h \cdot \|x\|^{\frac{1}{\nu-1}} + m_h.$$

In particular,

$$\|\alpha(x)\| \leqslant m_h \cdot \|x\|^{\frac{1}{\nu-1}} + m_h.$$

Further, for all $x, \xi \in \mathbb{R}^n$ such that $\|\xi - x\| \leqslant (1 + \|x\|)^{\frac{1}{1-\nu}}$ we have

$$\begin{aligned}
h^\star(\xi) - h^\star(x) &= \sup_{\alpha\in\mathbb{Z}^n} (\langle \xi, \alpha\rangle - h(\alpha)) - \sup_{\alpha\in\mathbb{Z}^n} (\langle x, \alpha\rangle - h(\alpha)) \\
&\leqslant (\langle \xi, \alpha(\xi)\rangle - h(\alpha(\xi))) - (\langle x, \alpha(\xi)\rangle - h(\alpha(\xi))) \\
&= \langle \xi - x, \alpha(\xi)\rangle \leqslant \|\xi - x\|\|\alpha(\xi)\| \leqslant (1 + \|x\|)^{\frac{1}{1-\nu}} m_h(1 + \|\xi\|^{\frac{1}{\nu-1}}) \\
&\leqslant \frac{2m_h(1 + \|x\|)^{\frac{1}{\nu-1}}}{(1 + \|x\|)^{\frac{1}{\nu-1}}} = 2m_h.
\end{aligned}$$

Similarly,

$$\begin{aligned} h^\star(x) - h^\star(\xi) &\leqslant \langle x-\xi, \alpha(x)\rangle \leqslant \|x-\xi\|\|\alpha(x)\| \\ &\leqslant \frac{m_h(1+\|x\|^{\frac{1}{\nu-1}})}{(1+\|x\|)^{\frac{1}{\nu-1}}} \leqslant \frac{2m_h(1+\|x\|)^{\frac{1}{\nu-1}}}{(1+\|x\|)^{\frac{1}{\nu-1}}} = 2m_h. \end{aligned}$$

From these estimates we get (6) with $M_h = 2m_h$. ■

Lemma 3.2. *Let the family φ satisfies the conditions* i_1) *and* i_2) *of Theorem 1.2. Then for each $m \in \mathbb{N}$:*

there exists a constant $K_m > 0$ such that

$$|\varphi_m^\star(\xi) - \varphi_m^\star(x)| \leqslant K_m \tag{7}$$

for all $x, \xi \in \mathbb{R}^n$ satisfying the condition $\|\xi - x\| \leqslant (1+\|x\|)^{-\frac{1}{\mu_m-1}}$;

$$\varphi_{m+1}^\star(x) - \varphi_m^\star(x) \geqslant \|x\| - d_m, \qquad x \in \mathbb{R}^n. \tag{8}$$

Proof. The inequality (7) holds in view of Lemma 3.1. So let us prove the inequality (8). For each $x \in \mathbb{R}^n$ and $m \in \mathbb{N}$ let $\alpha_m(x) \in \mathbb{Z}^n$ be a point where the supremum of the function $u_x : \alpha \in \mathbb{Z}^n \to \langle x, \alpha\rangle - \varphi_m(\alpha)$ over $\mathbb{Z}^n$ is attained. Let $\theta(x)$ be the point in $\mathbb{R}^n$ with coordinates θ_j defined as follows: $\theta_j = \frac{x_j}{|x_j|}$ if $x_j \neq 0$ and $\theta_j = 0$ if $x_j = 0$ $(j = 1, \dots, n)$. Then using the condition i_2) we have

$$\begin{aligned} \varphi_{m+1}^\star(x) - \varphi_m^\star(x) &\geqslant \langle x, \theta(x)\rangle - \varphi_{m+1}(\alpha_m(x) + \theta(x)) + \varphi_m(\alpha_m(x)) \\ &\geqslant |x_1| + \dots + |x_n| - d_m \geqslant \|x\| - d_m. \end{aligned}$$

Thus, the inequality (8) is proved. ■

Remark 3.1. For each $m \in \mathbb{N}$ and $x \in \mathbb{R}^n$ let $y_m(x)$ be a point where the supremum of the function $v_x : y \in \mathbb{R}^n \to \langle x, y\rangle - \varphi_m(y)$ over $\mathbb{R}^n$ is attained. Let $\alpha \in \mathbb{Z}^n$ be the nearest point to $y_m(x)$. Then

$$\varphi_m^*(x) = \langle x, y_m(x)\rangle - \langle x, \alpha\rangle + \langle x, \alpha\rangle - \varphi_{m+1}(\alpha) + \varphi_{m+1}(\alpha) - \varphi_m(y_m(x)).$$

