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A NOTE ON HECKE'S FUNCTIONAL EQUATION AND THE SELBERG CLASS

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Abstract: We study the solutions of Hecke's functional equation in the framework of the extended Selberg class. It turns out that, notwithstanding certain differences between the two theories, there are strong analogies in the behavior of the dimensions of the spaces of solutions. We also provide details to an old argument of Hecke, which is presented in a sketchy way both in Hecke's original works and in subsequent books on Hecke's theory.

Keywords: Selberg class; Hecke theory; Dirichlet series with functional equation.

1. Introduction

Well known work by Hecke (see Hecke [3],[4], Ogg [12], Berndt [1] and Berndt-Knopp [2]) deals with the solutions $\phi(s)$ of the functional equation

$$\left(\frac{\lambda}{2\pi}\right)^s \Gamma(s)\phi(s) = \omega\left(\frac{\lambda}{2\pi}\right)^{k-s} \Gamma(k-s)\phi(k-s) \tag{1.1}$$

($\lambda, k > 0$ and $\omega = \pm 1$) satisfying the following conditions:

(a) $\phi(s)$ is a Dirichlet series with finite abscissa of convergence;
(b) $(s-k)\phi(s)$ is an entire function of finite order.

Denoting by $\mathcal{D}(\lambda, k, \omega)$ the complex vector space of such Dirichlet series, Hecke proved that for every $k > 0$ and $\omega = \pm 1$

$$\dim_{\mathbb{C}} \mathcal{D}(\lambda, k, \omega) = \begin{cases} d(\lambda, k, \omega) & \text{if } 0 < \lambda \leqslant 2 \\ \infty & \text{if } \lambda > 2. \end{cases}$$

Moreover, Hecke gave the explicit value of the integer $d(\lambda, k, \omega)$. This is achieved by his well known correspondence theorem, giving an isomorphism between $\mathcal{D}(\lambda, k, \omega)$ and the complex vector space $\mathcal{M}(\lambda, k, \omega)$ of the modular forms of signature (λ, k, ω), i.e. the functions $f(z)$ holomorphic on the upper half-plane $\mathcal{H} = \{\Im z > 0\}$ satisfying the following conditions:

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(α) $f(z)$ has a Fourier series expansion of type

$$f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z/\lambda} \tag{1.2}$$

with $a_n \ll n^c$ for some $c > 0$;

(β) $f(z) = \omega(i/z)^k f(-1/z)$.

More precisely, Hecke's correspondence theorem states that given $\lambda, k > 0$, $\omega = \pm 1$ and $a_n \ll n^c$, writing $f(z)$ as in (1.2) and

$$\phi(s) = \sum_{n=1}^{\infty} a_n n^{-s},$$

then $f \in \mathcal{M}(\lambda, k, \omega)$ if and only if $\phi \in \mathcal{D}(\lambda, k, \omega)$ and $a_0 = \omega(\frac{\lambda}{2\pi})^k \Gamma(k) \operatorname{res}_{s=k} \phi(s)$. Note that a modular form of signature (λ, k, ω) is in fact a modular form for the group $G(\lambda)$ generated by the transformations $S : z \mapsto z + \lambda$ and $T : z \mapsto -1/z$, and that the dimension of $\mathcal{M}(\lambda, k, \omega)$ depends heavily on the geometry of the fundamental domain of $G(\lambda)$. Moreover, while the isomorphism between $\mathcal{D}(\lambda, k, \omega)$ and $\mathcal{M}(\lambda, k, \omega)$ is established by a simple application of Riemann's famous Mellin transform method for the functional equation of the Riemann zeta function, the dimension of $\mathcal{M}(\lambda, k, \omega)$ is computed by means of ingenious arguments. It is an interesting problem to obtain Hecke's type results for $\mathcal{D}(\lambda, k, \omega)$ dealing directly with the Dirichlet series.

