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FREE BOUNDARY MINIMAL ANNULI IN CONVEX THREE-MANIFOLDS

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Abstract

We prove the existence of free boundary minimal annuli inside suitably convex subsets of three-dimensional Riemannian manifolds of nonnegative Ricci curvature. This includes strictly convex domains in $\mathbb{R}^3$, thereby solving an open problem [**14**] of Jost.

1. Introduction

1.1. Definitions and results. Let M be a compact three-dimensional manifold with smooth boundary and g a Riemannian metric over M. We say that a smooth compact surface Σ in M with $\partial\Sigma \subseteq \partial M$ is *free boundary minimal* with respect to the metric g whenever it has zero mean curvature, and $T\Sigma$ is orthogonal to $T\partial M$ at every point of $\partial\Sigma$.

Free boundary minimal surfaces are precisely the critical points of the area functional for surfaces in M with boundary in ∂M. These surfaces were already studied in the nineteenth century, notably with Schwarz's work on Gergonne's problem (cf., for example, [**6**]), and have since attracted the interest of numerous mathematicians, including Courant [**4**], Lewy [**16**], Meeks and Yau [**18**], Smyth [**22**], Nitsche [**19**], Ros [**20**], and Fraser and Schoen [**8**, **9**], to name but a few.

The problem of existence of free boundary minimal disks in domains of $\mathbb{R}^3$ diffeomorphic to the three-ball was studied in the mid-eighties by Struwe [**23**], using the α-pertubed method of Sacks–Uhlenbeck for parametric surfaces, and by Grüter and Jost [**12**], using several ingredients from geometric measure theory, including the min–max theory of Almgren–Pitts. In particular, Grüter and Jost showed the existence of properly embedded free boundary minimal disks inside strictly convex subsets of $\mathbb{R}^3$. In both cases, the techniques used leave open the problem of existence of free boundary minimal surfaces of non-trivial prescribed topology. We prove existence for the case of annuli, which, in particular, solves the open problem [**14**] of Jost:

Theorem 1.1. *If $K \subseteq \mathbb{R}^3$ is a compact, strictly convex subset of $\mathbb{R}^3$ with smooth boundary, then there exists a properly embedded free boundary minimal annulus Σ in K.*

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Remark 1.2. In fact, the techniques of this paper also recover the result [**12**] of Grüter and Jost (cf. Remark 6.20).

We actually prove a more general existence result for free boundary minimal annuli inside suitably convex subsets of three-manifolds with nonnegative Ricci curvature, of which Theorem 1.1 is an immediate consequence. Existence results for free boundary minimal surfaces in general Riemannian manifolds have appeared in the literature before. Recently, in [**17**], using Almgren–Pitts' min–max theory, Li proved a general existence result for properly embedded free boundary minimal surfaces in arbitrary three-manifolds with boundary. This result assumes no curvature conditions on the boundary and, in addition, using recent ideas from [**5**] of De Lellis and Pellandini, provides genus bounds for the resulting surfaces. In particular, whenever the ambient manifold is diffeomorphic to the three-ball, Li's result implies the existence of an oriented free boundary minimal surface of genus zero, but it gives no information on the number of connected components of the boundary. We refer the interested reader to the introduction of [**17**] for a discussion on other existence results for free boundary minimal surfaces. Our general result can be stated as follows:

Theorem 1.3. *If (M, g) is a smooth, compact, functionally strictly convex Riemannian three-manifold of nonnegative Ricci curvature, then there exists a properly embedded annulus $\Sigma \subseteq M$ which is free boundary minimal with respect to g.*

We clarify the notion of convexity used here. (M, g) is said to be *functionally strictly convex* whenever there exists a smooth function $f : M \to [0, 1]$ which is strictly convex with respect to the metric g and whose restriction to ∂M is constant and equal to 1 (recall that f is said to be *strictly convex* with respect to a given metric whenever its Hessian is everywhere positive definite). In particular, if M is an open subset of $\mathbb{R}^3$ with smooth boundary, and if δ is the Euclidean metric over $\mathbb{R}^3$, then (M, δ) is functionally strictly convex if and only if it is strictly convex in the usual sense. Theorem 1.1 is, therefore, an immediate consequence of Theorem 1.3.

In more general manifolds, functional strict convexity trivially implies strict convexity in the usual sense although the converse does not in general hold. We use this concept because the space of functionally strictly convex manifolds is connected (cf. Proposition 2.1, below) and this is a necessary prerequisite for the degree theoretic techniques of this paper to be applied. Although other connected spaces of manifolds with locally strictly convex boundary can be constructed (using, for example, [**11**]), we feel the condition of functional strict convexity is the simplest (though cf. the comment added in proof, Section 1.4, below).

1.2. Idea of the proof. Theorem 1.3 is proven using a differential topological technique inspired by the work [**26**] of White. We reason as follows. Let Σ be a compact oriented surface with boundary. Let $\mathcal{E}$ be the space of equivalence classes $[e]$ of embeddings $e : \Sigma \to M$ modulo reparametrisation. Let $(g_x)_{x\in X}$ be a smooth family of Riemannian metrics with *positive* Ricci curvature parametrised by a compact, connected, finite-dimensional manifold X (possibly with non-trivial boundary). Let $\mathcal{Z}(X) \subseteq X \times \mathcal{E}$ be the set of all pairs $(x, [e])$ such that e is free boundary minimal with respect to g_x, and let $\Pi : \mathcal{Z}(X) \to X$ be the projection onto the first factor. Π is trivially continuous, and, by the compactness result of [**7**], Π is proper.

If $\mathcal{Z}(X)$ were a finite-dimensional differential manifold with the same dimension as X and if, moreover, Π were to map $\partial\mathcal{Z}(X)$ into ∂X, then it would follow from classical differential topology (cf. [**13**]) that Π would have a well-defined $\mathbb{Z}_2$-valued mapping degree. If, in addition, both X and $\mathcal{Z}(X)$ were shown to be orientable, then this degree could be taken to be integer-valued. Furthermore, this mapping degree would be independent of X, and since knowing $\Pi^{-1}(Y)$ for any subset Y of X amounts to knowing the space of free boundary minimal embeddings for any given metric, our existence result would then follow. We show that, although $\mathcal{Z}(X)$ might not necessarily have the aforementioned properties, X may be embedded into a higher dimensional manifold $\tilde{X}$ for which these properties do indeed hold. The proof of Theorem 1.3 for metrics with positive Ricci curvature follows by showing this degree to be non-zero when Σ is topologically an annulus. The result for non-negative Ricci curvature then follows by a straightforward limiting argument.

1.3. Overview of the paper. The reader familiar with the work [**26**] of White will notice both similarities and differences to his approach. The key observation in the current setting is that the Jacobi operator $\mathrm{J} := (\mathrm{J}^h, \mathrm{J}^\theta)$, which measures the perturbations of the mean curvature and of the boundary angle resulting from a normal perturbation of the embedding, actually defines a *Fredholm map of Fredholm index zero* (Proposition 2.15). This brings free boundary problems within the scope of White's analysis with minimal technical modifications. We have nonetheless chosen to further adapt White's ideas in two respects, which, although not strictly necessary in the current context, will be of use, we believe, for future applications. First, we have chosen a non-variational approach, treating free boundary minimal surfaces as zeroes vector fields over infinite-dimensional manifolds rather than as critical points of functionals. This allows one to study not only free boundary minimal surfaces (which are variational), but also other, non-variational, notions of curvature such as, for example, extrinsic curvature. Second, whereas White studies the problem by constructing infinite dimensional

Banach manifolds of solutions, we focus instead on finite dimensional sections of the solution space. This allows one to treat a larger class of functionals over the solution space (such as, for example, the weakly smooth functionals introduced by the third author in [**21**]). Finally, the explicit calculation of the degree carried out in Section 6 requires considerable modifications of White's argument in order to adapt it to the very different geometrical setting studied here.

The paper is structured as follows. We underline that we have preferred to sacrifice brevity in the interests of clarity and of obtaining a relatively self-contained text.

1.3.1. Section 2. We construct the framework to be used throughout the paper. We introduce the space $\mathcal{E}$ of reparametrisation equivalence classes of embeddings, e, of a given surface, Σ, into M such that $e(\partial\Sigma) \subseteq \partial M$. For any finite dimensional family, $X := (g_x)_{x\in X}$, of metrics, we define the solution space $\mathcal{Z}(X)$ as outlined above, and we define $\Pi : \mathcal{Z}(X) \longrightarrow X$ to be the projection onto the first factor. At this stage, we are only interested in $\mathcal{E}$ and $\mathcal{Z}(X)$ as topological spaces with the obvious topologies: more sophisticated structures will be introduced in Section 3. It follows that Π is continuous and, by recent work of Fraser and Li [**7**], Π is also proper. The formal construction of a $\mathbb{Z}$-valued mapping degree of Π and its explicit calculation in certain cases constitute the main aims of this paper.

The remainder of Section 2 is devoted to studying the infinitesimal theory of extremal embeddings. In Section 2.2, we calculate the Jacobi operator $\mathrm{J} := (\mathrm{J}^h, \mathrm{J}^\theta)$ of an embedding, where J^h is the usual Jacobi operator of mean curvature, and J^θ measures the perturbation of the boundary angle arising from a normal perturbation of the embedding. In Section 2.3, we calculate the perturbation operator $\mathrm{P} := (\mathrm{P}^h, \mathrm{P}^\theta)$ of an embedding, which measures the perturbations of mean curvature and of the boundary angle arising from perturbations of the ambient metric. In Section 2.4 we review the general theory of elliptic operators, and in Section 2.5 we show that J defines a Fredholm map of Fredholm index zero. As indicated above, this key observation allows us to extend the degree theory of [**26**] to the current context with minimal technical difficulty.

1.3.2. Section 3. We introduce the local theory of extremal embeddings. In Section 3.1 we introduce "graph charts" which map open subsets of $\mathcal{E}$ homeomorphically onto open subsets of $C^\infty(\Sigma)$. Viewing these charts as coordinate charts, we treat $\mathcal{E}$ formally as an infinite dimensional manifold. Within a given graph chart, we define the mean curvature and boundary angle functionals, H and Θ, respectively. The zero-set of the pair (H, Θ) coincides over each chart with the solution space $\mathcal{Z}(X)$, making $\mathcal{Z}(X)$ amenable to standard functional analytic techniques. In Section 3.2, we review the theories of Hölder spaces and of smooth maps over Banach spaces. In Section 3.3, we study the relationship between the functionals H and Θ and the perturbation and Jacobi operators introduced in Sections 2.2 and 2.3.

It is important to note the care required in carrying out this construction as, in contrast to the usual theory of differential manifolds, the transition maps between graph charts are not smooth. Fortunately, this does not present a serious problem in the current context, since it follows from elliptic regularity, as we shall see in Section 4, that the restrictions of the transition maps to the solution space are indeed smooth, justifying the differential manifold formalism used.

1.3.3. Section 4. We show how to extend X so that $\mathcal{Z}(X)$ carries the structure of a smooth compact oriented finite-dimensional differential manifold, possibly with boundary. In Section 4.1, we extend X so that the functional (H, Θ) defined over each chart in Section 3.1 has surjective derivative at every point of $\mathcal{Z}(X)$. In Section 4.2, we use ellipticity together with the standard theory of smooth functionals over Banach spaces to show that $\mathcal{Z}(X)$ then restricts to a smooth, finite-dimensional submanifold of every graph chart and that the transition maps are smooth, thus furnishing $\mathcal{Z}(X)$ with the structure of a finite dimensional differential manifold. Finally, in Section 4.3, we recall general results of functional analysis which allow us to furnish $\mathcal{Z}(X)$ with a canonical orientation form, from which it immediately follows that Π has a well-defined, integer-valued mapping degree, as desired.

1.3.4. Section 5. In order to calculate the mapping degree of Π, we should count algebraically the number of extremal embeddings for some generic, admissible metric g. The problem is that generic metrics are hard to find explicitly. In particular, in the case at hand, the natural candidate, being the Euclidean metric in a closed ball, is clearly not generic. Indeed, generic metrics are characterised by having finitely many extremal embeddings all of which are non-degenerate, but in the Euclidean case, the action of the rotation group yields a non-trivial continuum of extremal embeddings out of every extremal embedding.

In this section, we study the technique used to calculate the degree in the case where the metric g admits non-degenerate families of free boundary minimal embeddings. These are smooth families with the property that the Jacobi operator of each element of the family has kernel of dimension equal to that of the family itself. In Section 5.1, we show that if $[e]$ lies in a non-degenerate family, then for any infinitesimal perturbation δg of the metric, there exists a (more or less) unique infinitesimal perturbation δe of e such that the mean curvature of $e + \delta e$ lies in a fixed, finite-dimensional space which we identify with the cotangent space of the family at $[e]$. In Section 5.2, by perturbing the whole family we, therefore, obtain a smooth section of the cotangent bundle of this family whose zeroes correspond to free boundary minimal embeddings for the perturbed metric. In Sections 5.3 and 5.4, we show, moreover, how to choose the metric perturbation in such a manner that this section has non-degenerate zeroes, which in turn correspond to free boundary minimal embeddings with non-degenerate Jacobi operators. In short, upon perturbing the metric, we transform a non-degenerate family into a finite set of non-degenerate free boundary minimal embeddings corresponding to the zeroes of a generic section of the cotangent space of this family thus allowing us to determine its contribution to the degree.

1.3.5. Section 6. We apply the degree theory to the current setting in order to prove Theorem 1.3. Since, for topological reasons, the theory is developed for metrics of positive Ricci curvature, in Section 6.1 we use perturbation techniques to study rotationally symmetric free boundary minimal surfaces inside closed, strictly convex, geodesic balls in the three-dimensional sphere $\mathbb{S}^3(t)$. In Sections 6.2 and 6.3, by determining the dimensions of the kernels of the Jacobi operators of rotationally symmetric surfaces, we show that they define non-degenerate families of free boundary minimal embeddings, so that the results of Section 5 may be applied in order to calculate their contribution to the mapping degree. In Section 6.4, we adapt White's symmetry argument (cf. [**26**]) to the current context, showing that even though there may exist other extremal embeddings, their contribution to the mapping degree is zero. Finally, combining these results yields the mapping degree and the proof Theorem 1.3.

1.4. Comment added in proof. While this paper was under review, A. Ache, H. Wu and the second author proved in [**1**] that the space of smooth Riemannian manifolds with non-negative Ricci curvature and strictly convex boundary is path connected. Since the property of functional strict convexity was introduced precisely to ensure path-connectivity, it is no longer necessary in the statement of Theorem 1.3.

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2. The global and infinitesimal theories

2.1. The solution space. Let M be a compact three-manifold with boundary and let Σ be a compact surface with boundary. Throughout the sequel, we will assume that all manifolds are smooth and oriented. We denote by $\hat{\mathcal{E}}$ the space of all proper embeddings $e : \Sigma \to M$ with the properties that $e(\partial\Sigma) \subseteq \partial M$ and $e(\partial\Sigma) = e(\Sigma) \cap \partial M$. We furnish this space with the topology of C^∞ convergence. We say that two embeddings $e, e' \in \hat{\mathcal{E}}$ are *equivalent* whenever there exists an orientation-preserving diffeomorphism $\alpha : \Sigma \to \Sigma$ such that $e' = e \circ \alpha$. We denote by $\mathcal{E}$ the space of equivalence classes $[e]$ of elements e of $\hat{\mathcal{E}}$ furnished with the quotient topology.

A metric g over M is said to be *admissible* whenever it has positive Ricci curvature and there exists a smooth function $f : M \to [0,1]$ which is strictly convex with respect to g and whose restriction to ∂M is constant and equal to 1. We introduce this concept for the following reason:

Proposition 2.1. *The space of admissible metrics over M is connected.*

Proof. First suppose that M is a geodesic ball of small radius about some point p_0. Let d be the distance in M to p. Upon composing with the exponential map, we may consider M as a ball about the origin in $\mathbb{R}^3$. For $t \in (0,1]$, define the metric $g_t(x) := t^{-2}M_t^* g$ where $M_t x = tx$, and let g_0 be the Euclidean metric. For all $(s,t) \in [0,1]^2$, denote $g_{t,s} := e^{-2sd^2} g_t$ and let $\mathrm{Rc}^{t,s}$ be its Ricci curvature. Observe that:

$$\left.\frac{\partial}{\partial s}\right|_{s=0} \mathrm{Rc}^{t,s} = (n-2)\mathrm{Hess}(d^2) + (\Delta d^2) g_t.$$

When M has sufficiently small radius, d^2 is strictly convex with respect to g_t for all t, and there, therefore, exists $\epsilon > 0$ such that for all $(t,s) \in ([0,1] \times [0,\epsilon]) \setminus \{(0,0)\}$, $g_{t,s}$ has positive Ricci curvature. In particular, (M,g) lies in the same connected component as $(B_r(0), g_{0,s})$ for all sufficiently small $(r,s) \neq (0,0)$, and the result follows in this case.

Now let g be any admissible metric over M. Observe that since ∂M is locally strictly convex, the shortest curve in M joining any two interior points is a geodesic which does not touch the boundary. It follows by strict convexity that f has a unique global minimum, p, say, in M and no other critical point. Without loss of generality, we may assume that $f(p) = 0$. For all t, denote $M_t := f^{-1}([0,t])$. By uniqueness of p, there exists a smooth family of diffeomorphisms $\varphi_{1,t} : M \to M_t$. Now let $r > 0$ be such that the closure of $B_r(p)$ is contained in the interior of M and d^2 is strictly convex over this ball. For all $s \in [0,1]$, we define $f_s = (1-s)f + sd^2$, and for all t, we denote $M_{t,s} := f_s^{-1}([0,t])$. For sufficiently small ϵ, there exists a smooth family of diffeomorphisms $\varphi_s : M_\epsilon \to M_{\epsilon,s}$. In particular, (M,g) lies in the same connected component as $(M_{\epsilon,1}, g)$, and the result now follows by the preceding discussion. q.e.d.