From this representation using the condition i_2) we have

$$\varphi_m^*(x) \leqslant n\|x\| + \varphi_{m+1}^\star(x) + d_m.$$

Now by (8) we get for some $d_{m,n} > 0$

$$\varphi_m^*(x) \leqslant \varphi_{m+n+1}^\star(x) + d_{m,n}, \qquad x \in \mathbb{R}^n.$$

On the other hand, for each $m \in \mathbb{N}$ and $x \in \mathbb{R}^n$ $\varphi_m^\star(x) \leqslant \varphi_m^*(x)$.

Two last inequalities mean that the space $P_{\varphi^\star}$ coincides with the space P_{φ^*} which is an inductive limit of the spaces

$$P(\varphi_m^*) = \Big\{F \in H(\mathbb{C}^n) : F(z + 2\pi l) = F(z) \text{ for all } z \in \mathbb{C}^n,\ l \in \mathbb{Z}^n \\ \text{and such that } \|F\|_m = \sup_{z\in\mathbb{C}^n} \frac{|F(z)|}{e^{\varphi_m^*(Imz)}} < \infty\Big\}.$$

Here $\varphi^* = \{\varphi_m^*\}_{m=1}^\infty$.

3.2. Let the family φ satisfies the conditions of Theorem 1.2. In the first section we defined for each $f \in A_\varphi$ and $h \in \mathbb{Z}^n$ a sequence f_h by the rule: $f_h(\alpha) = f(\alpha + h),\ \alpha \in \mathbb{Z}^n$. Using the condition i_2) on φ for each $m \in \mathbb{N}$ we can find numbers $k \in \mathbb{N}$ and $d > 0$ such that $p_m(f_h) \leqslant dp_k(f)$ for all $f \in A_\varphi$. Thus, for each $h \in \mathbb{Z}^n$ the linear operator $S_h : f \in A_\varphi \to f_h$ acts from A_φ to A_φ and is continuous. So if H is a subset of $\mathbb{Z}^n$ consisting of finite number of elements and for $h \in H$ $\gamma_h \in \mathbb{C}$ then an operator $M : A_\varphi \to A_\varphi$ acting by the rule

$$M(f) = \sum_{h\in H} \gamma_h S_h(f), \qquad f \in A_\varphi,$$

is linear and continuous.

Let $g(z) := \sum_{h\in H} \gamma_h e^{-i\langle h,z\rangle}$ $(z \in \mathbb{C}^n)$. The function g is usually called a characteristic function of the operator M.

3.2.1. In the proof of Theorems 1.2 and 1.3 the following lemma by L. Ehrenpreis and B. Malgrange (see, for example, Lemma A.1 in [3]) is used.

Lemma 3.3. *Let P be a polynomial of degree m. Then there exists a constant $C > 0$ such that for every $r > 0, z \in \mathbb{C}^n$ and every function f which is defined for all $z' \in \mathbb{C}^n$ with $\|z' - z\| < r$ and is such that $\frac{f}{P}$ is holomorphic there, we have*

$$\left|\frac{f(z)}{p(z)}\right| \leqslant Cr^{-m} \sup_{z'\in B(z,r)} |f(z')|.$$

Proof of Theorem 1.2. We have to prove that the operator M is surjective on A_φ. First let us show that the image imM of the operator M is dense in A_φ. Let $N_g = \{z \in \mathbb{C}^n : g(z) = 0\}$. For $z \in \mathbb{C}^n \setminus N_g$ consider the equation $M(f) = f_z$. It has a solution $f = \frac{f_z}{g(z)}$ belonging to A_φ. From this and completeness of the system $\{f_z\}_{z\in\mathbb{C}^n\setminus N_g}$ we conclude that imM is dense in A_φ.

Let us show now that the image of the operator M is closed in A_φ. It is known that closedness of image of M in A_φ is equivalent to closedness of the image of the adjoint operator M^* in A^*_φ (see [2], Theorem 8.6.13).