From a rather different viewpoint, one of the aims of the Selberg class project is the classification of the solutions of Riemann type functonal equations, satisfying certain additional properties. We refer to Selberg [15], Kaczorowski-Perelli [9], Kaczorowski [6] and Perelli [13],[14] for definitions and basic properties of the Selberg class $\mathcal{S}$. Although the emphasis there is mainly on solutions with arithmetic properties (e.g. with Euler product), the classification results have been so far obtained in the framework of the extended Selberg class $\mathcal{S}^\sharp$, defined by the following properties:

(i) every $F \in \mathcal{S}^\sharp$ is an absolutely convergent Dirichlet series for $\sigma > 1$, and $(s-1)^m F(s)$ is an entire function of finite order for some integer m;

(ii) $F(s)$ satisfies a functional equation of type $\Phi(s) = \omega\bar{\Phi}(1-s)$ with

$$\Phi(s) = Q^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j) F(s),$$

$|\omega| = 1$, $Q > 0$, $\lambda_j > 0$ and $\Re\mu_j \geqslant 0$ ($\bar{f}(s) = \overline{f(\bar{s})}$).

We recall that the degree d_F and the conductor q_F of $F \in \mathcal{S}^\sharp$ are defined by

$$d_F = 2\sum_{j=1}^{r} \lambda_j, \qquad q_F = (2\pi)^{d_F} Q^2 \prod_{j=1}^{r} \lambda_j^{2\lambda_j}.$$

At present, the above mentioned classification is complete for all degrees $d < 2$, see Kaczorowski-Perelli [10], and the next open case, $d = 2$, looks quite difficult.

We normalize the solutions $\phi(s)$ of functional equation (1.1) writing

$$H(s) = \phi\Big(s + \frac{k-1}{2}\Big),$$

and denote by $D(\lambda, k, \omega)$ the space of the normalized functions from $\mathcal{D}(\lambda, k, \omega)$. Thus $D(\lambda, k, \omega)$ looks similar to the real vector space $\mathcal{S}_2^\sharp(Q, \mu, \omega)$ of the degree 2 functions in $\mathcal{S}^\sharp$ satisfying *(ii)* with

$$Q = \frac{\lambda}{2\pi}, \qquad r = 1, \qquad \lambda_1 = 1, \qquad \mu_1 = \mu = \frac{k-1}{2};$$

note that the conductor $q = q_F$ becomes in this case

$$q = (2\pi Q)^2 = \lambda^2. \tag{1.3}$$

However, there are differences between the definitions of the two spaces, such as the abscissa of absolute convergence, the conjugate in *(ii)* and the location and order of the possible pole in *(i)*. Indeed, the pole of $F \in \mathcal{S}_2^\sharp(Q, \mu, \omega)$ if present is at $s = 1$, while in the case of $H \in D(\lambda, k, \omega)$ the possible pole is at $s = \frac{k+1}{2}$. It is therefore of some interest to investigate the dimension of $\mathcal{S}_2^\sharp(Q, \mu, \omega)$, and in particular to see if the dimensions of $D(\lambda, k, \omega)$ and $\mathcal{S}_2^\sharp(Q, \mu, \omega)$ are finite/infinite for corresponding values of the parameters. We have

Theorem. *Let $Q > 0$, $\mu \geqslant 0$, $|\omega| = 1$ and let $q = (2\pi Q)^2$ denote the conductor. Then for every μ and ω, $\dim_{\mathbb{R}} \mathcal{S}_2^\sharp(Q, \mu, \omega)$ is finite if $0 < q \leqslant 4$, while there exist uncountably many linearly independent functions $F \in \mathcal{S}_2^\sharp(Q, \mu, \omega)$ if $q > 4$.*