Let X be a compact, finite-dimensional manifold possibly with non-trivial boundary, and let $g : X \times M \to \mathrm{Sym}^+ TM$ be a smooth function with the property that $g_x := g(x, \cdot)$ is an admissible metric for all $x \in X$. We henceforth refer to the pair (X, g) simply by X. We define $\mathcal{Z}(X) \subseteq X \times \mathcal{E}$ to be the set of all pairs $(x, [e])$ such that e is a free boundary minimal embedding with respect to the metric g_x. We describe $\mathcal{Z}(X)$ as the zero set of a functional. Indeed, for $(x, [e]) \in X \times \mathcal{E}$ we denote by $N : \Sigma \longrightarrow TM$ the unit normal vector field over e with respect to g_x which is compatible with the orientation and we denote by $A : \Sigma \longrightarrow \mathrm{End}(T\Sigma)$ and $H : \Sigma \longrightarrow \mathbb{R}$ the corresponding *shape operator* and *mean curvature*, respectively. That is, at each point $p \in \Sigma$:

$$H = \mathrm{tr}\, A.$$

We denote by ν the outward-pointing unit normal vector field over ∂M with respect to g_x, and we denote by $\Theta : \partial\Sigma \longrightarrow \mathbb{R}$ the *boundary angle* that $e(\Sigma)$ makes with ∂M with respect to this metric. That is, at each $p \in \partial\Sigma$:

$$\Theta = g\left(\nu, N\right).$$

Remark 2.2. The geometric quantities we have just defined depend on (x, e). To avoid confusion, we make this dependence explicit in our notation by writing $N_{x,e}, H_{x,e}, \Theta_{x,e}$, etc.

We define the *solution space* $\mathcal{Z}(X) \subseteq X \times \mathcal{E}$ by:

$$\mathcal{Z}(X) = \left\{(x, [e]) \in X \times \mathcal{E} \mid H_{x,e} = 0,\ \Theta_{x,e} = 0\right\},$$

and we define $\Pi : \mathcal{Z}(X) \to X$ to be the projection onto the first factor. Since both $H_{x,e}$ and $\Theta_{x,e}$ are equivariant under reparametrisation, this definition is consistent.

The main objective of this paper is to construct a $\mathbb{Z}$-valued mapping degree for the projection Π. A key element of this construction is the following compactness result:

Theorem 2.3 (Fraser–Li [**7**]). *Let $(g_m)_{m\in\mathbb{N}}$ be a sequence of metrics over M of nonnegative Ricci curvature. Let $(e_m)_{m\in\mathbb{N}} : \Sigma \longrightarrow M$ be a sequence of embeddings such that, for all m, e_m is a free boundary minimal embedding with respect to the metric g_m. If there exists a metric g_∞ over M towards which $(g_m)_{m\in\mathbb{N}}$ converges in the C^∞ sense, and if ∂M is strictly convex with respect to g_∞, then there exists an embedding $e_\infty : \Sigma \to M$ and a sequence $(\alpha_m)_{m\in\mathbb{N}} : \Sigma \to \Sigma$ of diffeomorphisms of Σ such that $(e_m \circ \alpha_m)_{m\in\mathbb{N}}$ subconverges towards e_∞ in the C^∞ sense. In particular, e_∞ is a free boundary minimal embedding with respect to the metric g_∞.*

In our current framework, this is restated (in slightly weaker form) as follows:

Proposition 2.4. *Let $\Pi : X \times \mathcal{E} \longrightarrow X$ be the projection onto the first factor. Then the restriction of Π to $\mathcal{Z}(X)$ is proper.*

If $\mathcal{Z}(X)$ were a finite-dimensional differential manifold with boundary of dimension equal to that of X and if, moreover, Π were to map $\partial\mathcal{Z}(X)$ into ∂X, then it would follow from classical differential topology that Π has a well-defined $\mathbb{Z}_2$-valued mapping degree. Furthermore, this degree would be independent of X, and if, in addition, both X and $\mathcal{Z}(X)$ were orientable, then it could be taken to be integer-valued. The main objective of Sections 3 and 4 below is to show that although $\mathcal{Z}(X)$ does not necessarily have the aforementioned properties, X may be embedded into a higher dimensional manifold $\tilde{X}$ for which these properties actually hold. This is summarised in Theorem 4.11 of Section 4. The existence result of Theorem 1.3 then follows upon showing this degree to be non-zero in the case treated there. To this end, we require in particular Theorem 5.12, which determines how smooth, non-degenerate families of solutions contribute to the degree. Theorems 4.11 and 5.12 together constitute the main results of Sections 3, 4 and 5, and the first-time reader may skim the rest, passing directly to Section 6 after completing Section 2 without losing much understanding.

We devote the remainder of this section to studying the infinitesimal theory of minimal embeddings with free boundary. Our goal is to prove that the Jacobi operator $\mathrm{J} := (\mathrm{J}^h, \mathrm{J}^\theta)$, which measures the perturbation of mean curvature as well as the perturbation of the boundary angle resulting from a normal perturbation of the embedding, defines a Fredholm map of Fredholm index zero.

2.2. Jacobi operators. Given $(x, [e]) \in \mathcal{Z}(X)$, we denote by $\mathrm{J}^h : C^\infty(\Sigma) \to C^\infty(\Sigma)$ and by $\mathrm{J}^\theta : C^\infty(\Sigma) \to C^\infty(\partial\Sigma)$, respectively, the *Jacobi operator of mean curvature* of e and the *Jacobi operator of the boundary angle* of e with respect to g_x. That is, J^h and J^θ are defined such that if $f : (-\delta, \delta) \times \Sigma \longrightarrow M$ is a smooth map with the properties that $e = f(0, \cdot)$, $e_t := f(t, \cdot)$ is an embedding for all t, and $\frac{\partial f}{\partial t}\big|_{t=0} = \varphi N$ for some $\varphi \in C^\infty(\Sigma)$, then

$$\mathrm{J}^h\varphi = \frac{\partial}{\partial t}\bigg|_{t=0} H_{x,e_t}, \text{ and } \mathrm{J}^\theta\varphi = \frac{\partial}{\partial t}\bigg|_{t=0} \Theta_{x,e_t}.$$

We denote by Ric the Ricci curvature of g_x and by Δ the Laplacian operator of $e^* g_x$ over Σ. We recall the second variation formula for the area:

Lemma 2.5. *Given $(x, [e]) \in \mathcal{Z}(X)$, for all $\varphi \in C^\infty(\Sigma)$:*

$$\mathrm{J}^h\varphi = -\Delta\varphi - \left(\mathrm{Ric}(N, N) + \|A\|^2\right)\varphi.$$

Let II denote the shape operator of ∂M with respect to g_x and the outward pointing normal ν. Since $(x, [e]) \in \mathcal{Z}(X)$, along the boundary of Σ, the vector N lies in the tangent space of ∂M, and we define $\kappa : \partial\Sigma \longrightarrow \mathbb{R}$ by:

$$\kappa = II(N, N).$$

Moreover, the vector field $\nu \circ e$ coincides with the conormal to $e(\partial\Sigma)$ inside $e(\Sigma)$ with respect to g_x, and we, therefore, define the operator $\partial_\nu : C^\infty(\Sigma) \longrightarrow C^\infty(\partial\Sigma)$ to be the derivative in the direction of the vector field $\nu \circ e$. That is, for all $f \in C^\infty(\Sigma)$ and at each $p \in \partial\Sigma$, $\partial_\nu f = \langle e^*\nu, df\rangle$. The following result is proven in [**2**]:

Proposition 2.6. *Given $(x, [e]) \in \mathcal{Z}(X)$, for all $\varphi \in C^\infty(\Sigma)$:*

$$\mathrm{J}^\theta\varphi = \kappa\varphi \circ \epsilon - \partial_\nu\varphi,$$

where $\epsilon : \partial\Sigma \to \Sigma$ is the canonical embedding.

Again, the geometric quantities we have just defined clearly depend on $(x, [e])$. To avoid confusion, we denote $\mathrm{J}_{x,e} := (\mathrm{J}^h_{x,e}, \mathrm{J}^\theta_{x,e})$. We refer to $\mathrm{J}_{x,e}$ as the *Jacobi operator* of $[e]$ with respect to the metric g_x.

2.3. Perturbation operators. For all $(x, [e]) \in \mathcal{Z}(X)$, we denote by $\mathrm{P}^h_{x,e} : T_xX \to C^\infty(\Sigma)$ and by $\mathrm{P}^\theta_{x,e} : T_xX \longrightarrow C^\infty(\partial\Sigma)$, respectively, the *perturbation operator* of *mean curvature* of e and the *perturbation operator of the boundary angle* of e with respect to changes in the metric. That is, if $\xi \in T_xX$, if $x : (-\delta, \delta) \to X$ is a smooth curve such that $x(0) = x$ and $\dot{x}(0) = \xi$, then we define:

$$\mathrm{P}^h_{x,e}\xi = \frac{\partial}{\partial t}\bigg|_{t=0} H_{x_t,e}, \text{ and } \mathrm{P}^\theta_{x,e}\varphi = \frac{\partial}{\partial t}\bigg|_{t=0} \Theta_{x_t,e}.$$

For all $(x, [e]) \in \mathcal{Z}(X)$, we denote $\mathrm{P}_{x,e} := \left(\mathrm{P}^h_{x,e}, \mathrm{P}^\theta_{x,e}\right)$, and we refer to $\mathrm{P}_{x,e}$ as the *perturbation operator* of e with respect to changes in the metric.

It turns out only to be necessary to consider conformal perturbations of the ambient metric. Let $g : (-\delta, \delta) \times M \to \mathrm{Sym}^+(TM)$ be a smooth family of metrics. Denote $g_t := g(t, \cdot)$ for all t and $g(0) = g$. Let $e : \Sigma \to M$ be an embedding and let $N : \Sigma \to TM$ be the normal vector field over e with respect to g which is compatible with the orientation. Since conformal perturbations leave angles invariant, we immediately have:

Proposition 2.7. *If $\dot{g}(0) = \varphi g$ for $\varphi \in C^\infty(M)$, then:*

$$\frac{\partial}{\partial t}\bigg|_{t=0} \Theta_{g_t,e} = 0.$$

Direct calculation yields:

Proposition 2.8. *If $\dot{g}(0) = \varphi g$ for $\varphi \in C^\infty(M)$, then:*

$$\frac{\partial}{\partial t}\bigg|_{t=0} H_{g_t,e} = d\varphi(N) - \frac{1}{2}\varphi H_{g(0),e}.$$

This yields the following surjectivity result:

Proposition 2.9. *For all $f \in C^\infty(\Sigma)$, there exists $\varphi \in C^\infty(M)$ such that if $\dot{g}(0) = \varphi g$, then:*

$$\frac{\partial}{\partial t}\bigg|_{t=0} H_{g_t,e} = f.$$

Moreover, for any neighbourhood U of $e(\mathrm{supp}(f))$ in M, φ may be chosen such that $\mathrm{supp}(\varphi) \subseteq U$.

Proof. Indeed, let φ be such that, near $e(\Sigma)$, $\varphi = \chi d(f \circ \pi)$, where d is the distance to $e(\Sigma)$ in M, $\pi : M \to e(\Sigma)$ is the closest-point projection and χ is a smooth function equal to 1 near $e(\Sigma)$. Since $e(\Sigma)$ is embedded, we may assume that φ is smooth, and the result now follows by Proposition 2.8 with a suitable choice of χ. q.e.d.

Proposition 2.9 is already sufficient for the proof of Theorem 4.2 of Section 4. However, the following refinement will prove useful:

Proposition 2.10. *Let $f_1, ..., f_m \in C^\infty(\Sigma)$ be a basis for $\mathrm{Ker}(\mathrm{J}_{g,e})$. For $p \in \Sigma$ and U a neighbourhood of $e(p)$ in M, there exist functions $\varphi_1, ..., \varphi_m \in C^\infty(M)$, all supported in U, such that, for all $1 \leqslant i, j \leqslant m$, if g(t) is a path of metrics with $\dot{g}(0) = \varphi_i g$, where $g(0) = g$, then:*

$$\left\langle \frac{\partial}{\partial t}\bigg|_{t=0} H_{g_t,e}, f_j \right\rangle = \delta_{ij},$$

where $\langle \cdot, \cdot \rangle$ is the L^2 inner product with respect to $e^* g$ over Σ.

Remark 2.11. We will see below that $\mathrm{Ker}(\mathrm{J}_{g,e})$ is finite dimensional.

Proof. We identify Σ with its image $e(\Sigma) \subseteq M$. Let $r : C^\infty(\Sigma) \longrightarrow C^\infty(\Sigma \cap U)$ be the restriction map. For any vector $p := (p_1, ..., p_m)$ of points in $\Sigma \cap U$, we define the map $L_p : C^\infty(\Sigma \cap U) \to \mathbb{R}^m$ by:

$$L_p(f) = (f(p_1), ..., f(p_n)).$$

Since $\mathrm{J}^h_{g,e}(f_k) = 0$ for all $1 \leqslant k \leqslant m$, and bearing in mind that $\mathrm{J}^h_{g,e}$ is a second-order elliptic linear partial-differential operator, it follows from Aronszajn's unique continuation theorem (cf. [**3**]) that r restricts to a linear isomorphism from $\mathrm{Ker}(\mathrm{J}_{g,e})$ to an m-dimensional subspace of $C^\infty(\Sigma \cap U)$. There, therefore, exists a vector p such that L_p defines a linear isomorphism from $\mathrm{Ker}(\mathrm{J}_{g,e})$ to $\mathbb{R}^m$.

Observe that, for all $1 \leqslant k \leqslant m$:

$$L_p(f)_k = \langle f, \delta_{p_k} \rangle,$$

where δ_{p_k} is the Dirac-delta distribution supported at k. For any vector $\psi := (\psi_1, ..., \psi_m)$ of smooth functions in $C^\infty_0(\Sigma \cap U)$, we define the map $L_\psi : C^\infty(\Sigma \cap U) \to \mathbb{R}^m$ such that for all $1 \leqslant k \leqslant m$:

$$L_\psi(f)_k = \langle f, \psi_k \rangle.$$

Observe that, as ψ converges to $(\delta_{p_1},...,\delta_{p_m})$ in the distributional sense, L_ψ converges to L_p. There, therefore, exists a vector ψ such that L_ψ is invertible. We may suppose, moreover, that for all $1 \leqslant k \leqslant m$, ψ_k is supported in $\Sigma \cap U$. In addition, upon replacing each of $\psi_1,...,\psi_n$ by a suitable linear combination of these functions if necessary, we may suppose that:

$$\langle \psi_i, f_j \rangle = \delta_{ij},$$

for each $1 \leqslant i,j \leqslant m$. By Proposition 2.9, there exist functions $\varphi_1,...,\varphi_m \in C^\infty(M)$ such that for all $1 \leqslant k \leqslant m$, $\mathrm{supp}(\varphi_k) \subseteq U$, and if $\dot{g}(0) = \varphi_k g$, then $\frac{\partial}{\partial t}\big|_{t=0} H_{g_t,e} = \psi_k$. Thus, for all $1 \leqslant i,j \leqslant m$, if $(\partial_t g)_0 = \varphi_i g_0$, then:

$$\left\langle \frac{\partial}{\partial t}\bigg|_{t=0} H_{g_t,e}, f_j \right\rangle = \delta_{ij},$$

as desired. q.e.d.

2.4. General elliptic theory. For $\lambda \in [0,\infty] \setminus \mathbb{N}$, that is, $\lambda = k + \alpha$ where $k \in \mathbb{N} \cup \{\infty\}$ and $\alpha \in (0,1)$, and for any compact manifold Ω, we denote by $C^\lambda(\Omega)$ the space of λ-times Hölder differentiable functions over Ω. For $\lambda < \infty$, we denote by $\|\cdot\|_\lambda$ the C^λ-Hölder norm of $C^\lambda(\Omega)$ and we denote by $C^{*,\lambda}(\Omega)$ the closure of $C^\infty(\Omega)$ in $C^\lambda(\Omega)$.

For $\varphi \in C^\infty(\partial\Omega)$, we define the *Robin operator* $R_\varphi : C^{*,\lambda+1}(\Omega) \longrightarrow C^{*,\lambda}(\partial\Omega)$ such that, for all $f \in C^{*,\lambda+1}(\Omega)$:

$$R_\varphi(f) = \varphi(f \circ \epsilon) + \partial_\nu f,$$

where $\epsilon : \partial\Omega \longrightarrow \Omega$ is the canonical embedding and $\partial_\nu f$ is the derivative of f in the outward pointing conormal direction. For all $\lambda \in [0,\infty] \setminus \mathbb{N}$, we define $C^{*,\lambda+1}_{\mathrm{rob}}(\Omega)$ to be the kernel of R_φ in $C^{*,\lambda+1}(\Omega)$.

Now let Δ be the Laplacian operator of Ω. The relevant elliptic theory is encapsulated in the following result:

Proposition 2.12. *For all $\lambda \in [0,\infty[\setminus\mathbb{N}$, $(\mathrm{Id} - \Delta, R_\varphi)$ defines an invertible, linear map from $C^{*,\lambda+2}(\Omega)$ into $C^{*,\lambda}(\Omega) \times C^{*,\lambda+1}(\partial\Omega)$.*

Proof. We first prove injectivity. For all $k \in \mathbb{N}$, let $H^k(\Omega)$ be the Sobolev space of k-times L^2-differentiable functions over Ω. For all k, by the Sobolev trace theorem, R_φ defines a continuous linear map from $H^{k+2}(\Omega)$ into $H^{k+1/2}(\partial\Omega)$, so that $H^{k+2}_{\mathrm{rob}}(\Omega) := \mathrm{Ker}(R_\varphi)$ is a well-defined subspace of $H^{k+2}(\Omega)$. For all k, by Exercise 3 of Section 5.7 of [**24**], $\mathrm{Id} - \Delta$ defines an invertible linear map from $H^{k+2}_{\mathrm{rob}}(\Omega)$ into $H^k(\Omega)$. In particular, if $f \in C^{*,\lambda+2}(\Omega)$ and if $((\mathrm{Id} - \Delta)f, R_\varphi f) = 0$, then $f \in H^{k+2}_{\mathrm{rob}}(\Omega)$ and $(\mathrm{Id} - \Delta)f = 0$, so that $f = 0$, and injectivity follows.

Now choose $(u,v) \in C^\infty(\Omega) \times C^\infty(\partial\Omega)$. Choose $g \in C^\infty(\Omega)$ such that $R_\varphi(g) = v$, and denote $w := u - (\mathrm{Id} - \Delta)g$. For all k, since $w \in H^k(\Omega)$, there exists $f_k \in H^{k+2}_{\mathrm{rob}}(\Omega)$ such that $(\mathrm{Id} - \Delta)f_k = w$. By uniqueness, for all $k \neq l$, $f_k = f_l =: f$, so that $f \in \cap_{k \geqslant 0} H^k_{\text{rob}}(\Omega) = C^\infty_{\text{rob}}(\Omega)$ and $(u, v) = (\text{Id} - \Delta, R_\varphi)(f + g)$ is in the image of $(\text{Id} - \Delta, R_\varphi)$. However, by the global Schauder estimates for the oblique derivative problem (Theorem 6.30 of [**10**]), there exists $C > 0$ such that, for all $f \in C^{*,\lambda+2}(\Omega)$:

$$\|f\|_{\lambda+2} \leqslant C(\|f\|_{L^\infty} + \|(\text{Id} - \Delta)f\|_\lambda + \|R_\varphi(f)\|_{\lambda+1}),$$

and it follows in the usual manner (cf. [**24**]) that the image of $C^{*,\lambda+2}(\Omega)$ under $(\text{Id} - \Delta, R_\varphi)$ is closed in $C^{*,\lambda}(\Omega) \times C^{*,\lambda+1}(\partial\Omega)$. Surjectivity now follows by the density of $C^\infty(\Omega) \times C^\infty(\partial\Omega)$ in $C^{*,\lambda}(\Omega) \times C^{*,\lambda+1}(\partial\Omega)$, and invertibility follows by the closed graph theorem. q.e.d.