Consider an operator $\widehat{M}^*$ on $P_{\varphi^\star}$ acting by the rule:

$$\widehat{M}^*(F) = \mathcal{F}(M^*(\mathcal{F}^{-1}(F))), \qquad F \in P_{\varphi^\star}.$$

Obviously, $\widehat{M}^*$ is a linear continuous operator on $P_{\varphi^\star}$. Taking into account Theorem 1.1 we see that closedness of the image of the operator M in A_φ is equivalent to closedness of the image $im\widehat{M}^*$ of the operator $\widehat{M}^*$ in $P_{\varphi^\star}$. Note that for each $F \in P_{\varphi^\star}$ and $z \in \mathbb{C}^n$

$$\begin{aligned}
\widehat{M}^*(F)(z) &= \mathcal{F}^{-1}(F)(M(f_z)) \\
&= \mathcal{F}^{-1}(F)\left(\sum_{h\in H} \gamma_h S_h(f_z)\right) = \mathcal{F}^{-1}(F)\left(\sum_{h\in H} \gamma_h (e^{-i\langle \alpha+h,z\rangle})_{\alpha\in\mathbb{Z}^n}\right) \\
&= \mathcal{F}^{-1}(F)(f_z) \sum_{h\in H} \gamma_h e^{-i\langle h,z\rangle} = F(z)g(z).
\end{aligned}$$

Thus, for each $F \in P_{\varphi^\star}$ we have $\widehat{M}^*(F)(z) = F(z)g(z),\ z \in \mathbb{C}^n$.

By Theorem 1 in [7] $im\widehat{M}^*$ is closed in $P_{\varphi^\star}$ iff $im\widehat{M}^* \cap P(\varphi_m^\star)$ is closed in $P(\varphi_m^\star)$ for each $m \in \mathbb{N}$. So let $m \in \mathbb{N}$ be arbitrary and F belongs to the closure of $im\widehat{M}^* \cap P(\varphi_m^\star)$ in $P(\varphi_m^\star)$. Then there exists a sequence $(F_k)_{k=1}^\infty$ of functions $F_k \in im\widehat{M}^* \cap P(\varphi_m^\star)$ converging to F in $P_{\varphi_m^\star}$. In particular, $F_k \to F$ uniformly on compact subsets of $\mathbb{C}^n$ as $k \to \infty$. So it is clear that the function $\psi(z) = \frac{F(z)}{g(z)}$ is holomorphic on $\mathbb{C}^n$. Obviously, $\psi(z + 2\pi l) = \psi(z)$ for all $z \in \mathbb{C}^n$ and $l \in \mathbb{Z}^n$. The functions F and ψ can be represented by the series

$$F(z) = \sum_{\alpha \in \mathbb{Z}^n} c_\alpha e^{-i\langle z, \alpha \rangle}, \qquad z \in \mathbb{C}^n,$$
$$\psi(z) = \sum_{\alpha \in \mathbb{Z}^n} b_\alpha e^{-i\langle z, \alpha \rangle}, \qquad z \in \mathbb{C}^n.$$

As we know (see Lemma 2.1) the first series converges to F in $P(\varphi_{m+k}^\star)$ for some $k \in \mathbb{N}$. The second series uniformly converges to ψ on compact subsets of $\mathbb{C}^n$. Let us show that $\psi \in P_{\varphi^\star}$. Obviously, the functions $F_0(\zeta) = \sum_{\alpha \in \mathbb{Z}^n} c_\alpha \zeta^\alpha$, $\psi_0(\zeta) = \sum_{\alpha \in \mathbb{Z}^n} b_\alpha \zeta^\alpha$, $g_0(\zeta) = \sum_{h \in H} \gamma_h \zeta^h$ are holomorphic in $(\mathbb{C} \setminus \{0\})^n$ and

$$\psi_0(\zeta) = \frac{F_0(\zeta)}{g_0(\zeta)}, \qquad \zeta \in (\mathbb{C} \setminus \{0\})^n.$$

Choose $N = (N_1, \ldots, N_n) \in \mathbb{Z}_+^n$ so that $P(\zeta) = g_0(\zeta)\zeta_1^{N_1} \cdots \zeta_n^{N_n}$ is a polynomial. Then

$$\psi_0(\zeta) = \frac{F_0(\zeta)\zeta^N}{P(\zeta)}, \qquad \zeta \in (\mathbb{C} \setminus \{0\})^n.$$

Let us estimate $|\psi_0(\zeta)|$ from above at points $\zeta = (\zeta_1, \ldots, \zeta_n) \in (\mathbb{C} \setminus \{0\})^n$. Let

$$a_m(\zeta) = (1 + \|(\ln|\zeta_1|, \ldots, \ln|\zeta_n|)\|)^{\frac{1}{1-\mu_m}},$$
$$r(\zeta) = \min\left(1, \left(1 - \exp\left(-\frac{a_m(\zeta)}{\sqrt{n}}\right)\right) \min_{1 \leqslant j \leqslant n} |\zeta_j|\right).$$