Similar results can be proved when the functional equation contains a shift, i.e. when μ is replaced by $\mu + i\theta$ with $\theta \in \mathbb{R}$. In other words, an analogous theorem holds for the general case of degree 2 functions from $\mathcal{S}^\sharp$ with exactly one Γ-factor. Note that the proof allows to obtain an explicit bound for the dimension when $0 < q \leqslant 4$, analogous to Hecke's case. Our Theorem is proved by a simple adaptation of Hecke's arguments, thus once again arguing on the side of modular forms. Note that, in view of (1.3), the dimensions of the two spaces behave exactly in the same way with respect to finiteness, notwithstanding the above mentioned differences in the definitions of the two spaces. Indeed, interesting members of $D(\lambda, k, \omega)$ like certain normalized Epstein zeta functions (which, having real coefficients, are potential members of $\mathcal{S}_2^\sharp(Q, \mu, \omega)$) do not belong to the corresponding $\mathcal{S}_2^\sharp(Q, \mu, \omega)$ due to the location of the pole, if $k \neq 1$. Of course, the same applies in general to all polar members of $D(\lambda, k, \omega)$ when $k \neq 1$. We also remark that apart from the functional equations of degree 1, where everything is known concerning dimensions (see Kaczorowski-Perelli [8]), as far as we know Hecke's functional equation is the only case where the dimension is known for all values of the conductor.

Finally, we take this opportunity to prove the folklore statement that for $\lambda > 2$ the dimension of $\mathcal{M}(\lambda, k, \omega)$ is uncountable. Moreover, still in the case $\lambda > 2$ we provide some details to an argument in Hecke's original paper [3] which in our opinion is a bit too vague, and which has not been clarified in the books on Hecke's theory (Hecke [4], Ogg [12], Berndt [1] and Berndt-Knopp [2]).

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2. The case $q > 4$

We start with Hecke's spaces $\mathcal{D}(\lambda, k, \omega)$. Let $\lambda > 2$, $z = x + iy$ and denote by $B = B_\lambda$ the region

$$B = \{z \in \mathbb{C} : -\frac{\lambda}{2} < x < 0, y > -2, |z| > 1\}.$$

The counterclockwise oriented boundary ∂B of B consists of the five sides $\gamma_1 = \{x = 0, y \geqslant 1\}$, $\gamma_2 = \{x = -\lambda/2\}$, $\gamma_3 = \{-\lambda/2 \leqslant x \leqslant 0, y = -2\}$, $\gamma_4 = \{x = 0, -2 \leqslant y \leqslant -1\}$ and $\gamma_5 = \{|z| = 1, x \leqslant 0\}$. The mapping $T(z) = -1/z$ is conformal and invertible on B, and maps B onto the simply connected region $T(B)$ contained in the unit disc. In fact, T maps the vertexes of ∂B to the vertexes of the counterclockwise oriented boundary $\partial T(B)$ of $T(B)$ as follows:

$$i \mapsto i, \qquad i\infty \mapsto 0, \qquad -\frac{\lambda}{2} - 2i \mapsto \xi_0, \qquad -2i \mapsto -\frac{i}{2}, \qquad -i \mapsto -i,$$

where ξ_0 is the non-zero intersection of the circles $|w - 1/\lambda| = 1/\lambda$ and $|w + i/4| = 1/4$; it is easily checked that $\partial T(B)$ is a piecewise smooth Jordan curve (see Sect.1.1 of Kodaira [11]).

Let $D \subset \mathbb{C}$ be a simply connected bounded open set whose (counterclockwise oriented) boundary ∂D is a piecewise smooth Jordan curve. We denote by $\prec$ the orientation on ∂D, by $\tilde{D}$ the closure of D and by $\tilde{\mathcal{H}} = \mathcal{H} \cup \mathbb{R} \cup \{\infty\}$ the closure of the upper half-plane $\mathcal{H}$ in the Riemann sphere. We need the following version of the Riemann mapping theorem.