2.5. The elliptic theory of Jacobi operators. Fix $(x, [e]) \in X \times \mathcal{E}$. To simplify notation, we will drop the $(x, [e])$ dependence of the geometric quantities and operators for the remainder of this section.

We define $\text{L} : C^{*,\lambda+2}(\Sigma) \to C^{*,\lambda}(\Sigma)$ such that, for all $\varphi \in C^{*,\lambda+2}(\Sigma)$:

$$\text{L}\varphi = -\varphi - (\text{Ric}(N, N) + \|A\|^2)\varphi,$$

so that, by Lemma 2.5:

$$\text{J}^h = (\text{Id} - \Delta) + \text{L}.$$

Proposition 2.13. *For all $\xi, \eta \in C^{*,\lambda+2}(\Sigma)$ such that $\text{J}^\theta\xi = \text{J}^\theta\eta = 0$:*

$$\int_\Sigma \eta \text{J}^h \xi \, dV = \int_\Sigma \xi \text{J}^h \eta \, dV,$$

where dV is the volume form of the metric e^*g.

Proof. Indeed, for all $\xi, \eta \in C^{*,\lambda+2}_{\text{rob}}(\Omega)$, the function $\eta\partial_\nu\xi - \xi\partial_\nu\eta$ vanishes along $\partial\Omega$. The result follows by Stokes' Theorem. q.e.d.

Proposition 2.14. *For all $(x, [e]) \in X \times \mathcal{E}$, and for all $\lambda \in [0, \infty[\setminus \mathbb{N}$, if $\varphi \in C^{*,\lambda+2}(\Sigma)$ and $\text{J}\varphi \in C^\infty(\Sigma) \times C^\infty(\partial\Sigma)$, then $\varphi \in C^\infty(\Sigma)$.*

Proof. Observe that:

$$\Big((\text{Id} - \Delta)\varphi, \text{J}^\theta\varphi\Big) = \text{J}\varphi - (\text{L}\varphi, 0) \in C^{*,\lambda+2}(\Sigma) \times C^{*,\lambda+3}(\partial\Sigma).$$

Thus, by Proposition 2.12, there exists $\varphi' \in C^{*,\lambda+4}(\Sigma)$ such that:

$$\Big((\text{Id} - \Delta)\varphi', \text{J}^\theta\varphi'\Big) = \Big((\text{Id} - \Delta)\varphi, \text{J}^\theta\varphi\Big).$$

By uniqueness, $\varphi = \varphi'$, and so $\varphi \in C^{*,\lambda+4}(\Sigma)$, and it follows by induction that $\varphi \in C^\infty(\Sigma)$, as desired. q.e.d.

Proposition 2.15. *For all $(x, [e]) \in X \times \mathcal{E}$, the operator* J *defines a Fredholm map from $C^{*,\lambda+2}(\Sigma)$ to $C^{*,\lambda}(\Sigma) \times C^{*,\lambda+1}(\partial\Sigma)$ of Fredholm index zero. Moreover:*

1) *if we denote by $\text{Ker}^{\lambda+2}(\text{J})$ and $\text{Ker}(\text{J})$ the kernels of* J *in $C^{*,\lambda+2}(\Sigma)$ and $C^\infty(\Sigma)$, respectively, then:*

$$\text{Ker}^{\lambda+2}(\text{J}) = \text{Ker}(\text{J}); \text{ and}$$

2) *if we denote by* $\mathrm{Im}^{\lambda+2}(\mathrm{J})$ *the image of* J *in* $C^{*,\lambda}(\Sigma)\times C^{*,\lambda+1}(\partial\Sigma)$, *then:*

$$\mathrm{Im}^{\lambda+2}(\mathrm{J})^{\perp} = \{(f, f\circ\epsilon)\ |\ f\in\mathrm{Ker}(\mathrm{J})\}\,,$$

where the orthogonal complement is taken with respect to the L^2 *inner-product of* e^*g.

Proof. By Proposition 2.12, $(\mathrm{Id}-\Delta, \mathrm{J}_\theta)$ defines an invertible map from $C^{*,\lambda+2}(\Sigma)$ into $C^{*,\lambda}(\Sigma)\times C^{*,\lambda+1}(\partial\Sigma)$. In particular, it is Fredholm of index 0. However, $(\mathrm{L},0) := \mathrm{J}-(\mathrm{Id}-\Delta,\mathrm{J}_\theta)$ maps $C^{*,\lambda+2}(\Sigma)$ into $C^{*,\lambda+2}(\Sigma)\times C^{*,\lambda+3}(\partial\Sigma)$. It, therefore, defines a compact map from $C^{*,\lambda+2}(\Sigma)$ into $C^{*,\lambda}(\Sigma)\times C^{*,\lambda+1}(\partial\Sigma)$, and so J is also Fredholm of index 0. Moreover, by Proposition 2.14, $\mathrm{Ker}^{\lambda+2}(\mathrm{J})\subseteq\mathrm{Ker}(\mathrm{J})$, and since the reverse inclusion is trivial, these two spaces, therefore, coincide. This proves (1).

Denote by $\langle\cdot,\cdot\rangle$ the L^2 inner-product of e^*g. Bearing in mind Stokes' Theorem, for all $\varphi\in C^{*,\lambda+2}(\Sigma)$ and for all $\psi\in\mathrm{Ker}(\mathrm{J})$:

$$\begin{aligned}\langle\mathrm{J}\varphi,(\psi,\psi\circ\epsilon)\rangle &= \int_\Sigma \psi\mathrm{J}^h\varphi\,dV + \int_{\partial\Sigma}\psi\mathrm{J}^\theta\varphi\,dV\\ &= \int_\Sigma \varphi\,\mathrm{J}^h\psi\,dV + \int_{\partial\Sigma}\varphi\,\mathrm{J}^\theta\psi\,dV\\ &= 0.\end{aligned}$$

It follows that $\{(f,f\circ\epsilon)\ |\ f\in\mathrm{Ker}(\mathrm{J})\}\subseteq\mathrm{Im}^{\lambda+2}(\mathrm{J})^\perp$. However, since J is Fredholm of index zero, the dimension of the orthogonal complement of $\mathrm{Im}^{\lambda+2}(\mathrm{J})$ equals that of $\mathrm{Ker}(\mathrm{J})$, and (2) follows. This completes the proof. q.e.d.

3. The local theory

3.1. Local charts I: The smooth case. Let Y be a compact neighbourhood in X. Let $e : Y\times\Sigma\to M$ be a smooth function such that, for all $y\in Y$, $e_y := e(y,\cdot)$ is an element of $\hat{\mathcal{E}}$ with the property that $e_y(\Sigma)$ meets ∂M orthogonally along $\partial\Sigma$ with respect to g_y. We refer to the triplet (Y,g,e) simply by Y. The following result is useful for constructing local charts of the space of embeddings with boundary in ∂M:

Theorem 3.1. *There exists a neighbourhood* U *of the zero section in* TM, *and a smooth map* $\mathrm{E}: U\to M$ *with the following properties:*

(1) *If* X_p *is a vertical vector over the point* $0_p\in TM$, *then:*

$$D\mathrm{E}(0_p)(X_p) = X_p;$$

(2) *If* $X_p\in U\cap T_p\partial M$, *then:*

$$\mathrm{E}(X_p)\in\partial M.$$

Proof. Let $\hat{M}$ be the manifold obtained by doubling M along ∂M. The exponential map of any smooth metric over $\hat{M}$ which is symmetric with respect to this doubling has the desired properties. q.e.d.

Let $N : Y \times \Sigma \to M$ be such that, for all $y \in Y$, $N_y := N(y, \cdot)$ is the unit, normal vector-field over e_y with respect to g_y which is compatible with the orientation. Define $\hat{\Phi}_Y : Y \times C^\infty(\Sigma) \to C^\infty(\Sigma, M)$ such that, for all $y \in Y$, for all $f \in C^\infty(\Sigma)$ and for all $p \in \Sigma$:

$$\hat{\Phi}_Y(y, f)(p) = \mathrm{E}(f(p)N_y(p)).$$

Proposition 3.2. *There exists $r > 0$ such that for all $y \in Y$, if $\|f\|_{L^\infty} < r$, then $\hat{\Phi}_Y(y, f)$ is an element of $\hat{\mathcal{E}}$.*

Proof. Consider the map $F : Y \times \Sigma \times \mathbb{R} \to M$ given by $F(y, p, t) = \mathrm{E}(tN_y(p))$. For all y, we denote $F_y := F(y, \cdot, \cdot)$. Observe that, for all $y \in Y$, and for all $p \in \partial\Sigma$, $N_y(p)$ is tangent to ∂M, so that, by definition of E, $F(y, p, t) \in \partial M$ for all t. Furthermore, for all $y \in Y$ and for all $p \in \Sigma$, DF_y is bijective at $(p, 0)$, and since every e_y is an embedding, there, therefore, exists $r > 0$ such that, for all $y \in Y$, the restriction of F_y to $\Sigma \times (-r, r)$ is also an embedding. Now, for $f \in C^\infty(\Sigma)$, we define $\hat{f} \in C^\infty(\Sigma, \Sigma \times \mathbb{R})$ by $\hat{f}(p) := (p, f(p))$. If $\|f\|_{L^\infty} < r$, then $\hat{f}$ trivially defines an embedding of Σ into $\Sigma \times (-r, r)$, and so, for all $y \in Y$, $\hat{\Phi}_Y(y, f) = F_y \circ \hat{f}$ defines an embedding of Σ into M. The result follows. q.e.d.

We define $\mathcal{U}_Y \subseteq Y \times C^\infty(\Sigma)$ by:

$$\mathcal{U}_Y = \{(y, f) \mid \|f\|_{L^\infty} < r\},$$

where r is as in Proposition 3.2. We define $\Phi_Y : \mathcal{U}_Y \to \mathcal{E}$ and $\Psi_Y : \mathcal{U}_Y \to Y \times \mathcal{E}$ such that for all $(y, f) \in \mathcal{U}_Y$:

$$\Phi_Y(y, f) = [\hat{\Phi}_Y(y, f)], \text{ and } \Psi_Y(y, f) = (y, [\hat{\Phi}_Y(y, f)]).$$

Proposition 3.3. *Ψ_Y is injective.*

Proof. Let $(y, f), (y', f') \in \mathcal{U}_Y$ be such that $\Psi_Y(y, f) = \Psi_Y(y', f')$. In particular, $y = y'$ and $\Phi_Y(y, f) = \Phi_Y(y', f')$. There, therefore, exists an orientation-preserving diffeomorphism α of Σ such that $\hat{\Phi}_Y(y, f') = \hat{\Phi}_Y(y, f) \circ \alpha$. Let r be as in Proposition 3.2 and define $\hat{f}, \hat{f}' : \Sigma \longrightarrow \Sigma \times (-r, r)$ by $\hat{f}(p) = (p, f(p))$ and $\hat{f}'(p) = (p, f'(p))$. Define $F_y : \Sigma \times (-r, r) \to M$ by $F_y(p, t) = \mathrm{E}(tN_y(p))$. By definition of $\hat{\Phi}_Y$:

$$F_y \circ \hat{f} \circ \alpha = \hat{\Phi}_Y(y, f) \circ \alpha = \hat{\Phi}_Y(y, f') = F_y \circ \hat{f}'.$$

Since F_y is an embedding, this yields, for all $p \in \Sigma$:

$$(\alpha(p), (f \circ \alpha)(p)) = (\hat{f} \circ \alpha)(p) = \hat{f}'(p) = (p, f'(p)).$$

It follows that α coincides with the identity and f' coincides with f, and Ψ_Y is, therefore, injective as desired q.e.d.

Proposition 3.4. *Ψ_Y is an open map.*

Proof. Choose $(y,f)\in\mathcal{U}_Y$ and let Ω be a neighbourhood of (y,f) in $\mathcal{U}_Y$. Denote $(y,[e])=\Psi_Y(y,f)$ and let $(y_m,[e_m])_{m\in\mathbb{N}}\in Y\times\mathcal{E}$ be a sequence converging to $(y,[e])$. In particular, $(y_m)_{m\in\mathbb{N}}$ converges to y. Let r be as in Proposition 3.2. We define $\hat{f}:\Sigma\to\Sigma\times(-r,r)$, by $\hat{f}(p)=(p,f(p))$, and $F:Y\times\Sigma\times(-r,r)\to M$, by $F(y,p,t):=\mathrm{E}(tN_y(p))$. By definition, $[e]=[F_y\circ\hat{f}]$. Since $([e_m])_{m\in\mathbb{N}}$ converges to $[e]$, there exists a sequence $(\alpha_m)_{m\in\mathbb{N}}$ of orientation-preserving diffeomorphisms of Σ such that $(e_m\circ\alpha_m)_{m\in\mathbb{N}}$ converges to $F_y\circ\hat{f}$. Bearing in mind that, in addition, $(y_m)_{m\in\mathbb{N}}$ converges to y, there exists $K\in\mathbb{N}$ such that for all $m\geqslant K$, $(e_m\circ\alpha_m)$ takes values in $F_{y_m}(\Sigma\times(-r,r))$ and that $(F_{y_m}^{-1}\circ e_m\circ\alpha_m)_{m\geqslant K}$ converges to $\hat{f}$.

Let $\pi_1:\Sigma\times\mathbb{R}\to\Sigma$ and $\pi_2:\Sigma\times\mathbb{R}\to\mathbb{R}$ be the canonical projections onto the first and second factors, respectively. For all $m\geqslant K$, we denote:

$$\beta_m:=\pi_1\circ F_{y_m}^{-1}\circ e_m\circ\alpha_m,\ \text{and}\ \widetilde{f}_m:=\pi_2\circ F_{y_m}^{-1}\circ i_m\circ\alpha_m.$$

Observe that $(\beta_m)_{m\geqslant K}$ converges to the identity map. Thus, upon increasing K if necessary, we may assume that β_m is a diffeomorphism for all m and that $(\beta_m)_{m\geqslant K}^{-1}$ also converges to the identity map. For all $m\geqslant K$, we denote:

$$f_m:=\tilde{f}_m\circ\beta_m^{-1}.$$

Since $(\tilde{f}_m)_{m\geqslant M}$ converges to f, so too does $(f_m)_{m\in\mathbb{N}}$. In particular, upon increasing K further if necessary, we may assume that $(y_m,f_m)\in\Omega$ for all m. However, for all m, $e_m\circ\alpha_m\circ\beta_m^{-1}=\hat{\Phi}_Y(y_m,f_m)$. In other words:

$$(y_m,[e_m])=(y_m,\Phi(Y)(y_m,f_m))=\Psi(Y)(y_m,f_m).$$

It follows that $(y_m,[e_m])\in\Psi_Y(\Omega)$ for all $m\geqslant K$, and we conclude that Ψ_Y is an open map as desired. q.e.d.

We denote the image $\Psi_Y(\mathcal{U}_Y)$ in $Y\times\mathcal{E}$ by $\mathcal{V}_Y$. By Proposition 3.4, $\mathcal{V}_Y$ is an open subset of $Y\times\mathcal{E}$. By Proposition 3.3, Ψ_Y defines a bijective map from $\mathcal{U}_Y$ into $\mathcal{V}_Y$, and by Proposition 3.4 again, this map is a homeomorphism. We thus refer to the triplet $(\Psi_Y,\mathcal{U}_Y,\mathcal{V}_Y)$ as the *graph chart* of $X\times\mathcal{E}$ *over* Y. When only $e_0:=e(x_0)$ is a-priori given, we refer to the triplet $(\Psi_Y,\mathcal{U}_Y,\mathcal{V}_Y)$ as a *graph chart* of $X\times\mathcal{E}$ *about* (x_0,e_0).

We define the *mean curvature function* $H_Y:\mathcal{U}_Y\to C^\infty(\Sigma)$ and the *boundary angle function* $\Theta_Y:\mathcal{U}_Y\to C^\infty(\partial\Sigma)$ such that, for all $(y,f)\in\mathcal{U}_Y$:

$$H_Y(y,f):=H_{y,\hat{\Phi}_Y(y,f)},\ \text{and}\ \Theta_Y(y,f):=\Theta_{y,\hat{\Phi}_Y(y,f)}.$$

We define $\mathcal{Z}_{Y,\mathrm{loc}}\subseteq\mathcal{U}_Y$ by:

$$\mathcal{Z}_{Y,\mathrm{loc}}=\left\{(y,f)\mid H_Y(y,f)=0,\Theta_Y(y,f)=0\right\},$$

and we call $\mathcal{Z}_{Y,\mathrm{loc}}$ the *local solution space* in the graph chart. Observe in particular that:

$$\mathcal{Z}_{Y,\mathrm{loc}} = \Psi_Y^{-1}(\mathcal{Z}(Y) \cap \mathcal{V}_Y).$$

In later sections, where no ambiguity arises, we will often suppress Y and simply write $\hat{\Phi}$, Φ, Ψ and so on, respectively, for $\hat{\Phi}_Y$, Φ_Y and Ψ_Y and so on.