By Lemma 3.3 there exists a constant $C > 0$ such that for all $\zeta \in (\mathbb{C} \setminus \{0\})^n$

$$\begin{aligned} |\psi_0(\zeta)| &\leqslant C(r(\zeta))^{-k} \sup_{w \in B(\zeta, r(\zeta))} (|F_0(w)| \|w\|^{|N|}) \\ &\leqslant C(r(\zeta))^{-k}(1 + \|\zeta\|)^{|N|} \sup_{w \in B(\zeta, r(\zeta))} |F_0(w)|. \end{aligned}$$

Since for $\zeta = (\zeta_1, \ldots, \zeta_n) \in (\mathbb{C} \setminus \{0\})^n$

$$|F_0(\zeta)| \leqslant \|F\|_m e^{\varphi_m^\star(\ln|\zeta_1|, \ldots, \ln|\zeta_n|)},$$

then

$$|\psi_0(\zeta)| \leqslant C(r(\zeta))^{-k}(1 + \|\zeta\|)^{|N|} \exp\left(\sup_{w \in B(\zeta, r(\zeta))} \varphi_m^\star(\ln|w_1|, \ldots, \ln|w_n|)\right). \quad (9)$$

For points $w = (w_1, \dots, w_n) \in B(\zeta, r(\zeta))$ we have

$$\exp(\left(-\frac{a_m(\zeta)}{\sqrt{n}}\right)|\zeta_j| \leqslant |w_j| \leqslant \exp\left(\frac{a_m(\zeta)}{\sqrt{n}}\right)|\zeta_j|(j = 1, \dots, n).$$

From this we get for $w \in B(\zeta, r(\zeta))$

$$|\ln|w_j| - \ln|\zeta_j|| \leqslant \frac{a_m(\zeta)}{\sqrt{n}},\ j = 1, \dots, n.$$

Thus, for points $w = (w_1, \dots, w_n) \in B(\zeta, r(\zeta))$

$$\|(\ln|w_1|, \dots, \ln|w_n|) - (\ln|\zeta_1|, \dots, \ln|\zeta_n|)\| \leqslant (1 + \|(\ln|\zeta_1|, \dots, \ln|\zeta_n|)\|)^{\frac{1}{1-\mu_m}}.$$

Now using the inequality (7) we have

$$\sup_{w \in B(\zeta, r(\zeta))} \varphi_m^\star(\ln|w_1|, \dots, \ln|w_n|) \leqslant \varphi_m^\star(\ln|\zeta_1|, \dots, \ln|\zeta_n|) + K_m.$$

From this and (9) we get

$$|\psi_0(\zeta)| \leqslant Ce^{K_m}(r(\zeta))^{-k}(1 + \|\zeta\|)^{|N|}e^{\varphi_m^\star(\ln|\zeta_1|, \dots, \ln|\zeta_n|)}$$

for all $\zeta = (\zeta_1, \dots, \zeta_n) \in (\mathbb{C}\backslash\{0\})^n$. Taking into account that for $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ and $y = Imz$

$$a_m(e^{-iz_1}, \dots, e^{-iz_n}) = (1 + \|y\|)^{\frac{1}{1-\mu_m}}$$

it is easy to show that $z \in \mathbb{C}^n$

$$|\psi(z)| \leqslant Ce^{K_m}\left(1 + \frac{1}{(1 - e^{-\frac{1}{\sqrt{n}}})e^{-\|y\|}}\right)^k (1 + \|(e^{Imz_1}, \dots, e^{Imz_n})\|)^{|N|}e^{\varphi_m^\star(Imz)}.$$

From this we have

$$|\psi(z)| \leqslant Ce^{K_m}\left(\frac{2}{1 - e^{-\frac{1}{\sqrt{n}}}}\right)^k e^{(|N|+k)\|Imz\|}(1 + n)^{|N|}e^{\varphi_m^\star(Imz)}.$$

Using the inequality (8) we obtain

$$|\psi(z)| \leqslant Ke^{\varphi_{m+k+|N|}^\star(Imz)}, \qquad z \in \mathbb{C}^n,$$

where K is some constant depending on $k, m, |N|, n$. Hence, $\psi \in P(\varphi_{m+k+|N|}^\star)$. Thus, $\psi \in P_{\varphi^\star}$. This means that $F \in im\widehat{M}^*$. Hence, $im\widehat{M}^* \cap P(\varphi_m^\star)$ is closed in $P(\varphi_m^\star)$ for each $m \in \mathbb{N}$. Thus, the image of the operator $\widehat{M}^*$ is closed in $P_{\varphi^\star}$. Therefore, the image of the operator M is closed in A_φ.