Lemma. *Let D be as above and let g be a conformal mapping from D onto $\mathcal{H}$. Then g can be extended to a homeomorphism $\tilde{g}$ from $\tilde{D}$ onto $\tilde{\mathcal{H}}$ that sends ∂D onto the positively oriented line $\mathbb{R} \cup \{\infty\}$. Moreover, if c_0, c_1, c_∞ are three distinct points on ∂D with $c_0 \prec c_1 \prec c_\infty$, then there exists exactly one conformal mapping g from D onto $\mathcal{H}$ such that $\tilde{g}(c_0) = 0$, $\tilde{g}(c_1) = 1$ and $\tilde{g}(c_\infty) = \infty$.*

Proof. This is Theorem 5.7 of Kodaira [11]. ■

Let $z_0 \in \gamma_3$ and write $w_0 = T(z_0)$. Since $T(B)$ satisfies the conditions on the region D in the Lemma, thanks to the Lemma we obtain a conformal mapping $g(w)$

on $T(B)$, depending on z_0, with continuous extension to $\partial T(B)$ as a real-valued function satisfying

$$g(i) = 0, \qquad g(0) = 1, \qquad g(w_0) = \infty.$$

Hence by composition with T we get a conformal mapping $h(z) = g(T(z))$ on B with continuous extension to ∂B as a real-valued function satisfying

$$h(i) = g(i) = 0, \qquad h(i\infty) = g(0) = 1, \qquad h(z_0) = g(w_0) = \infty.$$

Clearly, the mapping $h(z)$ depends on z_0. Let now consider the region

$$F = \{z \in \mathcal{H} : -\frac{\lambda}{2} < x < \frac{\lambda}{2}, |z| > 1\},$$

denote by $(B \cap \mathcal{H})^*$ the reflection of $B \cap \mathcal{H}$ with respect to the line $x = 0$ and let $\gamma_6 = \gamma_1 \setminus \{i\}$. Since

$$F = (B \cap \mathcal{H}) \cup \gamma_6 \cup (B \cap \mathcal{H})^*,$$

by the Schwarz reflection principle (see e.g. Sect.5.3, part a, of Kodaira [11]) the function $h(z)$ continues as a conformal mapping on F with continuous extension as a real-valued function on the boundary (excluding the part of the boundary on the real axis). Since F (union half of its boundary, excluding the part on the real axis) is a fundamental domain for $G(\lambda)$, by repeated applications of the reflection principle we obtain the analytic continuation of $h(z)$ to the whole upper half-plane $\mathcal{H}$. From the above construction it is clear that:

- $h(z)$ is $G(\lambda)$-invariant,
- $h(z)$ tends to 1 as $z \to i\infty$ and hence $h(z) \neq 1$ on $\mathcal{H}$,
- $h(z)$ is bounded on $\mathcal{H}$,
- $h'(z)$ never vanishes apart from a simple zero at $z = i$ and at its $G(\lambda)$-orbit,
- $h(z)$ has analytic continuation to the lower part of B and tends to ∞ only as $z \to z_0$.

In particular, as z_0 varies the functions $h(z)$ are linearly independent. Moreover, thanks to the fact that $z = i$ is a double zero and to the monodromy theorem, the function $h(z)$ is the square of a holomorphic function on $\mathcal{H} \cup B$, denoted by $\sqrt{h(z)}$ and satisfying $\sqrt{h(z+\lambda)} = \sqrt{h(z)}$ and $\sqrt{h(-1/z)} = -\sqrt{h(z)}$; see p.28 of [2] (with a different notation).

The above construction clearly follows Hecke's arguments in [3], but provides some missing details concerning his construction of a function analogous to $h(z)$. Indeed, Hecke's construction starts with the region

$$B' = \{z \in \mathbb{C} : -\frac{\lambda}{2} < x < 0, |z| > 1\},$$

which is unbounded at $\pm i\infty$. However, since $\pm i\infty$ coincide on the Riemann sphere, the boundary of $T(B')$ is not homeomorphic to $\mathbb{R} \cup \{\infty\}$, hence in particular the Lemma does not apply. No reference is given in [3] to an appropriate version of the Riemann mapping theorem in order to justify all steps of the construction; subsequent treatments do not clarify this point, and we were unable to trace such a reference. Our truncation of the region simplifies the picture and allows to use standard references.