3.2. Local charts II: The Hölder case. We consider families of Hölder spaces parametrised by $\lambda \in [0,\infty) \setminus \mathbb{N}$ (cf. Section 2.4). Let $r > 0$ be as in Proposition 3.2 and define $\mathcal{U}_Y^{\lambda+1} \subseteq Y \times C^{*,\lambda+1}(\Sigma)$ by:

$$\mathcal{U}_Y^{\lambda+1} := \{(y,f) \mid \|f\|_{L^\infty} < r\}.$$

We define $H_Y^{\lambda+2} : \mathcal{U}_Y^{\lambda+2} \to C^{*,\lambda}(\Sigma)$ as in Section 3.1. Since it may be written in terms of a finite combination of addition, multiplication, differentiation and post-composition by smooth functions, $H_Y^{\lambda+2}$ defines a smooth function between Banach spaces. For each k, we denote by $D_k H_Y^{\lambda+2}$ its partial derivative with respect to the k'th component in $\mathcal{U}_Y^{\lambda+2} \subseteq Y \times C^{*,\lambda+2}(\Sigma)$. In particular, by definition of the Jacobi operator of mean curvature:

$$D_2 H_Y^{\lambda+2}(x,0) = \mathrm{J}^h_{x,e}. \tag{3.1}$$

We also define the *boundary angle function* $\Theta_Y^{\lambda+1} : \mathcal{U}_Y^{\lambda+1} \to C^{*,\lambda}(\partial\Sigma)$ as in Section 3.1. $\Theta_Y^{\lambda+1}$ likewise defines a smooth function between Banach spaces. For each k, we denote by $D_k\Theta_Y^{\lambda+1}$ its partial derivative with respect to the k'th component in $\mathcal{U}_Y^{\lambda+1} \subseteq Y \times C^{*,\lambda+1}(\Sigma)$. In particular, that by definition of the Jacobi operator of the boundary angle:

$$D_2 \Theta_Y^{\lambda+1}(x,0) = \mathrm{J}^\theta_{x,e}. \tag{3.2}$$

Finally, we define $\mathcal{Z}_{Y,\mathrm{loc}}^{\lambda+2} \subseteq \mathcal{U}_Y^{\lambda+2}$ by:

$$\mathcal{Z}_{Y,\mathrm{loc}}^{\lambda+2} := \left\{(y,f) \mid H_Y^{\lambda+2}(y,f) = 0,\ \Theta_Y^{\lambda+2}(y,f) = 0\right\}.$$

We recall the following classical result concerning the regularity of embeddings of prescribed mean curvature:

Theorem 3.5. *Let g be a smooth metric over M, let $h : M \to \mathbb{R}$ be a smooth function, and let $\Sigma \subseteq M$ be an embedded compact submanifold of M of class $C^{\lambda+2}$ such that $\partial\Sigma \subseteq \partial M$ and $\partial\Sigma = \Sigma \cap \partial M$. If Σ meets ∂M orthogonally along $\partial\Sigma$ with respect to the metric g and if the mean curvature of Σ is at every point $p \in \Sigma$ equal to $h(p)$, then Σ is smooth.*

Proof. This follows by applying, for example, Schauder estimates [**10**]. q.e.d.

Expressed in terms of graph charts, this immediately yields:

Proposition 3.6. *If $(y, f) \in \mathcal{U}_Y^{\lambda+2}$ is such that $H_Y^{\lambda+2}(y, f) \in C^\infty(\Sigma)$, then $f \in C^\infty(\Sigma)$. In particular, for all $\lambda \in [0, \infty[\setminus\mathbb{N}$, $\mathcal{Z}_{Y,\mathrm{loc}}^{\lambda+2} = \mathcal{Z}_{Y,\mathrm{loc}}$.*

3.3. Conjugations. We finish this section by describing the relationship between functionals H and Θ and the perturbation and Jacobi operators introduced in Sections 2.2 and 2.3. Thus choose $(y, f) \in \mathcal{Z}_{Y,\mathrm{loc}}$ and denote $e' = \hat{\Phi}_Y(y, f)$. We define the vector field $V_{y,f}$ over e' by:

$$V_{y,f}(p) := \partial_t \hat{\Phi}_Y(y, f + t)(p)|_{t=0}.$$

We define the function $\lambda_{y,f} : \Sigma \to \mathbb{R}$ by:

$$\lambda_{y,f} := g_y(N_{y,e'}, V_{y,f}).$$

Observe that both $V_{y,f}$ and $\lambda_{y,f}$ are smooth. Furthermore, since $V_{y,f}$ is at no point tangent to $e'(\Sigma)$, the function $\lambda_{y,f}$ never vanishes. The next proposition follows immediately from the definition of P:

Proposition 3.7. *For all $(y, f) \in \mathcal{Z}_{Y,\mathrm{loc}} = \mathcal{Z}_{Y,\mathrm{loc}}^{\lambda+2}$, and all $\xi_y \in T_yY$:*

$$D_1 H_Y^{\lambda+2}(y, f)(\xi_y) = \mathrm{P}_{y,e'}^h(\xi_y), \text{ and } D_1\Theta_Y^{\lambda+1}(y, f)(\xi_y) = \mathrm{P}_{y,e'}^\theta(\xi_y).$$

For the partial derivatives with respect to the second component, we have:

Proposition 3.8. *For all $(y, f) \in \mathcal{Z}_{Y,\mathrm{loc}} = \mathcal{Z}_{Y,\mathrm{loc}}^{\lambda+2}$, and all $\varphi \in C^{*,\lambda+2}(\Sigma)$:*

$$D_2 H_Y^{\lambda+2}(y, f)(\varphi) = \mathrm{J}_{y,e'}^h(\lambda_{y,f}\varphi).$$

Proof. Denote $e' = \hat{\Phi}_Y(y, f)$. Let Y' be a compact neighbourhood of y in Y and let $(\Psi_{Y'}, \mathcal{U}_{Y'}, \mathcal{V}_{Y'})$ be a graph chart of $X \times \mathcal{E}$ about (y, e') over Y'. Choose $\varphi \in C^\infty(\Sigma)$. There exists $\delta > 0$ and smooth mappings $\alpha : (-\delta, \delta) \times \Sigma \to \Sigma$ and $\psi : (-\delta, \delta) \times \Sigma \to \mathbb{R}$ such that $\alpha(0, \cdot)$ coincides with the identity map, $\psi_0 = 0$, and, for all $t \in (-\delta, \delta)$, $\alpha_t := \alpha(t, \cdot)$ is a smooth diffeomorphism of Σ and $\hat{\Phi}_{Y'}(y, \psi_t) \circ \alpha_t = \hat{\Phi}_Y(y, f + t\varphi)$, where $\psi_t := \psi(t, \cdot)$. Bearing in mind the definition of $V_{y,f}$, differentiating with respect to t yields $D_2\hat{\Phi}_Y(y, f)(\varphi) = \varphi V_{y,f}$. Likewise:

$$D_2\tilde{\Phi}_{Y'}(y, 0)((\partial_t\psi)_0) = (\partial_t\psi)_0 N_{y,e'}.$$

By the chain rule, this yields $\varphi V_{y,f} = (\partial_t\psi)_0 N_{y,e'} + W$, where W is tangent to $e(\Sigma)$, and so $(\partial_t\psi)_0 = \varphi g_y(N_{y,e'}, V_{y,f}) = \lambda_{y,f}\varphi$. Now let $H_{Y'}$ be the mean curvature function in the chart $(\Psi_{Y'}, \mathcal{U}_{Y'}, \mathcal{V}_{Y'})$. Since $(y, f) \in \mathcal{Z}(Y)$, $H_{Y'}(y, 0) = H_Y(y, f) = 0$. Furthermore, $H_{Y'}(y, \psi_t) \circ \alpha_t = H_Y(x, f + t\varphi)$, for all t, and differentiating at $t = 0$, therefore, yields:

$$\begin{aligned} D_2H_Y(x, f)(\varphi) &= D_2H_{Y'}(y, 0)((\partial_t\psi)_0) \\ &= D_2H_{Y'}(y, 0)(\lambda_{y,f}\varphi) \\ &= \mathrm{J}_{y,e'}^h(\lambda_{y,f}\varphi). \end{aligned}$$

The result follows by continuity and density of $C^\infty(\Sigma)$ in $C^{*,\lambda+2}(\Sigma)$. q.e.d.

Proposition 3.9. *For all $(y,f)\in\mathcal{Z}_{Y,\mathrm{loc}}=\mathcal{Z}_{Y,\mathrm{loc}}^{\lambda+1}$, and for all $\varphi\in C^{*,\lambda+1}(\Sigma)$:*

$$D_2\Theta_Y^{\lambda+1}(y,f)(\varphi)=\mathrm{J}^\theta_{y,e'}(\lambda_{y,f}\varphi).$$

Proof. Choose $\varphi\in C^\infty(\Sigma)$. We use the same construction as in the proof of Proposition 3.8. Let $\Theta_{Y'}$ be the boundary angle function in the chart generated by Y'. As before, $\Theta_{Y'}(y,0)=\Theta_Y(y,f)=0$ and $\Theta_{Y'}(y,\psi_t)\circ\alpha_t=\Theta_Y(x,f+t\varphi)$ for all t, so that:

$$\begin{aligned}D_2\Theta_Y(x,f)(\varphi) &= D_2\Theta_{Y'}(y,0)((\partial_t\psi)_0)\\ &= D_2\Theta_{Y'}(y,0)(\lambda_{y,f}\varphi)\\ &= \mathrm{J}^\theta_{y,e'}(\lambda_{y,f}\varphi).\end{aligned}$$

The result follows by continuity and density of $C^\infty(\Sigma)$ in $C^{*,\lambda+1}(\Sigma)$. q.e.d.

4. The differential structure of the solution space

4.1. Extensions and surjectivity. Let $\widetilde{X}$ be another smooth, compact, finite-dimensional manifold. Let $\widetilde{g}:\widetilde{X}\times M\to\mathrm{Sym}^+(TM)$ be a smooth function such that for all $x\in\widetilde{X}$, the metric $\widetilde{g}_x:=\widetilde{g}(x,\cdot)$ is admissible. We say that $\widetilde{X}$ is an *extension* of X whenever $X\subseteq\widetilde{X}$, and the restriction of $\widetilde{g}$ to X coincides with g. In this section, we show the smoothness of the solution space $\mathcal{Z}(\widetilde{X})$ for a suitable extension $\widetilde{X}$ of X. Upon furnishing $\widetilde{X}$ with a canonical orientation, we then define a canonical orientation of $\mathcal{Z}(\widetilde{X})$. In particular, this yields a canonical $\mathbb{Z}$-valued mapping degree of $\Pi:\mathcal{Z}(\widetilde{X})\to\tilde{X}$ which we denote by $\mathrm{Deg}(\Pi)$. We will see in Sections 5.1 and 6.4 that it is also useful to define a local degree. We, therefore, denote for any open subset $\Omega\subseteq\mathcal{E}$:

$$\mathcal{Z}(X|\Omega):=\mathcal{Z}(X)\cap(X\times\Omega),\text{ and }\partial_\omega\mathcal{Z}(X|\Omega):=\mathcal{Z}(X)\cap(X\times\partial\Omega).$$

Since $\mathcal{Z}(X|\Omega)$ is an open subset of $\mathcal{Z}(X)$, $\mathcal{Z}(\widetilde{X}|\Omega)$ is also smooth for a suitable extension $\widetilde{X}$ of X. If, in addition, $\partial_\omega\mathcal{Z}(X|\Omega)=\emptyset$, then we may suppose also that $\partial_\omega\mathcal{Z}(\widetilde{X}|\Omega)=\emptyset$, and, upon furnishing $\tilde{X}$ with an orientation form, we obtain as before a $\mathbb{Z}$-valued mapping degree of $\Pi:\mathcal{Z}(\widetilde{X}|\Omega)\to\widetilde{X}$, which we denote by $\mathrm{Deg}(\Pi|\Omega)$. We recall from Section 3.3 that $P_{x,e}+J_{x,e}$ is conjugate to the derivative of (H,Θ) in any graph chart about (x,e).

Proposition 4.1. *If $\mathrm{P}_{x,e}+\mathrm{J}_{x,e}$ is surjective at $(x,[e])\in\mathcal{Z}(X)$, then there exists a neighbourhood $W_{x,e}$ of $(x,[e])$ in $\mathcal{Z}(X)$ such that $\mathrm{P}_{x',e'}+\mathrm{J}_{x',e'}$ is surjective for all $(x',[e'])\in W$.*

Proof. Suppose the contrary. There exists $(x,[e])\in\mathcal{Z}(X)$ such that $\mathrm{P}_{x,e}+\mathrm{J}_{x,e}$ is surjective and a sequence $(x_m,[e_m])_{m\in\mathbb{N}}\in\mathcal{Z}(X)$ which converges to $(x,[e])$ such that $\mathrm{P}_{x_m,e_m}+\mathrm{J}_{x_m,e_m}$ is not surjective. Choose $\lambda\in[0,\infty[\setminus\mathbb{N}$. By Proposition 2.15, $\mathrm{P}_{x,e}+\mathrm{J}_{x,e}$ defines a surjective, Fredholm map from $T_xX\times C^{*,\lambda+2}(\Sigma)$ into $C^{*,\lambda}(\Sigma)\times C^{*,\lambda+1}(\partial\Sigma)$. Since $(\mathrm{P}_{x_m,e_m},\mathrm{J}_{x_m,e_m})_{m\in\mathbb{N}}$ converges to $\mathrm{P}_{x,e}+\mathrm{J}_{x,e}$ in the operator norm, and since the property of being a surjective, Fredholm map is open, there exists $M\in\mathbb{N}$ such that for each $m\geqslant M$, $\mathrm{P}_{x_m,e_m}+\mathrm{J}_{x_m,e_m}$ also defines a surjective map from $T_{x_m}X\times C^{*,\lambda+2}(\Sigma)$ into $C^{*,\lambda}(\Sigma)\times C^{*,\lambda+1}(\partial\Sigma)$. By Propositions 2.14 and 2.15, it follows that for all $m\geqslant M$, $\mathrm{P}_{x_m,e_m}+\mathrm{J}_{x_m,e_m}$ defines a surjective map from $T_{x_m}X\times C^\infty(\Sigma)$ into $C^\infty(\Sigma)\cap C^\infty(\partial\Sigma)$, and this completes the proof. q.e.d.

Theorem 4.2. *For every open set $\Omega\subseteq\mathcal{E}$ such that $\partial_\omega\mathcal{Z}(X|\Omega)=\emptyset$, there exists an extension $\widetilde{X}$ of X such that $\partial_\omega\mathcal{Z}(\tilde{X}|\Omega)=\emptyset$ and, for all $(x,[e])\in\mathcal{Z}(\tilde{X}|\Omega)$, the operator $\mathrm{P}_{x,e}+\mathrm{J}_{x,e}$ defines a surjective map from $T_x\widetilde{X}\times C^\infty(\Sigma)$ into $C^\infty(\Sigma)\times C^\infty(\partial\Sigma)$.*

Proof. We define the map $\widetilde{g}:C^\infty(M)\times X\times M\to\mathrm{Sym}^+(TM)$ such that, for all $f\in C^\infty(M)$ and for all $x\in X$:

$$\widetilde{g}_{f,x}:=\widetilde{g}(f,x,\cdot)=e^f g_x.$$

Let E be a finite-dimensional, linear subspace of $C^\infty(M)$ and for $r>0$, let E_r be the closed ball of radius r about 0 in E with respect to some metric. Observe that for sufficiently small r, and for all $(f,x)\in E_r\times X$, the metric $\tilde{g}_{f,x}$ is also admissible. We denote $\widetilde{X}:=E_r\times X$, and we will show that $\widetilde{X}$ has the desired properties for suitable choices of E and r.

Choose $(x,[e])\in\mathcal{Z}(X|\Omega)$. We claim that there exists a finite dimensional subspace $E_{x,e}\subseteq C^\infty(M)$ with the property that if E contains $E_{x,e}$, then:

$$C^\infty(\Sigma)\times C^\infty(\partial\Sigma)=\mathrm{Im}(\mathrm{P}_{(0,x),e})+\mathrm{Im}(\mathrm{J}_{(0,x),e}).$$

Indeed, let $f_1,...,f_m$ be a basis of $\mathrm{Ker}(\mathrm{J}_{(0,x),e})$. Let U be an open subset of M intersecting $e(\Sigma)$ non-trivially, let $\varphi_1,...,\varphi_m$ be as in Proposition 2.10, and let $E_{x,e}\subseteq C^\infty(M)$ be their linear span. For $1\leqslant k\leqslant m$, we think of φ_k as a tangent vector to $E_{x,e}$ at 0 and we denote $\psi_k=\mathrm{P}^h_{(0,x),e}(\varphi_k)$. For all $1\leqslant k\leqslant m$, by Proposition 2.7, $\mathrm{P}^\theta_{(0,x),e}(\varphi_k)=0$ and so $\mathrm{P}_{(0,x),e}(\varphi_k)=(\psi_k,0)$. Let $F_{x,e}$ be the linear span of $(\psi_1,0),...,(\psi_m,0)$ in $C^\infty(\Sigma)\times C^\infty(\partial\Sigma)$. We claim that:

$$C^\infty(\Sigma)\times C^\infty(\partial\Sigma)\subseteq F_{x,e}+\mathrm{Im}(\mathrm{J}_{(0,x),e}).$$

Indeed, let π be the orthogonal projection from $C^\infty(\Sigma)\times C^\infty(\partial\Sigma)$ onto $\mathrm{Im}(\mathrm{J}_{(0,x),e})$ with respect to the L^2 inner-product of e^*g_x and denote $\pi^\perp=\mathrm{Id}-\pi$. By Proposition 2.15, $\mathrm{Im}(\pi^\perp)$ is spanned by $(f_q,f_q\circ\epsilon)_{1\leqslant q\leqslant m}$, where $\epsilon : \partial\Sigma \to \Sigma$ is the canonical embedding. However, denoting the volume form of $e^* g_x$ by $dV_{x,e}$, for all $1 \leqslant p, q \leqslant m$, we have:

$$\langle \pi^\perp(\psi_p, 0), (f_q, f_q \circ \epsilon)\rangle = \langle (\psi_p, 0), (f_q, f_q \circ \epsilon)\rangle = \int_\Sigma \psi_p f_q \, dV_{x,e} = \delta_{pq}.$$

The restriction of $\pi^\perp$ to $F_{x,e}$, therefore, defines a linear isomorphism onto $\mathrm{Im}(\pi^\perp)$, and so:

$$F_{x,e} \cap \mathrm{Im}(\mathrm{J}_{(0,x),e}) = F_{x,e} \cap \mathrm{Ker}(\pi^\perp) = \{0\}\,.$$

Since the dimension of $F_{x,e}$ is equal to the codimension of $\mathrm{Im}(\mathrm{J}_{(0,x),e})$ in $C^\infty(\Sigma) \times C^\infty(\partial\Sigma)$, it follows that $F_{x,e}$ and $\mathrm{Im}(\mathrm{J}_{(0,x),e})$ are complementary subspaces so that:

$$C^\infty(\Sigma) \times C^\infty(\partial\Sigma) \subseteq F_{x,e} \oplus \mathrm{Im}(\mathrm{J}_{(0,x),e}),$$

as asserted, and so, if E contains $E_{x,e}$, then $\mathrm{J}_{(0,x),e} + \mathrm{P}_{(0,x),e}$ is surjective.