Thus, the image of the operator M is dense and closed in A_φ. Therefore, $imM = A_\varphi$. The proof is complete. ■

3.2.2. Consider the equation $M(f) = \mathbf{0}$ in A_φ. Note that if f belongs to the kernel W of the operator M then for each $\beta \in \mathbb{Z}^n$ $S_\beta(f) \in W$.

Recall that for $\mu \in \mathbb{Z}_+^n$ and $\zeta \in \mathbb{C}^n$ we defined the sequence $E_{\mu,\zeta}$ by the rule: $\alpha \in \mathbb{Z}^n \to \alpha^\mu e^{-i\langle\alpha,\zeta\rangle}$.

Lemma 3.4. *The sequence $E_{\mu,\zeta}$ is in W iff $(D^\beta g)(\zeta) = 0$ for all $\beta \in \mathbb{Z}_+^n$ such that $\beta \leqslant \mu$.*

Proof. Consider more complicated case $n \geqslant 2$. Let $\mu \in \mathbb{Z}_+^n$ and $\zeta \in \mathbb{C}^n$ be such that $(D^\beta g)(\zeta) = 0$ for all $\beta \in \mathbb{Z}_+^n$ such that $\beta \leqslant \mu$. Then for each $\alpha \in \mathbb{Z}^n$

$$\begin{aligned} M(E_{\mu,\zeta})(\alpha) &= \sum_{h\in H} \gamma_h (\alpha+h)^\mu e^{-i\langle\alpha+h,\zeta\rangle} \\ &= e^{-i\langle\alpha,\zeta\rangle} \sum_{h\in H} \gamma_h (\alpha+h)^\mu e^{-i\langle h,\zeta\rangle} \\ &= e^{-i\langle\alpha,\zeta\rangle} \sum_{\beta\in\mathbb{Z}_+^n:\beta\leqslant\mu} C_\mu^\beta \alpha^{\mu-\beta} \sum_{h\in H} \gamma_h h^\beta e^{-i\langle h,\zeta\rangle} \\ &= e^{-i\langle\alpha,\zeta\rangle} \sum_{\beta\in\mathbb{Z}_+^n:\beta\leqslant\mu} C_\mu^\beta \alpha^{\mu-\beta} i^{|\beta|} (D^\beta g)(\zeta) = 0. \end{aligned}$$

Thus, $E_{\mu,\zeta} \in W$.

Now let for some $\mu \in \mathbb{Z}_+^n$ and $\zeta \in \mathbb{C}^n$ $E_{\mu,\zeta} \in W$. Then for each $\alpha \in \mathbb{Z}^n$

$$\sum_{h\in H} \gamma_h (\alpha+h)^\mu e^{-i\langle\alpha+h,\zeta\rangle} = 0$$

Hence, for each $\alpha \in \mathbb{Z}^n$

$$\sum_{h\in H} \gamma_h (\alpha+h)^\mu e^{-i\langle h,\zeta\rangle} = 0. \tag{10}$$

In particular, $\sum_{h\in H} \gamma_h h^\mu e^{-i\langle h,\zeta\rangle} = 0$. This means that $(D^\mu g)(\zeta) = 0$. Further, let $\nu = (\nu_1, \ldots, \nu_n) \in \mathbb{Z}_+^n$ be such that $(D^\beta g)(\zeta) = 0$ for all $\beta \in \mathbb{Z}_+^n$ such that $\nu \leqslant \beta \leqslant \mu$. If $\nu = (0, \ldots, 0)$ then Lemma holds. If $\nu \neq (0, \ldots, 0)$ then there exists $j \in \{1, \ldots, n\}$ such that $\nu_j \geqslant 1$. For simplicity suppose $j = 1$ and show that $(D^{(\beta_1,\nu_2,\ldots,\nu_n)} g)(\zeta) = 0$ for all $\beta_1 = 0, \ldots, \nu_1 - 1$. For each $\alpha \in \mathbb{Z}^n$