Next we follow Hecke's arguments, keeping track of certain properties of the Fourier coefficients. From the above properties we deduce that $h(z)$ has the Fourier expansion

$$h(z) = \sum_{n=0}^{\infty} a_n e^{2\pi inz/\lambda}, \qquad a_0 = 1,\ a_1 \neq 0,\ y > 0.$$

By the construction of $h(z)$ we have

$$h(-\lambda/2 + iy) = \sum_{n=0}^{\infty} (-1)^n a_n e^{-2\pi ny/\lambda} \in \mathbb{R}$$

and hence $a_n \in \mathbb{R}$. Moreover, since $h(z)$ is bounded on $\mathcal{H}$, a standard argument (see e.g. Lemma 2.2 of [2]) shows that the coefficients a_n are bounded as well. In order to construct the required modular forms, consider the holomorphic function on $\mathcal{H}$

$$\begin{aligned} k(z) = \frac{h'(z)}{h(z)-1} &= \frac{2\pi i}{\lambda} \frac{\sum_{n=1}^{\infty} n a_n e^{2\pi inz/\lambda}}{\sum_{n=1}^{\infty} a_n e^{2\pi inz/\lambda}} = \frac{2\pi i}{\lambda} \frac{\sum_{n=1}^{\infty} n a_n e^{2\pi i(n-1)z/\lambda}}{\sum_{n=1}^{\infty} a_n e^{2\pi i(n-1)z/\lambda}} \\ &= \sum_{n=0}^{\infty} b_n e^{2\pi inz/\lambda}, \end{aligned}$$

where $b_n \in i\mathbb{R}$, $b_0 \neq 0$ and $b_n = O(n^c)$ for some $c > 0$. Indeed, the inverse of a power series with coefficients of polynomial growth still has coefficients of polynomial growth, and the same holds for the product of two such power series. Since $h(z)$ is $G(\lambda)$-invariant, $\sqrt{h(z)}$ has also a Fourier expansion, which we may assume to be of type

$$\sqrt{h(z)} = \sum_{n=0}^{\infty} c_n e^{2\pi inz/\lambda}, \qquad c_0 = 1,\ c_n \in \mathbb{R}$$

with $c_n = O(n^c)$ for some $c > 0$ (this can be checked by a recursive argument). Hence the function $\ell(z) = k(z)/\sqrt{h(z)}$ is holomorphic and non-vanishing on $\mathcal{H}$, and has an expansion of type

$$\ell(z) = \frac{k(z)}{\sqrt{h(z)}} = i \sum_{n=0}^{\infty} d_n e^{2\pi inz/\lambda}$$

with $d_n \in \mathbb{R}$, $d_0 \neq 0$ and $d_n = O(n^c)$ for some $c > 0$. Finally, for $k > 0$ we consider

$$f(z) = \exp(\frac{k}{2} \log \ell(z)) = i^{k/2} \sum_{n=0}^{\infty} e_n e^{2\pi inz/\lambda}$$

with $e_n \in \mathbb{R}$, $e_0 \neq 0$ and $e_n = O(n^c)$ for some $c > 0$. Clearly, $f(z)$ is holomorphic on $\mathcal{H}$ and satisfies condition *(α)*. Moreover, thanks to the above properties of $\sqrt{h(z)}$ and to the fact that $h'(-1/z) = z^2 h'(z)$, we see that $f(z)$ also satisfies condition *(β)* with $\omega = 1$, hence $f \in \mathcal{M}(\lambda, k, 1)$. To get functions $\tilde{f} \in \mathcal{M}(\lambda, k, -1)$, in view of $\sqrt{h(-1/z)} = -\sqrt{h(z)}$ one has simply to multiply by $\sqrt{h(z)}$, thus getting

$$\tilde{f}(z) = \sqrt{h(z)} f(z) = i^{k/2} \sum_{n=0}^{\infty} f_n e^{2\pi i n z/\lambda}$$