We conclude using compactness. By Proposition 4.1, there is a neighbourhood $W_{x,e}$ of $(x,[e])$ in $\mathcal{Z}(X|\Omega)$ such that if E contains $E_{x,e}$, then, for all $(x',[e']) \in W_{x,e}$, $\mathrm{P}_{(0,x'),e'} + \mathrm{J}_{(0,x'),e'}$ defines a surjective map from $T_{(0,x')}(E_r \times X) \times C^\infty(\Sigma)$ into $C^\infty(\Sigma) \times C^\infty(\partial\Sigma)$. However, since $\partial_\omega \mathcal{Z}(X|\Omega) = \emptyset$, $\mathcal{Z}(X|\Omega)$ is a closed subset of $\mathcal{Z}(X)$, and, by Proposition 2.4, it is, therefore, compact. There, therefore, exist finitely many points $(x_k, [e_k])_{1 \leqslant k \leqslant m}$ such that:

$$\mathcal{Z}(X|\Omega) \subseteq \bigcup_{k=1}^{m} W_{x_k, e_k}.$$

We define $E = E_{x_1,e_1} + ... + E_{x_m,e_m}$ and we see that for all $(x,[e]) \in \mathcal{Z}(X|\Omega)$, $\mathrm{P}_{(0,x),e} + \mathrm{J}_{(0,x),e}$ defines a surjective map from $T_{(0,x)}(E \times X) \times C^\infty(\Sigma)$ into $C^\infty(\Sigma) \times C^\infty(\partial\Sigma)$. Finally, by compactness again, for sufficiently small r we have $\partial_\omega \mathcal{Z}(\widetilde{X}|\Omega) = \mathcal{Z}(E_r \times X) \cap (E_r \times X \times \partial\Omega) = \emptyset$, and $\mathrm{P}_{x,e} + \mathrm{J}_{x,e}$ defines a surjective map from $T_x\tilde{X} \times C^\infty(\Sigma)$ into $C^\infty(\Sigma) \times C^\infty(\partial\Sigma)$ for all $(x,[e]) \in \mathcal{Z}(\widetilde{X}|\Omega)$. This completes the proof. q.e.d.

4.2. Surjectivity and smoothness.

Proposition 4.3. *Let $\Omega \subseteq \mathcal{E}$ be such that $\partial_\omega \mathcal{Z}(X|\Omega) = \emptyset$. If $\mathrm{P}_{x,e} + \mathrm{J}_{x,e}$ is surjective for all $(x,[e]) \in \mathcal{Z}(X|\Omega)$, then for every compact neighbourhood Y of X with smooth boundary and for every graph chart $(\Psi, \mathcal{U}, \mathcal{V})$ of $X \times \mathcal{E}$ over Y, $\mathcal{Z}_{\mathrm{loc}} \cap \Psi^{-1}(X \times \Omega) = \mathcal{Z}^{\lambda+2}_{\mathrm{loc}} \cap \Psi^{-1}(X \times \Omega)$ is a smooth, embedded submanifold of $\mathcal{U}^{\lambda+2}$ with smooth boundary and of finite dimension equal to* $\mathrm{Dim}(X)$. *Moreover:*

1) *the differential structure induced over $\mathcal{Z}_{\mathrm{loc}} \cap \Psi^{-1}(X \times \Omega)$ by the canonical embedding into $\mathcal{U}^{\lambda+2}$ is independent of λ; and*
2) *Π defines a smooth map from $\mathcal{Z}_{\mathrm{loc}} \cap \Psi^{-1}(X \times \Omega)$ into Y with the property that $\Pi(\partial \mathcal{Z}_{\mathrm{loc}}) \subseteq \partial Y$.*

Proof. Choose $(x,[e])\in\mathcal{Z}(X|\Omega)$ and choose $\lambda\in[0,\infty[\setminus\mathbb{N}$. By Proposition 2.15, $P_{x,e}+J_{x,e}$ defines a Fredholm map of index $\mathrm{Dim}(X)$ from $T_xX\times C^{*,\lambda+2}(\Sigma)$ into $C^{*,\lambda}(\Sigma)\times C^{*,\lambda+1}(\partial\Sigma)$. It also follows by the hypotheses and Proposition 2.15 again that this map is surjective.

Now let Y be a compact neighbourhood of x in X with smooth boundary, let $(\Psi,\mathcal{U},\mathcal{V})$ be a graph chart of $X\times\mathcal{E}$ over Y and let $H^{\lambda+2}$ and $\Theta^{\lambda+1}$ be, respectively, the mean curvature function and the boundary angle function in this chart (cf. Section 3.1). By Propositions 3.7, 3.8 and 3.9, for all $(y,f)\in\mathcal{Z}^{\lambda+2}_{\mathrm{loc}}\cap\Psi^{-1}(X\times\Omega)$, the map $D(H^{\lambda+2},\Theta^{\lambda+2})(y,f)$ is conjugate to $\mathrm{P}_{y,e}+\mathrm{J}_{y,e}$, where $e=\hat{\Phi}(y,f)$. It, therefore, defines a surjective, Fredholm map of index equal to $\mathrm{Dim}(X)$ from $T_xX\times C^{*,\lambda+2}(\Sigma)$ into $C^{*,\lambda}(\Sigma)\times C^{*,\lambda+1}(\partial\Sigma)$, and it follows from the submersion theorem for Banach manifolds that $\mathcal{Z}^{\lambda+2}_{\mathrm{loc}}\cap\Psi^{-1}(X\times\Omega)$ is a smooth, embedded submanifold of $\mathcal{U}^{\lambda+2}$ of finite dimension equal to $\mathrm{Dim}(X)$ and, moreover, that $\Pi(\partial\mathcal{Z}^{\lambda+2}_{\mathrm{loc}})\subseteq\partial Y$.

It remains to show independence. However, by the preceding discussion, for all $\mu\geqslant\lambda$, $\mathcal{Z}^{\mu+2}_{\mathrm{loc}}\cap\Psi^{-1}(X\times\Omega)$ and $\mathcal{Z}^{\lambda+2}_{\mathrm{loc}}\cap\Psi^{-1}(X\times\Omega)$ are smooth, embedded, submanifolds of $\mathcal{U}^{\mu+2}$ and $\mathcal{U}^{\lambda+2}$, respectively, both of finite dimension equal to $\mathrm{Dim}(X)$. Let $i_{\mu,\lambda}:Y\times C^{*,\mu+2}(\Sigma)\longrightarrow Y\times C^{*,\lambda+2}(\Sigma)$ be the canonical embeddings. The map $i_{\mu,\lambda}$ is smooth and injective with injective derivative at every point, and, therefore, restricts to a diffeomorphism from $\mathcal{Z}^{\mu+2}_{\mathrm{loc}}\cap\Psi^{-1}(X\times\Omega)$ to $\mathcal{Z}^{\lambda+2}_{\mathrm{loc}}\cap\Psi^{-1}(X\times\Omega)$. The differential structure induced over $\mathcal{Z}_{\mathrm{loc}}\cap\Psi^{-1}(X\times\Omega)$ by the canonical embedding into $\mathcal{U}^{\lambda+2}$ is, therefore, independent of λ, and this completes the proof. q.e.d.

We recall the following technical result:

Proposition 4.4. *Let N_1,N_2 be smooth, finite-dimensional manifolds and suppose that N_2 is compact. Let Φ be a map from N_1 into $C^\infty(N_2)$, and define the function $\varphi:N_1\times N_2\to\mathbb{R}$ such that for all $(p,q)\in N_1\times N_2$:*

$$\varphi(p,q)=\Phi(p)(q).$$

Φ defines a smooth map from N_1 into $C^{*,\lambda}(N_2)$ for all $\lambda\in[0,\infty[\setminus\mathbb{N}$ if and only if φ is smooth.

Proof. For $k\in\{1,2\}$, denote by D_k the partial derivative with respect to the k'th component. Choose $m\in\mathbb{N}$ and $\lambda>m$. If Φ defines a smooth map from N_1 into $C^{*,\lambda}(N_2)$, then $D_1^pD_2^q\varphi$ exists and is continuous for all $p,q\in\mathbb{N}\times\{0,...,m\}$. It follows that if Φ defines a smooth map from N_1 into $C^{*,\lambda}(N_2)$ for all $\lambda\in[0,\infty[\setminus\mathbb{N}$, then φ is smooth. The reverse implication is trivial, and this completes the proof. q.e.d.

Theorem 4.5. *Let $\Omega\subseteq\mathcal{E}$ be such that $\partial_\omega\mathcal{Z}(X|\Omega)=\emptyset$. If $\mathrm{P}_{x,e}+\mathrm{J}_{x,e}$ is surjective for all $(x,[e])\in\mathcal{Z}(X|\Omega)$, then $\mathcal{Z}(X|\Omega)$ carries the* *canonical structure of a smooth, compact manifold with boundary of finite dimension equal to* $\mathrm{Dim}(X)$. *Moreover,* Π *defines a smooth map from* $\mathcal{Z}(X|\Omega)$ *to* X *such that:*

$$\Pi(\partial\mathcal{Z}(X|\Omega)) \subseteq \partial X,$$

where $\partial\mathcal{Z}(X|\Omega)$ *here denotes the manifold boundary of* $\mathcal{Z}(X|\Omega)$.

Proof. Since $\partial_\omega\mathcal{Z}(X|\Omega) = \emptyset$, $\mathcal{Z}(X|\Omega)$ is a closed subset of $\mathcal{Z}(X)$. Since $\mathcal{Z}(X)$ is compact, by Proposition 2.4, so too is $\mathcal{Z}(X|\Omega)$. In addition, Proposition 4.3 yields an atlas of smooth charts of $\mathcal{Z}(X|\Omega)$, and it thus remains to prove that the transition maps are also smooth. Choose $(x,[e]) \in \mathcal{Z}(X|\Omega)$. Let Y be a compact neighbourhood of x in X and let $\tilde{e} : Y \times \Sigma \longrightarrow M$ be such that $\widetilde{e}(x) = e$ and, for all $y \in Y$, $\widetilde{e}_y := \widetilde{e}(y,\cdot)$ is an embedding such that $\tilde{e}_y(\Sigma)$ meets ∂M orthogonally along $\partial\Sigma$ with respect to g_y. Let $N : Y \times \Sigma \longrightarrow TM$ be such that, for all $y \in Y$, $N_y := N(y,\cdot)$ is the unit, normal vector field over e_y with respect to g_y which is compatible with the orientation. We define the map $F : Y \times \Sigma \times \mathbb{R} \longrightarrow M$ by:

$$F(y,p,t) = \mathrm{E}(tN_y(p)),$$

where E is the modified exponential map. Let Y' be another compact neighbourhood of x in X and define $\tilde{e}'$, N' and F' in the same manner. For all y, we denote $F_y := F(y,\cdot,\cdot)$ and $F'_y := F'(y,\cdot,\cdot)$.

Let $(\Psi,\mathcal{U},\mathcal{V})$ and $(\Psi',\mathcal{U}',\mathcal{V}')$ be the graph charts of $X \times \mathcal{E}$ generated by $(Y,\tilde{e})$ and $(Y',\widetilde{e}')$, respectively. Denote $Z_0 = \mathcal{Z}_{Y,\mathrm{loc}} \cap \Psi^{-1}(X \times \Omega)$ and let $B := (\eta,\varphi) : Z_0 \longrightarrow Y \times C^\infty(\Sigma)$ be the canonical embedding. By definition (η,φ) defines a smooth map from Z_0 into $Y \times C^{*,\lambda+2}(\Sigma)$ for all λ. It follows that η is smooth and, by Proposition 4.4, the function $\tilde{\varphi} : Z_0 \times \Sigma \to \mathbb{R}$ given by $\tilde{\varphi}(z,p) := \varphi(z)(p)$ is smooth. Observe that, for all $(z,p) \in Z_0 \times \Sigma$:

$$(\hat{\Phi} \circ B)(z)(p) = F_{\eta(z)}(p,\tilde{\varphi}(z,p)).$$

Let $\pi_1 : \Sigma \times \mathbb{R} \longrightarrow S$ and $\pi_2 : \Sigma \times \mathbb{R} \to \mathbb{R}$ be the projections onto the first and second factors, respectively. We define $\alpha : Z_0 \times \Sigma \to S$ and $\psi : Z_0 \times \Sigma \to \mathbb{R}$ by:

$$\alpha(z,p) := (\pi_1 \circ (F'_{\eta(z)})^{-1} \circ F_{\eta(z)})(p,\tilde{\varphi}(z,p)),$$
$$\psi(z,p) := (\pi_1 \circ (F'_{\eta(z)})^{-1} \circ F_{\eta(z)})(p,\tilde{\varphi}(z,p)).$$

Observe that both α and ψ are smooth mappings. Moreover, for all z sufficiently close to $z_0 := (x,0)$, $\alpha_z := \alpha(z,\cdot)$ is a diffeomorphism. We, therefore, define $\beta : Z_0 \times \Sigma \to \Sigma$ such that for all $z \in Z_0$, $\beta_z := \beta(z,\cdot) = \alpha_z^{-1}$, and we see that β is also a smooth map. However, for all $z \in Z_0$:

$$((\Psi')^{-1} \circ \Psi \circ B)(z) = (\eta(z),\psi_z \circ \beta_z).$$

Since the map $(z,p) \mapsto (\psi_z \circ \beta_z)(p)$ is smooth, it follows from Proposition 4.4 again that $((\Psi')^{-1} \circ \Psi \circ B)$ is also a smooth map, and the transition maps are, therefore, smooth as desired. q.e.d.

4.3. Surjectivity and orientation. We first review some basic spectral theory. Let E and F be Hilbert spaces. Let $i : E \to F$ be a compact, injective map with dense image. Let $A : E \to F$ be a Fredholm map of index 0. We say that A is *self-adjoint* whenever it has the property that for all $u, v \in E$:

$$\langle A(u), i(v)\rangle = \langle i(u), A(v)\rangle.$$

We henceforth identify E with its image $i(E)$. Let $K \subseteq E \subseteq F$ be the kernel of A, let $R_f \subseteq F$ be its orthogonal complement and denote $R_e := R_f \cap E$. Observe that R_e and R_f are closed subspaces of E and F, respectively. Moreover:

$$E = K \oplus R_e, \text{ and } F = K \oplus R_f.$$

By the closed graph theorem, A restricts to an invertible, linear map from R_e to R_f. We define $B : R_f \to R_e$ to be the inverse of this restriction, and we extend B to an operator from F into E by composing with the orthogonal projection of F onto R_f, so that B then defines a self-adjoint, compact operator from F to itself. By the Sturm–Liouville Theorem, the (non-zero) spectrum of B, which we denote by $\mathrm{Spec}(B)$ is a discrete subset of $\mathbb{R} \setminus \{0\}$ and every eigenvalue has finite multiplicity. We recall that the *spectrum* of A, which we denote by $\mathrm{Spec}(A)$, is defined to be the set of all $\lambda \in \mathbb{R}$ such that $A - \lambda$ is not invertible, and we see that:

$$\mathrm{Spec}(A) \setminus \{0\} = \left\{\lambda \in \mathbb{R} \setminus \{0\} \ \middle|\ \lambda^{-1} \in \mathrm{Spec}(B)\right\},$$

from which it follows, in particular, that $\mathrm{Spec}(A)$ is a discrete subset of $\mathbb{R}$, and every eigenvalue has finite multiplicity.

We define the *nullity* of A, denoted by $\mathrm{Null}(A)$, to be the dimension of its kernel. Since A is Fredholm, $\mathrm{Null}(A)$ is finite. We define the *index* of A, denoted by $\mathrm{Ind}(A)$ (and not to be confused with its Fredholm index), to be the sum of the multiplicities of its negative eigenvalues. That is:

$$\mathrm{Ind}(A) = \sum_{\lambda \in \mathrm{Spec}(A) \cap (-\infty, 0)} \mathrm{Mult}(\lambda).$$

When $\mathrm{Ind}(A)$ is finite, we define the *signature* of A by:

$$\mathrm{Sig}(A) = (-1)^{\mathrm{Ind}(A)}.$$

We define $\mathcal{F}^+(E,F)$ to be the set of all self-adjoint, Fredholm maps $A : E \longrightarrow F$ such that, for all non-zero $v \in E$:

$$\frac{\langle Av, v\rangle}{\langle v, v\rangle} \geqslant B, \tag{4.1}$$

for some $B \in \mathbb{R}$, where $\langle \cdot, \cdot \rangle$ is the inner-product of F. Observe that $\mathcal{F}^+(E, F)$ is convex and that, for all $A \in \mathcal{F}^+(E, F)$, $\mathrm{Ind}(A) < \infty$, so that $\mathrm{Sig}(A)$ is well defined. We recall (cf. [**26**]):

Proposition 4.6. *Let $C \subseteq \mathcal{F}^+(E, F)$ be connected. If* Null *is constant over C, then so too is* Ind.

Proof. By classical spectral theory (cf. [**15**]), Ind defines a lower semi-continuous function over $\mathcal{F}^+(E, F)$, whilst $(\mathrm{Ind} + \mathrm{Null})$ defines an upper-semicontinuous function over this set. Consequently, if Null is continuous (i.e., locally constant), then so too is Ind, and the result follows. q.e.d.

Let X be a finite-dimensional vector space with orientation form τ. Let $\mathcal{M} := \mathcal{M}(X, E, F)$ be the space of all pairs (M, A) such that:

1) $M : X \to F$ is a linear map;
2) $A : E \to F$ is an element of $\mathcal{F}^+(E, F)$; and
3) $M + A$ is surjective.

Observe, in particular, that $\mathrm{Ker}(M + A)$ has constant dimension equal to $\mathrm{Dim}(X)$.

Proposition 4.7. *If $\pi : X \times E \to X$ is the projection onto the first component, then π restricts to a linear isomorphism from* $\mathrm{Ker}(M + A)$ *into X if and only if A is invertible.*

Proof. Since $\mathrm{Dim}(\mathrm{Ker}(M + A)) = \mathrm{Dim}(X)$, this restriction is bijective if and only if it is injective, and the result follows since $\mathrm{Ker}(M + A) \cap \mathrm{Ker}(\pi) = \{0\} \times \mathrm{Ker}(A)$. q.e.d.