$$\begin{aligned} \sum_{h\in H} \gamma_h (\alpha+h)^\mu e^{-i\langle h,\zeta\rangle} &= \sum_{h\in H} \gamma_h \sum_{(0,\ldots,0)\leqslant\beta\leqslant\mu} C_\mu^\beta \alpha^{\mu-\beta} h^\beta e^{-i\langle h,\zeta\rangle} \\ &= \sum_{(0,\ldots,0)\leqslant\beta\leqslant\mu} C_\mu^\beta \alpha^{\mu-\beta} \sum_{h\in H} \gamma_h h^\beta e^{-i\langle h,\zeta\rangle} \\ &= \sum_{(0,\ldots,0)\leqslant\beta\leqslant\mu} C_\mu^\beta \alpha^{\mu-\beta} i^{|\beta|} (D^\beta g)(\zeta) \\ &= \sum_{(0,\ldots,0)\leqslant\beta\leqslant\nu,\beta\neq\nu} C_\mu^\beta \alpha^{\mu-\beta} i^{|\beta|} (D^\beta g)(\zeta). \end{aligned}$$

In view of (10)

$$\sum_{(0,\dots,0)\leqslant\beta\leqslant\nu,\beta\neq\nu} C_\mu^\beta \alpha^{\mu-\beta} i^{|\beta|}(D^\beta g)(\zeta)=0, \qquad \alpha\in\mathbb{Z}^n.$$

From this we have for all $\alpha\in\mathbb{Z}^n$

$$\sum_{\beta_1=0}^{\nu_1} C_{\mu_1}^{\beta_1}\alpha_1^{\nu_1-\beta_1} i^{\beta_1} \sum_{(0,\dots,0)\leqslant\beta'\leqslant\nu'} {\alpha'}^{\nu'-\beta'} i^{|\beta'|}(D^\beta g)(\zeta)=0.$$

Putting here $\alpha_2=\cdots=\alpha_n=0$ we get for all $\alpha_1\in\mathbb{Z}$

$$\sum_{\beta_1=0}^{\nu_1-1} C_{\mu_1}^{\beta_1}\alpha_1^{\nu_1-\beta_1} i^{\beta_1}(D^{(\beta_1,\nu_2,\dots,\nu_n)} g)(\zeta)=0.$$

From this it follows that $(D^{(\beta_1,\nu_2,\dots,\nu_n)}g)(\zeta)=0$ for all $\beta_1=0,\dots,\nu_1-1$. Obviously, applying these arguments so on we will obtain that $(D^\beta g)(\zeta)=0$ for all $\beta\in\mathbb{Z}_+^n$ such that $\beta\leqslant\mu$.

The proof of Lemma is complete. ∎

Let $\mathcal{A}=\{(\mu,\zeta)\in\mathbb{Z}_+^n\times\mathbb{C}^n : E_{\mu,\zeta}\in W\}$. Recall that $\mathcal{E}$ is a set of all solutions of the form $E_{\mu,\zeta}$ of the equation $\sum_{h\in H}\gamma_h f_h=\mathbf{0}$.

Proof of Theorem 1.3. Let S be a linear continuous functional on A_φ such that $S(f)=0$ for each $f\in\mathcal{E}$. If we will show that $S(f)=0$ for each $f\in W$ then a closure of a linear envelope of $\mathcal{E}$ will coincide with W. Using Lemma 3.4 and taking account that for all $(\mu,\zeta)\in\mathcal{A}$ $(D^\beta\hat{S})(\zeta)=0$ for $\beta\in\mathbb{Z}_+^n$ such that $\beta\leqslant\mu$ it is easy to check that the function $\psi:=\frac{\hat{S}}{g}$ is entire. From the proof of Theorem 1.2 it follows that $\psi\in P_{\varphi^\star}$. Hence, by Theorem 1.1 there exists a functional $\Psi\in A'_\varphi$ such that $\hat{\Psi}=\psi$. Obviously, the functional $\Psi\circ M$ is in A'_φ too. And for each $z\in\mathbb{C}^n$ we have $\mathcal{F}(\Psi\circ M)(z)=\Psi(M(f_z))=\Psi(g(z)f_z)=\psi(z)g(z)=\hat{S}(z)$. By theorem 1 $\Psi\circ M=S$. Now for each $f\in W$ we have

$$S(f)=(\Psi\circ M)(f)=\Psi(M(f))=0.$$

Thus, a linear envelope of $\mathcal{E}$ is dense in W. ∎

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