with $f_n \in \mathbb{R}$, $f_0 \neq 0$ and $f_n = O(n^c)$ for some $c > 0$. An easy way to obtain functions in $\mathcal{M}(\lambda, k, \pm 1)$ that are linearly independent as z_0 varies is to multiply the above $f(z)$ and $\tilde{f}(z)$ by a sufficiently large integer power of $h(z)$, in order to "create a pole" at $z = z_0$. In such a way, for every $k > 0$ and $\omega = \pm 1$ we get uncountably many linearly independent functions $f \in \mathcal{M}(\lambda, k, \omega)$, thus proving the folklore statement that the dimension of $\mathcal{M}(\lambda, k, \omega)$ (or equivalently of $\mathcal{D}(\lambda, k, \omega)$ and $D(\lambda, k, \omega)$) is uncountable when $\lambda > 2$. We denote by $\mathcal{F}(\lambda, k, \omega)$ the set of such functions. Note that every $f \in \mathcal{F}(\lambda, k, \omega)$ has an expansion of type (1.2) with $a_n = i^{k/2} g_n$, $g_n \in \mathbb{R}$, $g_0 \neq 0$ and $g_n = O(n^c)$ for some $c > 0$.

Now we show how the second part of the Theorem follows from the above result. Choose $\omega = 1$ and λ, k such that $Q = \lambda/2\pi$ and $\mu = (k-1)/2$. It is well known that a function $f \in \mathcal{M}(\lambda, k, 1)$ gives rise to an entire Dirichlet series in $D(\lambda, k, 1)$ if and only if its 0-th Fourier coefficient a_0 vanishes. Let $f_0 \in \mathcal{F}(\lambda, k, 1)$ be fixed and for every $f \in \mathcal{F}(\lambda, k, 1) \setminus \{f_0\}$ let $\alpha \in \mathbb{R}$ be such that

$$u(z) = f(z) - \alpha f_0(z)$$

has vanishing 0-th coefficient. The functions $u(z)$ form an uncountable set of linearly independent elements of $\mathcal{M}(\lambda, k, 1)$. Indeed, if

$$0 = c_1 u_1(z) + \ldots + c_N u_N(z) = c_1 f_1(z) + \ldots + c_N f_N(z) - (c_1\alpha_1 + \ldots + c_N \alpha_N) f_0(z)$$

identically, then $c_1 = \ldots = c_N = 0$. We now prove that the functions $i^{-k/2} u(z)$ give rise to an uncountable set of linearly independent functions in $\mathcal{S}_2^\sharp(Q, \mu, 1)$. Indeed, the Hecke correspondence theorem associates to

$$i^{-k/2} u(z) = \sum_{n=1}^{\infty} h_n e^{2\pi i n z/\lambda}, \qquad (h_n \in \mathbb{R} \text{ and } h_n = O(n^c) \text{ for some } c > 0)$$

the Dirichlet series

$$\phi(s) = \sum_{n=1}^{\infty} \frac{h_n}{n^s}.$$

Therefore, the normalized function $H(s) = \phi(s + (k-1)/2)$ satisfies the functional equation in *(ii)* (with $r = 1$, $\lambda_1 = 1$, $\mu_1 = \mu$ and $\omega = 1$) since the conjugate has no effect on real coefficients. Moreover, $H(s)$ is an entire function of finite order. Further, $H(s)$ converges absolutely for $\sigma > 1$. This follows by a standard argument (see e.g. the proofs of Theorem 5.1 and Corollary 5.2 in Iwaniec [5]), which we very briefly sketch for completeness. As $y \to 0^+$ we have $u(z) \ll y^{-k/2}$, hence by Parseval's formula we get

$$\sum_{n=1}^{\infty} |h_n|^2 e^{-4\pi n y/\lambda} \ll y^{-k}, \qquad y \to 0^+.$$

Therefore, taking $y = 1/X$ we obtain

$$\sum_{n \leqslant X} |h_n|^2 \ll X^k,$$

and the result follows by the Cauchy-Schwarz inequality and partial summation (see p.188–189 of [7] for more details). Finally, given $|\omega| = 1$ and $H \in \mathcal{S}_2^\sharp(Q, \mu, 1)$ it is easy to find $|\eta| = 1$ such that $\eta H \in \mathcal{S}_2^\sharp(Q, \mu, \omega)$, hence the second part of the Theorem follows.