When A is invertible, we, therefore, define the *orientation form* $\sigma(M, A)$ over $\mathrm{Ker}(M + A)$ by:

$$\sigma(M, A) = \mathrm{Sig}(A)(\pi^* \tau).$$

Identifying orientation forms that differ only by a positive factor, we obtain (cf. Proposition 4 of [**25**]):

Proposition 4.8. *$\sigma(M, A)$ extends continuously to define an orientation form over* $\mathrm{Ker}(M + A)$ *for all $(M, A) \in \mathcal{M}$.*

Proof. Consider first the simpler case where $E = F$ is finite-dimensional and A vanishes. For convenience, we furnish X with a positive-definite inner-product. Since M is now surjective, we identify $\mathrm{Ker}(M)^\perp$ with F and suppose that the restriction of M to this subspace is the identity. Let dV_K and dV_F be the volume forms of $\mathrm{Ker}(M)$ and F, respectively. Identifying dV_K and dV_F with their pull-backs through orthogonal projection, we have:

$$\tau = dV_K \wedge dV_F.$$

Now choose $(\tilde{M},\tilde{A})$ near $(M,0)$. Let π_X and π_F be the canonical projections of $X\times F$ onto the first and second factors, respectively. For $v\in\mathrm{Ker}(\tilde{M}+\tilde{A})$, $(\tilde{M}\circ\pi_X)(v)=-(\tilde{A}\circ\pi_F)(v)$, so that, over $\mathrm{Ker}(\tilde{M}+\tilde{A})$:

$$\pi_X^*\tilde{M}^*dV_F=\pi_F^*(-\tilde{A})^*dV_F=(-1)^{\mathrm{Dim}(F)}\mathrm{Det}(\tilde{A})\pi_F^*dV_F.$$

However, if $\tilde{M}_1$ denotes the restriction of $\tilde{M}$ to $\mathrm{Ker}(M)^\perp$, then:

$$dV_K\wedge\tilde{M}^*dV_F=dV_K\wedge\tilde{M}_1^*dV_F=\mathrm{Det}(\tilde{M}_1)dV_K\wedge dV_F=\mathrm{Det}(\tilde{M}_1)\tau,$$

so that, taking the exterior product with $\pi_X^*dV_K$ yields:

$$\mathrm{Det}(\tilde{M}_1)\pi_X^*\tau=(-1)^{\mathrm{Dim}(F)}\mathrm{Det}(\tilde{A})\pi_X^*dV_K\wedge\pi_F^*dV_F.$$

Since the restriction of $\pi_X^*dV_K\wedge\pi_F^*dV_F$ to $\mathrm{Ker}(M+A)$ is non-zero, and since $\mathrm{Sig}(\tilde{A})=\mathrm{Sign}(\mathrm{Det}(\tilde{A}))$, the result follows in this case.

For the general case, let U be a neighbourhood of (M,A) in $\mathcal{M}$ and let $(\tilde{M},\tilde{A})$ be a point of U. Upon conjugating with suitable smooth families of unitary operators (cf. [**15**]), we may suppose that $\tilde{A}$ preserves both K and R. In particular, upon reducing U further if necessary, we obtain:

$$\mathrm{Sig}(\tilde{A})=\mathrm{Sig}(\tilde{A}|_K)+\mathrm{Sig}(\tilde{A}|_R)=\mathrm{Sig}(\tilde{A}|_K)+\mathrm{Sig}(A|_R)=\mathrm{Sig}(\tilde{A}|_K)+\mathrm{Sig}(A).$$

Now let $p:F\to K$ be the orthogonal projection. Since the projection $(x,f)\to(x,p(f))$ maps $\mathrm{Ker}(M+A)$ isomorphically onto $\mathrm{Ker}(p\circ M)\oplus K$, we may suppose that $F=E=K$ is finite dimensional and that $A=0$. The result now follows by the preceding discussion. q.e.d.

Proposition 4.9. *Let $\Omega\subseteq\mathcal{E}$ be such that $\partial_\omega\mathcal{Z}(X|\Omega)=\emptyset$. If $\mathrm{P}_{x,e}+\mathrm{J}_{x,e}$ is surjective for all $(x,[e])\in\mathcal{Z}(X|\Omega)$ then $(x,[e])\in\mathcal{Z}(X|\Omega)$ is a regular point of the restriction of Π to $\mathcal{Z}(X|\Omega)$ if and only if $\mathrm{J}_{x,e}$ is invertible.*

Proof. Choose $(x,[e])\in\mathcal{Z}(X|\Omega)$. Let Y be a compact neighbourhood of x in X and let $(\Psi,\mathcal{U},\mathcal{V})$ be a graph chart of $X\times\mathcal{E}$ about $(x,[e])$ over Y. Let H and Θ be the mean curvature function and the boundary angle function in this chart. Let $\Pi':Y\times C^\infty(\Sigma)\to Y$ be the projection onto the first factor. The point $(x,[e])$ is a regular point of Π if and only if it is a regular point of Π'. However:

$$\begin{aligned}T_{(x,0)}\mathcal{Z}_{\mathrm{loc}}\cap\mathrm{Ker}(D_{(x,0)}\Pi')&=\mathrm{Ker}(D_{(x,0)}(H,\Theta))\cap(\{0\}\times C^\infty(\Sigma))\\&=\mathrm{Ker}(\mathrm{P}_{x,e}+\mathrm{J}_{x,e})\cap(\{0\}\times C^\infty(\Sigma))\\&=\mathrm{Ker}(\mathrm{J}_{x,e}).\end{aligned}$$

We conclude that $(x,[e])$ is a regular value of Π if and only if $\mathrm{J}_{x,e}$ is invertible, as desired. q.e.d.

Combining these results yields:

Theorem 4.10. *Let $\Omega\subseteq\mathcal{E}$ be such that $\partial_\omega\mathcal{Z}(X|\Omega)=\emptyset$. If X is orientable with orientation form τ, and if $\mathrm{P}_{x,e}+\mathrm{J}_{x,e}$ is surjective for all $(x,[e])\in\mathcal{Z}(X|\Omega)$, then $\mathcal{Z}(X|\Omega)$ carries a canonical orientation σ.*

Moreover, $(x,[e])$ is a regular point of the restriction of Π to $\mathcal{Z}(X|\Omega)$ if and only if $\mathrm{J}_{x,e}$ is non-degenerate, and in this case:

$$\sigma(x,[e]) \sim \mathrm{Sig}(\mathrm{J}_{x,e})\Pi^*\tau,$$

where $\mathrm{Sig}(\mathrm{J}_{x,e})$ is defined to be the signature of the restriction of $\mathrm{J}^h_{x,e}$ to the kernel of $\mathrm{J}^\theta_{x,e}$.

Proof. By Theorem 4.5, $\mathcal{Z}(X|\Omega)$ is a smooth manifold of finite dimension equal to $\mathrm{Dim}(X)$. Choose $(x,[e]) \in \mathcal{Z}(X|\Omega)$. Observe that $T_{(x,[e])}\mathcal{Z}(X|\Omega)$ identifies canonically with $\mathrm{Ker}(\mathrm{P}_{x,e}+\mathrm{J}_{x,e})$. Let $H^2(\Sigma)$ be the Sobolev space of twice L^2-differentiable functions over Σ. As in the proof of Proposition 2.12, let $H^2_{\mathrm{rob}}(\Sigma)$ be the kernel of $\mathrm{J}^\theta_{x,e}$ in this space. Observe that the canonical embedding of $H^2_{\mathrm{rob}}(\Sigma)$ into $L^2(\Sigma)$ is compact with dense image, and by Proposition 2.13, the restriction of $\mathrm{J}^h_{x,e}$ to $H^2_{\mathrm{rob}}(\Sigma)$ is self-adjoint. The preceding discussion, therefore, applies, and we define the orientation form σ over $T_{(x,[e])}\mathcal{Z}(X|\Omega) = \mathrm{Ker}(\mathrm{P}_{x,e}+\mathrm{J}_{x,e})$ as in Proposition 4.8 using the restriction of $\mathrm{J}^h_{x,e}$ to $H^2_{\mathrm{rob}}(\Sigma)$. We thereby obtain a continuous family of orientation forms over $\mathcal{Z}(X|\Omega)$. Finally, by Proposition 4.9, $(x,[e]) \in \mathcal{Z}(X|\Omega)$ is a regular point of Π if and only if $\mathrm{J}_{x,e}$ is invertible, and, by definition, $\sigma(x,[e]) \sim \mathrm{Sig}(\mathrm{J}_{x,e})\Pi^*\tau$. This completes the proof. q.e.d.

The results of this section may be summarised as follows:

Theorem 4.11. *Let $\Omega \subseteq \mathcal{E}$ be such that $\partial_\omega\mathcal{Z}(X|\Omega) = \emptyset$. There exists an extension $\tilde{X}$ of X, which we may take to be orientable, such that $\partial_\omega\mathcal{Z}(\tilde{X}|\Omega) = \emptyset$, $\mathcal{Z}(\tilde{X}|\Omega)$ carries canonically the structure of a smooth orientable manifold of finite dimension equal to that of $\tilde{X}$, and $\Pi(\partial\mathcal{Z}(\tilde{X}|\Omega)) \subseteq \partial\tilde{X}$. In particular, the restriction of Π to $\mathcal{Z}(X|\Omega)$ has a well-defined $\mathbb{Z}$-valued degree. Moreover, a point $x \in \tilde{X}$ is a regular value of this restriction if and only if $\mathrm{J}_{x,e}$ is non-degenerate for all $(x,[e]) \in \mathcal{Z}(\{x\}|\Omega)$, and in this case:*

$$\mathrm{Deg}(\Pi|\Omega) = \sum_{(x,[e])\in\mathcal{Z}(\{x\}|\Omega)} \mathrm{Sig}(\mathrm{J}_{x,e}),$$

where $\mathrm{Sig}(\mathrm{J}_{x,e})$ is defined to be the signature of the restriction of $\mathrm{J}^h_{x,e}$ to the kernel of $\mathrm{J}^\theta_{x,e}$.

Proof. It follows from Theorem 4.2 that there exists an extension $\tilde{X}$ of X with $\partial_\omega\mathcal{Z}(\tilde{X}|\Omega) = \emptyset$ such that the operator $\mathrm{P}_{x,e}+\mathrm{J}_{x,e}$ defines a surjective map from $T_x\tilde{X} \times C^\infty(\Sigma)$ into $C^\infty(\Sigma) \times C^\infty(\partial\Sigma)$ for all $(x,[e]) \in \mathcal{Z}(\tilde{X}|\Omega)$. Upon extending $\tilde{X}$ further if necessary, we may assume that it is orientable with orientation form, τ, say. By Theorems 4.5 and 4.10, $\mathcal{Z}(\tilde{X}|\Omega)$ carries the structure of a smooth, compact manifold with boundary, of finite dimension equal to that of $\tilde{X}$ and with orientation form σ. Furthermore, $\Pi(\partial\mathcal{Z}(\tilde{X}|\Omega)) \subseteq \partial\tilde{X}$. By Proposition 2.4, Π defines a proper map from $\mathcal{Z}(\tilde{X}|\Omega)$ into $\tilde{X}$, and so, by classical differential topology (cf. [**13**]), its restriction to $\mathcal{Z}(\tilde{X}|\Omega)$ has a well-defined $\mathbb{Z}$-valued degree. By Theorem 4.10 again, $x \in \tilde{X}$ is a regular value of Π if and only if $\mathrm{J}_{x,e}$ is non-degenerate for all $(x,[e]) \in \mathcal{Z}(\{x\}\,|\Omega)$, and, in this case:

$$\mathrm{Deg}(\Pi|\Omega) = \sum_{(x,[e])\in\mathcal{Z}(\{x\}|\Omega)} \mathrm{Sig}(\mathrm{J}_{x,e}),$$

as desired. q.e.d.

5. Non-degenerate families

5.1. Non-degenerate families. Let Z be a closed, finite-dimensional manifold. Let $\mathcal{F} : Z \to \mathcal{E}$ be a continuous map. We say that $\mathcal{F}$ is *smooth* whenever it has the property that for all $z \in \mathcal{Z}$, there exists a compact neighbourhood Z_0 of z in Z and a smooth function $e : Z_0 \times \Sigma \to M$ such that for all $w \in Z_0$, $e_w := e(w,\cdot)$ is an element of $\hat{\mathcal{E}}$ and $\mathcal{F}(w) = [e_w]$. We refer to the pair (Z_0, e) as a *local parametrisation* of $(Z,\mathcal{F})$ about z. We say that $\mathcal{F}$ is an *immersion* whenever it has the property that for all $z \in \mathcal{Z}$, for every local parametrisation (Z_0, e) of $(Z,\mathcal{F})$ about z, for all $w \in Z_0$ and for all non-zero $\xi_w \in T_wZ_0$, the vector field $(D_1e)_w(\xi_w)$ is not tangent to $e_w(\Sigma)$ at at least one point, where D_1e is the partial derivative of e with respect to the first component in $Z_0 \times \Sigma$. We say that $\mathcal{F}$ is an *embedding* whenever it is, in addition, injective.

Proposition 5.1. *Let g_0 be an admissible metric over M and let $\mathcal{F} : Z \to \mathcal{E}$ be a smooth embedding. If $\mathcal{F}(z)$ is free boundary minimal with respect to g_0 for all $z \in Z$, then for all $z \in Z$:*

$$\mathrm{Null}(\mathrm{J}_{g_0,\mathcal{F}(z)}) = \mathrm{Dim}(\mathrm{Ker}(\mathrm{J}_{g_0,\mathcal{F}(z)})) \geqslant \mathrm{Dim}(Z).$$

Proof. Let n be the dimension of Z. Choose $z \in Z$. Observe that $\mathcal{F}$ defines an n-dimensional family of non-trivial, free boundary minimal perturbations of $\mathcal{F}(z)$, from which it follows that the derivative of $\mathcal{F}$ defines an injective map from T_zZ into $\mathrm{Ker}(\mathrm{J}_{g_0,\mathcal{F}(z)})$. More formally, this injection is explicitly described in the proof of Proposition 5.4 (below). In particular, $\mathrm{Null}(\mathrm{J}_{g_0,\mathcal{F}(z)}) \geqslant n$, and the result follows. q.e.d.

Proposition 5.1 motivates the following definition: if g_0 is an admissible metric over M, and if $\mathcal{F}(z)$ is free boundary minimal with respect to g_0 for all $z \in Z$, then $(Z,\mathcal{F})$ is said to be a *non-degenerate family* whenever it has in addition the property that for all $z \in Z$:

$$\mathrm{Null}(\mathrm{J}_{g_0,\mathcal{F}(z)}) = \mathrm{Dim}(\mathrm{Ker}(\mathrm{J}_{g_0,\mathcal{F}(z)})) = \mathrm{Dim}(Z).$$

We recall from Proposition 4.6 that if $\mathrm{Null}(\mathrm{J}_{g_0,\mathcal{F}(z)})$ is constant, then so too is $\mathrm{Ind}(\mathrm{J}_{g_0,\mathcal{F}(z)})$, and we, therefore, define the *index* of the family Z, which we denote by $\mathrm{Ind}(Z)$ to be equal to $\mathrm{Ind}(\mathrm{J}_{g_0,\mathcal{F}(z)})$ for all $z \in Z$.

Let (Z_0, e) be a local parametrisation of $(Z, \mathcal{F})$. Let X be another smooth, compact, finite-dimensional manifold. Let x_0 be an element of X and let $g : X \times M \to \mathrm{Sym}^+(TM)$ be a smooth function such that $g_x := g(x, \cdot)$ is admissible for all $x \in X$ and $g(x_0, \cdot) = g_0$. We extend e and g to functions defined over $X \times Z_0$ by setting e to be constant in the X direction and by setting g to be constant in the Z_0 direction. Let $(\Psi, \mathcal{U}, \mathcal{V})$ be the graph chart of $X \times Z_0 \times \mathcal{E}$ generated by $(X \times Z_0, e)$ and let H and Θ be, respectively, the mean curvature function and the boundary angle function in this chart. We define $\mathcal{K} \subseteq X \times Z_0 \times C^\infty(\Sigma)$ by:

$$\mathcal{K} = \{(x, z, f) \mid f \in \mathrm{Ker}(\mathrm{J}_{g_0,e_z})\},$$

and for all $(x, z) \in X \times Z_0$, we denote the fibre over (x, z) by $\mathcal{K}_{x,z}$. Observe that $\mathcal{K}$ is a $\mathrm{Dim}(Z)$-dimensional vector bundle over $X \times Z_0$. We shall see presently that $\mathcal{K}$ is smooth, and is, in fact, canonically isomorphic to TZ_0. We also define $\mathcal{K}^\perp \subseteq X \times Z_0 \times C^\infty(\Sigma)$ such that for all $(x, z) \in X \times Z_0$ the fibre $\mathcal{K}^\perp_{x,z}$ is the orthogonal complement of $\mathcal{K}_{x,z}$ in $C^\infty(\Sigma)$ with respect to the L^2-inner-product of $e_z^* g_0$.

Proposition 5.2. *There exists a connected, compact neighbourhood Y of x_0 in X and a unique continuous function $F : Y \times Z_0 \to C^\infty(\Sigma)$ such that $F(0, z) = 0$ for all z and, for all $(x, z) \in Y \times Z_0$:*

1) *$F_{x,z} := F(x, z)$ is an element of $\mathcal{K}^\perp_{x,z}$;*
2) *$\Theta(x, z, F_{x,z}) = 0$; and*
3) *$H(x, z, F_{x,z})$ is an element of $\mathcal{K}_{x,z}$.*

Moreover, the function $f : Y \times Z_0 \times S \to \mathbb{R}$ given by $f(x, z, p) = F(x, z)(p)$ is smooth.

Proof. Choose $\lambda \in [0, \infty[\setminus\mathbb{N}$. Observe that $(H^\lambda, \Theta^\lambda)$ is a smooth, Fredholm map of index equal to $\mathrm{Dim}(Z) = \mathrm{Dim}(\mathcal{K})$. Furthermore, for all $z \in Z_0$, $D(H^\lambda, \Theta^\lambda)(x_0, z, 0) = \mathrm{J}_{g_0,e_z}$, and so, by Proposition 2.15:

$$\mathrm{Ker}(D(H^\lambda, \Theta^\lambda)) = \mathrm{Ker}(\mathrm{J}_{g_0,e_z}) = \mathcal{K}_{x_0,z}.$$

In particular, it follows from that the Fredholm property that, upon reducing X if necessary, $(H^\lambda, \Theta^\lambda)$ is a submersion and, by the submersion theorem for Banach manifolds, $\mathcal{K}$ defines a smooth $\mathrm{Dim}(Z)$-dimensional Banach sub-bundle of $X \times Z_0 \times C^{*,\lambda}(\Sigma)$.

Let $\mathcal{K}^{\lambda,\perp} \subseteq X \times Z_0 \times C^{*,\lambda}(\Sigma)$ be the Banach sub-bundle whose fibre over any point $(x, z) \in X \times Z_0$ is the orthogonal complement of $\mathcal{K}_{x,z}$ in $C^{*,\lambda}(\Sigma)$ with respect to the L^2 inner-product of $e_z^* g_0$. We define $\Pi^\lambda : X \times Z_0 \times C^{*,\lambda}(\Sigma) \longrightarrow \mathcal{K}^{\lambda,\perp}$ such that for all $(x, z) \in X \times Z_0$, $\Pi^\lambda_{x,z} := \Pi^\lambda(x, z, \cdot)$ is the projection along the fibre $\mathcal{K}_{x,z}$. Observe that Π^λ is a smooth Banach bundle map.

We now define $\overline{H}^{\lambda+2}:\mathcal{U}^{\lambda+2}\to\mathcal{K}^{\lambda,\perp}$ by:

$$\overline{H}^{\lambda+2}(x,z,f) := \Pi^{\lambda}_{x,z}\circ H^{\lambda+2})(x,z,f).$$

Let $D_3\overline{H}^{\lambda+2}$ be the partial derivative of $\overline{H}^{\lambda+2}$ with respect to the third component in $X\times Z_0\times C^{*,\lambda+2}(\Sigma)$. We claim that, for all $z\in Z_0$, the restriction of $D_3(\overline{H}^{\lambda+2},\Theta^{\lambda+2})(x_0,z,0) = (\Pi^{\lambda}_{x,z}\circ \mathrm{J}^h_{g_0,e_z},\mathrm{J}^{\theta}_{g_0,e_z})$ to $\mathcal{K}^{\lambda+2,\perp}_{x_0,z}$ defines a linear isomorphism onto $\mathcal{K}^{\lambda,\perp}_{x_0,z}\times C^{*,\lambda+1}(\partial\Sigma)$. Indeed, by definition, J_{g_0,e_z} restricts to a linear isomorphism from $\mathcal{K}^{\lambda+2,\perp}_{x_0,z}$ to $\mathrm{Im}^{\lambda+2}(\mathrm{J}_{g_0,e_z})$, and it thus suffices show that the restriction of $(\Pi^{\lambda}_{x_0,z},\mathrm{Id})$ to $\mathrm{Im}^{\lambda+2}(\mathrm{J}_{g,e_z})$ defines a linear isomorphism onto $\mathcal{K}^{\lambda,\perp}_{x_0,z}\times C^{*,\lambda+1}(\partial\Sigma)$. However, since J_{g_0,e_z} is Fredholm of index zero:

$$\mathrm{Dim}(\mathrm{Ker}(\Pi^{\lambda}_{x_0,z},\mathrm{Id})) = \mathrm{Dim}(\mathcal{K}) = \mathrm{Codim}(\mathrm{Im}^{\lambda+2}(\mathrm{J}_{g_0,e_z})).$$

Consequently, if $\mathrm{Ker}(\Pi^{\lambda}_{x_0,z},\mathrm{Id})\cap\mathrm{Im}^{\lambda+2}(\mathrm{J}_{g_0,e_z})=\{0\}$, then:

$$C^{\lambda}(\Sigma)\times C^{\lambda+1}(\partial\Sigma) = \mathrm{Ker}(\Pi^{\lambda}_{x_0,z},\mathrm{Id})\oplus\mathrm{Im}^{\lambda+2}(\mathrm{J}_{g_0,e_z}),$$

and the assertion would follow. It thus suffices to show that this intersection is trivial. However, if $(\psi,0)\in\mathrm{Ker}(\Pi^{\lambda}_{x_0,z},\mathrm{Id})\cap\mathrm{Im}^{\lambda+2}(\mathrm{J}_{g_0,e_z})$, then there exists $\varphi\in C^{*,\lambda+2}(\Sigma)$ such that $\mathrm{J}^h_{g_0,e_z}(\varphi)=\psi$ and $\mathrm{J}^{\theta}_{g_0,e_z}(\varphi)=0$. Moreover, since $\psi\in\mathrm{Ker}(\mathrm{J}_{g_0,e_z})$, $\mathrm{J}^{\theta}_{g_0,e_z}(\psi)=0$. Thus, denoting by dV the volume form of $e_z^*g_0$, and bearing in mind Proposition 2.13, we have:

$$\int_\Sigma \psi^2\,dV = \int_\Sigma(\mathrm{J}^h_{g_0,e_z}\varphi)\psi\,dV = \int_\Sigma\varphi(\mathrm{J}^h_{g_0,e_z}\psi)\,dV = 0,$$

and the intersection is, therefore, trivial, as desired. Since Z_0 is compact, it now follows from the implicit function theorem for Banach manifolds that there exists a compact neighbourhood Y of x_0 and a unique continuous (in fact, smooth) map $F:Y\times Z_0\to\mathcal{K}^{\lambda+2,\perp}$ such that, for all $z\in Z_0$, $F(x_0,z)=0$ and for all $(x,z)\in Y\times Z_0$:

$$(\overline{H}^{\lambda+2},\Theta^{\lambda+2})(x,z,F(x,z)) = (0,0).$$

It remains to prove that $f:Y\times Z_0\times\Sigma\longrightarrow\mathbb{R}$ is smooth. We first show that F defines a smooth map into $\mathcal{K}^{\mu+2,\perp}$ for all $\mu\in[0,\infty[\setminus\mathbb{N}$. Indeed, choose $(x,z)\in Y\times Z$ and choose $\mu\in[0,\infty[\setminus\mathbb{N}$ such that $\mu>\lambda$. Since invertibility is an open property, upon reducing Y if necessary, we may suppose that $D_3\overline{H}^{\lambda+2}(x,z,f(x,z))$ maps $\mathcal{K}^{\lambda+2,\perp}_{x,z}$ invertibly into $\mathcal{K}^{\lambda,\perp}_{x,z}\times C^{*,\lambda+1}(\partial\Sigma)$. However, by Proposition 2.15, $\mathrm{Ker}(\mathrm{J}_{g_0,e_z})\subseteq C^{\infty}(\Sigma)$, so that $H^{\lambda+2}(x,z,F(x,z))\in C^{\infty}(\Sigma)$, and, by Proposition 3.6, $F(x,z)\in C^{\infty}(\Sigma)\subseteq C^{*,\mu+2}(\Sigma)$. Furthermore, by Proposition 2.15 again, $D_3\overline{H}^{\mu+2}(x,z,F(x,z))$ maps $\mathcal{K}^{\mu+2,\perp}_{x,z}$ invertibly into $\mathcal{K}^{\mu,\perp}_{x,z}\times C^{*,\mu+1}(\partial\Sigma)$. Thus, by the implicit function theorem for Banach manifolds, there exists a neighbourhood, Ω, of $(x,z)\in Y\times Z$ and a smooth map $F':\Omega\to$

$\mathcal{K}^{\mu+2,\perp} \subseteq \mathcal{K}^{\lambda+2,\perp}$ such that $F'(x,z) = F(x,z)$ and for all $(x',z') \in \Omega$:

$$(\tilde{H}^{\mu+2}, \Theta^{\mu+2})(x', z', F'(x', z')) = (0, 0).$$

However, by uniqueness, F' coincides with F, and so F defines a smooth map from Ω into $C^{*,\mu+2}(\Sigma)$, as asserted. Since μ is arbitrary, it follows by Proposition 4.4 that the function $f : Y \times Z \times \Sigma \to \mathbb{R}$ given by:

$$f(y,z,p) = F(y,z)(p)$$

is smooth, and this completes the proof. q.e.d.

5.2. Global sections over non-degenerate families. Let $Y \subseteq X$ and $F : Y \times Z_0 \longrightarrow C^\infty(\Sigma)$ be as in Proposition 5.2. Recalling that $(\Psi, \mathcal{U}, \mathcal{V})$ is the graph chart of $X \times Z_0 \times \mathcal{E}$ generated by $(X \times Z_0, e)$, we define $\widetilde{e} : Y \times Z_0 \times \Sigma \to M$ by:

$$\widetilde{e}_{x,z} := \widetilde{e}(x,z,\cdot) = \Psi(x,z,F_{x,z}).$$

We define $\mathcal{A} : Y \times Z_0 \to \mathbb{R}$ by:

$$\mathcal{A}(x,z) = \mathrm{Vol}(\widetilde{e}_{x,z}) = \int_\Sigma dV_{x,z},$$

where $dV_{x,z}$ is the volume form of $\widetilde{e}^*_{x,z}\widetilde{g}_{x,z}$, and, for all $x \in Y$, we denote $\mathcal{A}_x := \mathcal{A}(x,\cdot)$. As in Section 3.3, we define $\widetilde{\lambda} : Y \times TZ_0 \times \Sigma \to \mathbb{R}$ such that for all $(x,z) \in Y \times Z_0$ and for all $\xi_z \in T_z Z_0$:

$$\widetilde{\lambda}_{x,z}(\xi_z) := \widetilde{\lambda}(x,z,\xi_z,\cdot) = \widetilde{g}_z((D_2\widetilde{e})_{x,z}(\xi_z), \widetilde{N}_{x,z}),$$

where $D_2\widetilde{e}$ is the partial derivative of $\widetilde{e}$ with respect to the second component in $Y \times Z_0 \times \Sigma$, and $\widetilde{N}_{x,z}$ is the unit, normal vector field over $\widetilde{e}_{x,z}$ with respect to $\widetilde{g}_{x,z}$ which is compatible with the orientation. Observe that $\widetilde{\lambda}_{x,z}$ defines a linear map from $T_z Z_0$ to $C^\infty(\Sigma)$. Finally, we define $\widetilde{h} : Y \times Z_0 \times \Sigma \to \mathbb{R}$ such that for all $(x,z) \in Y \times Z_0$:

$$\widetilde{h}_{x,z} := \widetilde{h}(x,z,\cdot) = H(x,z,F_{x,z}).$$

The first variation formula for area immediately yields:

Proposition 5.3. *For all $(x,z) \in Y \times Z_0$ and for all $\xi_z \in T_z Z$:*

$$d\mathcal{A}_x(\xi_z) = \int_\Sigma \widetilde{h}_{x,z}\widetilde{\lambda}_{x,z}(\xi_z) dV_{x,z}.$$

Proposition 5.4. *Upon reducing Y if necessary, for all $(x,z) \in Y \times Z_0$, the pairing:*

$$T_z Z_0 \times \mathrm{Ker}(\mathrm{J}_{g_0,e_z}) \longrightarrow \mathbb{R}; (\xi_z, \varphi) \mapsto \int_\Sigma \varphi \widetilde{\lambda}_{x,z}(\xi_z) dV_{x,z}$$

is non-degenerate.

Proof. Choose $z \in Z_0$. There exists a neighbourhood Ω of z in Z_0 and smooth mappings $\alpha : \Omega \times \Sigma \longrightarrow \Sigma$ and $\psi : \Omega \times \Sigma \to \mathbb{R}$ such that $\alpha(z,\cdot)$ coincides with the identity map, $\psi(z,\cdot) = 0$, and, for all $w \in \Omega$, $\alpha_w := \alpha(w,\cdot)$ is a smooth diffeomorphism of Σ and $\Psi(x_0, z, \psi_w) \circ \alpha_w = \tilde{e}_{x_0,w}$, where $\psi_w := \psi(w,\cdot)$. In particular, for all $w \in \Omega$, $(H,\Theta)(x_0, z, \psi_w) = (0,0)$, and so, for all $\xi_z \in T_z Z$:

$$\mathrm{J}_{g,e_z}(D_1\psi)_z(\xi_z) = D_3(H,\Theta)(x_0,z,0)(D_1\psi)_z(\xi_z) = 0. \tag{5.1}$$

However, as in the proof of Proposition 3.8, $(D_1\psi)_z(\xi_z) = \widetilde{\lambda}_{x_0,z}(\xi_z)$, from which it follows that $\widetilde{\lambda}_{x_0,z}$ maps $T_z Z_0$ into $\mathrm{Ker}(\mathrm{J}_{g_0,e_z})$. Moreover, since $\mathcal{F}$ is an immersion, $\widetilde{\lambda}_{x_0,z}$ is injective for all $z \in Z_0$, and since $\mathcal{F}$ is non-degenerate, $\mathrm{Dim}(TZ_0) = \mathrm{Dim}(\mathrm{Ker}(\mathrm{J}_{g_0,e_z}))$, so that $\tilde{\lambda}_{x_0,z}$ is a linear isomorphism. The pairing (5.1) is, therefore, non-degenerate at this point, and the result now follows by compactness of Z_0. q.e.d.

Combining Propositions 5.3 and 5.4 immediately yields:

Proposition 5.5. *For all $(x,z) \in Y \times Z_0$, $\widetilde{h}_{x,z} = 0$ if and only if $d\mathcal{A}_x(z) = 0$.*

Gluing together sections over different charts, we now obtain:

Proposition 5.6. *There exists a compact neighbourhood Y of x_0 in X, a smooth map $\widetilde{\mathcal{F}} : Y \times Z \longrightarrow \mathcal{E}$ and a smooth family of sections $\sigma : Y \times Z \to T^*Z$ such that:*

1) *the restriction of $\tilde{\mathcal{F}}$ to $\{x_0\} \times Z$ coincides with $\mathcal{F}$; and*
2) *for all $(y,z) \in Y \times Z$, $(y, \tilde{\mathcal{F}}(y,z))$ is an element of $\mathcal{Z}(Y \times Z)$ if and only if $\sigma(y,z) = 0$.*

Proof. Since Z is compact, there exists a finite family $(Z_i, e_i)_{1 \leqslant i \leqslant m}$ of local parameterisations of $(Z, \mathcal{F})$ which covers Z. Choose $1 \leqslant i \leqslant m$. Let $Y_i \subseteq X$ and $F_i : Y_i \times Z_i \longrightarrow C^\infty(\Sigma)$ be as in Propositions 5.2 and 5.4. Define F_i, $\widetilde{e}_i$ and $\mathcal{A}_i$ as above. Define $\widetilde{\mathcal{F}}_i : Y_i \times Z_i \to \mathcal{E}$ by $\widetilde{\mathcal{F}}_i(x,z) := [\widetilde{e}_{i,x,z}]$ and define $\sigma_i : Y_i \times Z_i \to T^*Z_i$ by $\sigma_i(x,z) := d\mathcal{A}_{i,x}(z)$.

Denote $Y = Y_1 \cap ... \cap Y_m$. Choose $z \in Z_i \cap Z_j$. Since $[e_{i,z}] = \mathcal{F}(z) = [e_{j,z}]$, there exists a smooth, orientation-preserving diffeomorphism $\alpha : \Sigma \longrightarrow \Sigma$ such that $e_{i,z} \circ \alpha = e_{j,z}$. By uniqueness, for all $x \in Y$, $F_{i,x,z} \circ \alpha = F_{j,x,z}$, so that $\widetilde{e}_{i,x,z} \circ \alpha = \widetilde{e}_{j,x,z}$ and $\widetilde{\mathcal{F}}_i(x,z) = \widetilde{\mathcal{F}}_j(x,z)$. We thus define $\widetilde{\mathcal{F}} : Y \times Z \to \mathcal{E}$ to be equal to $\tilde{\mathcal{F}}_i$ over $Y \times Z_i$, and since every $\tilde{\mathcal{F}}_i$ is smooth, so too is $\tilde{\mathcal{F}}$. Finally, we define $\mathcal{A} : Y \times Z \to \mathbb{R}$ by $\mathcal{A}(x,z) := \mathrm{Vol}(\tilde{\mathcal{F}}(x,z))$. $\mathcal{A}$ is a smooth function and, by Proposition 5.5, $\tilde{\mathcal{F}}(x,z)$ is minimal with respect to g_x if and only if $\sigma(x,z) = d\mathcal{A}_x(z)$ vanishes. Since $\tilde{\mathcal{F}}(x,z)$ always meets ∂M orthogonally, it follows that $(x, \tilde{\mathcal{F}}(x,z))$ is an element of $\mathcal{Z}(Y \times Z)$ if and only if $\sigma(x,z) = 0$, and this completes the proof. q.e.d.

5.3. Non-degenerate sections. We briefly consider the following general result for sections of bundles over finite-dimensional manifolds. Let N_1 and N_2 be two Riemannian manifolds, let V be a smooth vector bundle over N_2 and let $\sigma : N_1 \times N_2 \longrightarrow V$ be a smooth family of sections of V parametrised by N_1. We say that σ is *non-degenerate* whenever $D_1\sigma(p,q)$ defines a surjective map from T_pN_1 onto V_qN_2 for all $(p,q) \in \sigma^{-1}(\{0\})$. Non-degenerate families of sections are of interest due to the following result:

Proposition 5.7. *If $\sigma : N_1 \times N_2 \longrightarrow V$ is a non-degenerate family of sections, then $W := \sigma^{-1}(\{0\})$ is a smooth, embedded submanifold of $N_1 \times N_2$ of dimension equal to* $\mathrm{Dim}(N_1)+\mathrm{Dim}(N_2)-\mathrm{Dim}(V)$*. Moreover, if N_2 is compact, then there exists an open, dense subset $N_1^0 \subseteq N_1$ such that for all $p \in N_1^0$, every zero of the section $\sigma_p := \sigma(p,\cdot)$ is non-degenerate.*

Proof. The first assertion follows from the implicit function theorem. Let $\pi : N_1 \times N_2 \to N_1$ be the canonical projection onto the first factor. Let $N_1^0 \subseteq N_1$ be the set of regular values of the restriction of π to W. By Sard's Theorem, N_1^0 is a dense subset of N_1, and by compactness of N_2 it is open. Choose $p \in N_1^0$. It now follows by basic linear algebra that every zero of the section σ_p is non-degenerate, and this completes the proof. q.e.d.

In the present framework, we have the following result:

Proposition 5.8. *There exists an extension $\widetilde{X}$ of X and a compact neighbourhood Y of x_0 in $\widetilde{X}$ with the property that if $\widetilde{h} : Y \times Z_0 \times S \to \mathbb{R}$ is defined as in the preceding section, then $\widetilde{h}$ defines a non-degenerate family of sections of $\mathcal{K}$ over Z_0 parametrised by Y.*

Proof. We define the map $\widetilde{g} : C^\infty(M) \times X \times M \to \mathrm{Sym}^+(TM)$ such that, for all $\varphi \in C^\infty(M)$ and for all $x \in X$:

$$\widetilde{g}_{\varphi,x} := \widetilde{g}(\varphi, x, \cdot) = e^{\varphi} g_x.$$

Let E be a finite-dimensional, linear subspace of $C^\infty(M)$, and for $r > 0$, let E_r be the closed ball of radius r about 0 in E with respect to some metric. Since X is compact, for sufficiently small r, and for all $(\varphi, x) \in E_r \times X$, the metric $g_{\varphi,x}$ is admissible. We denote $\widetilde{X} = E_r \times X$ and we will show that $\widetilde{X}$ has the desired properties for suitable choices of E and r.

Choose $z \in Z_0$. Let $\{\psi_1, ..., \psi_m\} \subset \mathrm{Ker}(\mathrm{J}_{g_0,e_z})$ be a basis. Let $\varphi_1, ..., \varphi_m \in C^\infty(M)$ be as in Proposition 2.9 with $U = M$ and let E_z be the linear span of $\varphi_1, ..., \varphi_m$ in $C^\infty(M)$.