3. The case $0 < q \leqslant 4$

We start with a simple observation about the possible pole at $s = 1$. By the functional equation, a pole at $s = 1$ of $F \in \mathcal{S}_2^\sharp(Q, \mu, \omega)$ induces a pole of the same order of $\Phi(s)$ at $s = 0$, hence $\Gamma(s + \mu)$ must have a pole at $s = 0$. Therefore, if $\mu > 0$ then $F(s)$ must be entire, while if $\mu = 0$ the pole at $s = 1$, if present, must be of order 1.

If $F \in \mathcal{S}_2^\sharp(Q, \mu, \omega)$ we write

$$\phi(s) = F(s - \mu), \qquad k = 2\mu + 1, \;\; \lambda = 2\pi Q$$

and note that $\phi(s)$ satisfies

$$\Big(\frac{\lambda}{2\pi}\Big)^s \Gamma(s)\phi(s) = \omega \Big(\frac{\lambda}{2\pi}\Big)^{k-s} \Gamma(k - s)\bar{\phi}(k - s). \tag{3.1}$$

Moreover, by the above observation $(s - k)\phi(s)$ is an entire function of finite order for every $\mu \geqslant 0$. An inspection of the Mellin transform argument leading to the Hecke correspondence theorem (see Theorem 1 of Ch.I of Hecke [4]) shows that (3.1) implies the modified modular relation

$$f(z) = \omega \Big(\frac{i}{z}\Big)^k f^*\Big(-\frac{1}{z}\Big). \tag{3.2}$$

Here, denoting by a_n $(n \geqslant 1)$ the Dirichlet coefficients of $\phi(s)$ and writing

$$a_0 = \omega\overline{\rho}, \qquad \rho = \mathrm{res}_{s=k} \Big(\frac{\lambda}{2\pi}\Big)^s \Gamma(s)\phi(s),$$

$f(z)$ is defined as in (1.2) and is holomorphic on $\mathcal{H}$, while $f^*(z)$ is the series in (1.2) with $\overline{a_n}$ in place of a_n for $n \geqslant 0$. Since $f^*(z) = \overline{f(-\overline{z})}$, from (3.2) we deduce that

$$f^*(z) = \overline{\omega} \Big(\frac{i}{z}\Big)^k f\Big(-\frac{1}{z}\Big),$$

and hence writing $g(z) = f(z)f^*(z)$ we have that $g(z)$ is holomorphic over $\mathcal{H}$, has a Fourier series of type (1.2) with coefficients of polynomial growth and satisfies

$$g(z) = \Big(\frac{i}{z}\Big)^{2k} g\Big(-\frac{1}{z}\Big).$$

Thus $g \in \mathcal{M}(\lambda, 2k, 1)$.

Now we recall that in Hecke's theory the computation of the dimension of the spaces $\mathcal{M}(\lambda, k, \omega)$ with $\lambda \leqslant 2$ is reduced to counting the multiplicity of the zero at $i\infty$ of $f \in \mathcal{M}(\lambda, k, \omega)$ (see Ch. 3 and 4 of [4]). Clearly, this idea carries over to the case of the solutions of functional equation (3.1) and its associated functions $f(z)$ satisfying (3.2). However, in order to perform the actual computations it is technically convenient to deal with functions satisfying the original modular relation *(β)* in the Introduction, rather than (3.2). Hence we deal with $g(z)$ instead of $f(z)$. Since the zeros of $g(z)$ contain those of $f(z)$, and $g(z)$ satisfies the modular relation *(β)*, we deduce that $\dim_{\mathbb{R}} \mathcal{S}_2^\sharp(Q, \mu, \omega)$ is finite for all values of Q for which the corresponding modular space is finite, i.e. $0 < q \leqslant 4$. Moreover, explicit bounds similar to those of Hecke's case can be obtained in this case.

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