Let $Y_z \subseteq E_{z,r} \times X$ and $f_z : Y_z \times Z_0 \times \Sigma \longrightarrow \mathbb{R}$ be, respectively, a compact neighbourhood of $(0, x_0)$ and a smooth function as in Proposition 5.2. We define $\tilde{h}_z : Y_z \times Z_0 \to C^\infty(\Sigma)$ by:

$$\widetilde{h}_{z,(\varphi,x,w)} := \widetilde{h}_z(\varphi, x, w) = H(\varphi, x, w, f_{z,(\varphi,x,w)}).$$

Let $D_1\widetilde{h}_z$ be the partial derivative of $\widetilde{h}_z$ with respect to the first component in $E_{z,r} \times X \times Z_0$. We claim that $(D_1\widetilde{h}_z)_{(0,x_0,z)}$ defines a surjective map from E_z onto $\mathcal{K}_{(0,x_0,z)}$. First, since $f_{z,(0,x_0,z)} = 0$, and since $f_{z,(\varphi,x_0,z)} \in \mathcal{K}_z^\perp$ for all $\varphi \in E_{z,r}$, it follows that $(D_1 f_z)_{(0,x_0,z)}\varphi_k \in \mathcal{K}_z^\perp = \mathrm{Ker}(\mathrm{J}_{g_0,e_z})^\perp$ for all $1 \leqslant k \leqslant m$. On the other hand, $(\Pi_z \circ \mathrm{J}^h_{g_0,e_z}, \mathrm{J}^\theta_{g_0,e_z})(D_1 f_z)_{(0,x_0,z)}\varphi_k = 0$, where $\Pi_z : C^\infty(\Sigma) \longrightarrow \mathcal{K}_z^\perp$ is the orthogonal projection with respect to the L^2-inner-product of $e_z^* g_0$. However, as in the proof of Proposition 5.2, the restriction of $(\Pi_z \circ \mathrm{J}^h_{g_0,e_z}, \mathrm{J}^\theta_{g_0,e_z})$ to $\mathcal{K}_z^\perp$ is injective, and so $(D_1 f_z)_{(0,x_0,z)}\varphi_k = 0$ for all $1 \leqslant k \leqslant m$. Applying the chain rule now yields:

$$(D_1\widetilde{h}_z)_{(0,x_0,z)}\varphi_k = \psi_k,$$

for all $1 \leqslant k \leqslant m$, so that $\mathrm{Im}((D_1\tilde{h}_z)_{(0,x_0,z)}) = \mathrm{Ker}(\mathrm{J}_{g_0,e_z}) = \mathcal{K}_{(0,x_0),z}$ and $(D_1\tilde{h}_z)_{(0,x_0,z)}$ thus defines a surjective map from E_z onto $\mathcal{K}_{(0,x_0,z)}$ as asserted. Now observe that if E contains E_z then, by uniqueness, the restrictions of f and $\tilde{h}$ to $E_z \times X \times Z \times S$ coincide with f_z and $\tilde{h}_z$, respectively. Since surjectivity of Fredholm maps is an open property, there, therefore, exists a neighbourhood W_z of z in Z_0 such that if E contains E_z, then $(D_1\tilde{h})_{(0,x_0,z)}$ defines a surjective map from E onto $\mathcal{K}_{(0,x_0,w)}$ for all $w \in W$. By compactness of Z_0, there exists a finite collection $z_1, ..., z_m$ of points in Z_0 such that:

$$Z_0 = \bigcup_{k=1}^{m} W_{z_k}.$$

We denote $E = E_{z_1} + ... + E_{z_k}$, so that $(D_1\widetilde{h})_{x_0,z}$ defines a surjective map from $T_{x_0}\widetilde{X}$ onto $\mathrm{Ker}(\mathrm{J}_{g_0,e_z})$ for all $z \in Z_0$, and the result now follows by compactness of Z_0 again. q.e.d.

Proposition 5.9. *There exists an extension $\widetilde{X}$ of X and a compact neighbourhood Y of x_0 in $\widetilde{X}$ such that if $\sigma : Y \times Z \to T^*Z$ is defined as in Proposition* 5.6*, then σ defines a non-degenerate family of sections of T^*Z over Z parametrised by Y.*

Proof. We use the notation of the proof of Proposition 5.6. We denote $\widetilde{X}_0 = X$. For $1 \leqslant i \leqslant m$, having defined $\widetilde{X}_{i-1}$, we extend it to $\widetilde{X}_i$ so that it satisfies the conclusion of Proposition 5.8 with $Z_0 = Z_i$. We denote $\widetilde{X} = \widetilde{X}_m$. By compactness, for $1 \leqslant i \leqslant m$, there exists a compact neighbourhood Y_i of x_0 in $\widetilde{X}$ such that $\widetilde{X}$ satisfies the conclusion of Proposition 5.8 with $Y = Y_i$ and $Z_0 = Z_i$. We denote $Y = Y_1 \cap ... \cap Y_m$.

Choose $1 \leqslant i \leqslant m$. Choose $(x, z) \in Y \times Z_i$ such that $\sigma_i(x, z) = 0$. By Proposition 5.5, $\widetilde{h}_{i,x,z} = 0$. Now choose $\alpha \in T^*Z_i$. By Proposition 5.4, there exists $\psi \in \mathcal{K}_{i,x,z}$ such that for all $\xi_z \in T_z Z_i$:

$$\alpha(\xi_z) = \int_\Sigma \psi \widetilde{\lambda}_{i,x,z}(\xi_z)\, dV_{i,x,z}.$$

However, by construction, there exists $\eta_x \in T_x \tilde{X}$ so that $(D_1 \tilde{h}_{i,x,z})(\eta_x) = \psi$, and so, for all $\xi_z \in TZ_i$:

$$\begin{aligned} D_1\sigma_{i,x,z}(\eta_x)(\xi_z) &= \int_\Sigma D_1\widetilde{h}_{i,x,z}(\eta_x)\widetilde{\lambda}_{i,x,z}(\xi_z) dV_{x,z} \\ &= \int_\Sigma \psi \widetilde{\lambda}_{i,x,z}(\xi_z)\, dV_{x,z} \\ &= \alpha(\xi_z). \end{aligned}$$

It follows that $D_1\sigma_{i,x,z}$ is surjective, and σ_i, therefore, defines a non-degenerate family of sections of T^*Z_i over Z_i parametrised by Y. Since i is arbitrary, the result follows. q.e.d.

5.4. Determining the index. The following result is proven in [**25**]:

Lemma 5.10. *Let A be an element of $\mathcal{F}^+(E, F)$. Let $K \subseteq E$ be the kernel of A. There exists a neighbourhood U of A in $\mathcal{F}^+(E, F)$ such that if $A' \in U$ and if A' maps K' into K for some $K' \subseteq E$ of dimension equal to that of K, then:*

$$\mathrm{Null}(A') = \mathrm{Null}(A'|_{K'}), \qquad \mathrm{Ind}(A') = \mathrm{Ind}(A) + \mathrm{Ind}(A'|_{K'}),$$

where $A'|_{K'}$ denotes the restriction of the bilinear form $\langle A'\cdot, \cdot\rangle$ to K'.

Proposition 5.11. *For all $(x, z) \in Y \times Z_0$ such that $\sigma(x, z) = 0$ and for all $\xi_z \in T_z Z$:*

$$(\mathrm{J}^h_{(x,z),\widetilde{e}_{x,z}} \circ \widetilde{\lambda}_{x,z})(\xi_z) \in \mathrm{Ker}(\mathrm{J}_{g_0,e_z}),$$

and, for all $\xi_z, \eta_z \in T_z Z_0$:

$$D\sigma_x(z)(\xi_z, \eta_z) = \int_\Sigma (\mathrm{J}^h_{(x,z),\widetilde{e}_{x,z}} \circ \widetilde{\lambda}_{x,z})(\xi_z)\widetilde{\lambda}_{x,z}(\eta_z)\, dV_{x,z},$$

where $dV_{x,z}$ is the volume form of $\widetilde{e}^*_{x,z}\widetilde{g}_{x,z}$.

Proof. Since $\sigma(x, z) = 0$, by Proposition 5.5, $\widetilde{h}_{x,z} = 0$. Thus, for all $\xi_z \in T_z Z$, as in the proof of Proposition 3.8:

$$(D_2\widetilde{h})_{x,z}(\xi_z) = (\mathrm{J}^h_{(x,z),\widetilde{e}_{x,z}} \circ \widetilde{\lambda}_{x,z})(\xi_z),$$

from which it follows that for all $\xi_z, \eta_z \in T_z Z_0$:

$$\begin{aligned} D\sigma_x(z)(\xi_z, \eta_z) &= \int_\Sigma (D_2\widetilde{h})_{x,z}(\xi_z)\widetilde{\lambda}_{x,z}(\eta_z)\, dV_{x,z} \\ &= \int_\Sigma (\mathrm{J}^h_{(x,z),\widetilde{e}_{x,z}} \circ \widetilde{\lambda}_{x,z})(\xi_z)\widetilde{\lambda}_{x,z}(\eta_z)\, dV_{x,z}, \end{aligned}$$

and the second assertion follows. Moreover, since $\widetilde{h}_{x,z}$ is an element of $\mathrm{Ker}(\mathrm{J}_{g_0,e_z})$ for all $(x,z)\in Y\times Z$, when $\widetilde{h}_{x,z}=0$:

$$(\mathrm{J}^h_{(x,z),\widetilde{e}_{x,z}}\circ\lambda_{x,z})(\xi_z)=(D_2\widetilde{h})_{x,z}(\xi_z)\in\mathrm{Ker}(\mathrm{J}_{g_0,e_z}).$$

This proves the first assertion, and completes the proof. q.e.d.

Combining the above results yields:

Theorem 5.12. *If $\mathcal{Z}(\{x_0\})$ contains a closed, non-degenerate family Z, then there exists a neighbourhood Ω of Z in $\mathcal{E}$ such that:*

$$\mathcal{Z}(\{x_0\})\cap\overline{\Omega}=Z.$$

Moreover, for any such neighbourhood Ω, there exists a compact neighbourhood Y of x_0 in X such that $\partial_\omega\mathcal{Z}(Y|\Omega)=\emptyset$ and the local mapping degree of the restriction of Π to $\mathcal{Z}(Y|\Omega)$ is given by:

$$\mathrm{Deg}(\Pi|\Omega)=(-1)^{\mathrm{Ind}(Z_0)}\chi(Z_0),$$

where $\mathrm{Ind}(Z_0)$ and $\chi(Z_0)$ are, respectively, the index and Euler characteristic of Z_0.

Proof. Let $\mathcal{F}:Z\to\mathcal{E}$ be the canonical embedding. In what follows, $\tilde{X}$ will be a suitable extension of X and $Y\subseteq\tilde{X}$ will be a suitably small compact neighbourhood of x_0 in $\tilde{X}$. By Theorems 4.2 and 4.5, $\mathcal{Z}(\tilde{Y})$ is a smooth, compact $\mathrm{Dim}(\tilde{X})$-dimensional manifold. By Proposition 4.4, $\mathcal{F}$ defines a smooth map from Z_0 into $\mathcal{Z}(\tilde{Y})$, and since $(Z,\mathcal{F})$ is non-degenerate, this map is an embedding. Let $\tilde{\mathcal{F}}:\tilde{Y}\times Z\to\mathcal{E}$ be a smooth map extending $\mathcal{F}$ and let $\sigma:\tilde{Y}\times Z\to T^*Z$ be a smooth family of sections as in Proposition 5.6. By Proposition 5.9, we may suppose that σ is non-degenerate. We, therefore, define $W\subseteq\tilde{Y}\times Z$ by $W:=\sigma^{-1}(\{0\})$, and, by Proposition 5.7, W is a smooth, $\mathrm{Dim}(\tilde{X})$-dimensional embedded submanifold of $Y\times Z$. We define $\tilde{G}:W\longrightarrow\tilde{Y}\times\mathcal{E}$ by $\tilde{G}(y,z):=(y,\widetilde{\mathcal{F}}(y,z))$. Observe that $\widetilde{G}$ defines a smooth map from W into $\mathcal{Z}(Y)$.

Since $D\tilde{G}$ is a linear isomorphism at (x_0,z) for all $z\in Z$, we may assume that it is also a linear isomorphism at every point of $\tilde{Y}\times Z$. Since $\mathcal{F}$ is a diffeomorphism and since Z is compact, we may, in fact, assume that $\tilde{G}$ is a diffeomorphism onto its image in $\mathcal{Z}(Y)$. In particular, $Z=\tilde{G}(\{x_0\}\times Z)$ is an isolated subset of $\mathcal{Z}(\{x_0\})$, and there, therefore, exists a neighbourhood Ω of Z in $\mathcal{E}$ such that $Z=\mathcal{Z}(\{x_0\})\cap\overline{\Omega}$. This proves the first assertion. Furthermore, by elementary point-set-topological arguments, we may also assume that $\tilde{G}(W)=\mathcal{Z}(\tilde{Y}|\Omega)$.

Now, for all $y\in\tilde{Y}$:

$$\mathcal{Z}(\{y\}\,|\Omega)=\tilde{G}(\{(y,z)\mid z\in\sigma_y^{-1}(\{0\})\}),$$

and, since σ is a non-degenerate family, it follows from Proposition 5.7 that there exists $y\in\tilde{Y}_0$ such that the zeroes of the section $d\mathcal{A}_y=$

σ_y are non-degenerate. We claim that y is also a regular value of the restriction of Π to $\mathcal{Z}(\widetilde{Y}|\Omega)$. Indeed, by Proposition 4.9 it suffices to show that $\mathrm{J}_{(y,z),\tilde{e}_{y,z}}$ is invertible for all $z \in \sigma_y^{-1}(\{0\})$. However, choose $z \in \sigma_y^{-1}(\{0\})$. By Lemma 5.10 and Proposition 5.11, we may suppose that $\mathrm{Null}(\mathrm{J}_{(y,z),\widetilde{e}_{y,z}}) = \mathrm{Null}(\mathrm{J}_{(y,z),\widetilde{e}_{y,z}}|_{E_{y,z}})$, where $E_{y,z} = \left\{\widetilde{\lambda}_{y,z}(\xi_z) \mid \xi_z \in T_z Z_0\right\}$, and, by Proposition 5.11 again:

$$\mathrm{Null}(\mathrm{J}_{\widetilde{g}_{y,z},\widetilde{e}_{y,z}}|_{E_{y,z}}) = \mathrm{Null}(D\sigma_y(z)) = 0, \tag{5.2}$$

since σ_y is non-degenerate. The point y is thus a regular value of the restriction of Π to $\mathcal{Z}(\widetilde{Y}|\Omega)$, as asserted.

By Lemma 5.10 and Proposition 5.11 again:

$$\begin{aligned}\mathrm{Ind}(\mathrm{J}_{(y,z),\widetilde{e}_{y,z}}) &= \mathrm{Ind}(\mathrm{J}_{g_0,e_z}) + \mathrm{Ind}(\mathrm{J}_{(y,z),\widetilde{e}_{y,z}}|_{E_{y,z}})\\ &= \mathrm{Ind}(Z_0) + \mathrm{Ind}(d\sigma_y(z)),\end{aligned}$$

so that the mapping degree of the restriction of Π to $\mathcal{Z}(\tilde{Y}|\Omega)$ is given by:

$$\begin{aligned}\mathrm{Deg}(\Pi|\Omega) &= \sum_{(y,[e])\in\mathcal{Z}(\{y\}|\Omega)} \mathrm{Sig}(\mathrm{J}_{y,e})\\ &= \sum_{z\in\sigma_y^{-1}(\{0\})} \mathrm{Sig}(\mathrm{J}_{(y,z),\widetilde{e}_{y,z}})\\ &= (-1)^{\mathrm{Ind}(Z_0)} \sum_{z\in\sigma_y^{-1}(\{0\})} \mathrm{Sig}(\mathrm{Hess}(\mathcal{A}_y)(z))\\ &= (-1)^{\mathrm{Ind}(Z_0)}\chi(Z_0),\end{aligned}$$

where the last equality follows from classical Morse theory. This completes the proof. q.e.d.

6. Free boundary minimal surfaces inside convex domains

6.1. Rotationally invariant free boundary minimal surfaces. Let δ be the Euclidean metric over $\mathbb{R}^3$ and let $B := B^3 \subset \mathbb{R}^3$ be the unit Euclidean three-ball. In order to apply degree theoretic techniques, it is preferable to work with metrics of strictly positive curvature. Thus, for $-1 < t < 1$, define the metric g_t over B by:

$$g_t(x) = \delta + \frac{t^2}{1 - t^2\|x\|^2} x \otimes x, \tag{6.1}$$

where δ is the standard Euclidean metric. Observe that $g_0 = \delta$ and, for all $t \neq 0$, g_t is the metric of a spherical cap of radius $1/|t|$. In particular, for all $t \in (-1,1)$, g_t has positive constant sectional curvature equal to t^2, and (B, g_t) is functionally strictly convex.

For every unit vector v in $\mathbb{R}^3$ and for all $\theta \in \mathbb{R}$, we define $R_{v,\theta} \in \mathrm{SO}(3)$ to be the rotation about v by θ radians in the positive direction (with respect to the canonical orientation of $\mathbb{R}^3$). In this section, we consider embedded surfaces in B mainly as subsets of B (rather than as equivalence classes of embeddings). We recall that an embedded surface $\Sigma \subseteq B$ is said to be *invariant by rotation* about v whenever:

$$R_{v,\theta}(\Sigma) = \Sigma,$$

for all $\theta \in \mathbb{R}$. For $f : \mathbb{R} \to]0,\infty[$ a positive function, recall that the surface of revolution of f about v is defined by:

$$\Sigma_{v,f} = \{R_{v,\theta}(tv + f(t)w) \mid \theta, t \in \mathbb{R}\},$$

where $w \in \mathbb{R}^3$ is any unit vector orthogonal to v. Solving ODEs, we readily obtain:

Proposition 6.1. *For every unit vector $v \in \mathbb{R}^3$, the unique (unoriented) properly embedded free boundary minimal surfaces in (B,δ) which are invariant under rotation about v are:*

(1) *the disk obtained by intersecting B with the equatorial plane normal to v; and*
(2) *the annulus obtained by intersecting B with the catenoid $\Sigma_{v,f}$, where $f(t) = r_0^{-1}\cosh(r_0 t)$, $r_0 = t_0\cosh(t_0)$ and $t_0 > 0$ is the unique positive solution of $t_0 = \coth(t_0)$.*

Remark 6.2. An elementary calculation shows that $r_0 > t_0 > 1$.

Proposition 6.3. *For all $t \neq 0$ and for every vector $v \in \mathbb{R}^3$, the unique (unoriented) properly embedded free boundary minimal disk in (B, g_t) which is invariant under rotation about v is the disk obtained by intersecting with the equatorial Euclidean plane normal to v.*

Proof. Choose $t \neq 0$. Define the foliation $\{\mathcal{C}_s\}_{s\in(-1,1)}$ of $\partial B \setminus \{v,-v\}$ by $\mathcal{C}_s = \{w \in \partial B : \langle v, w\rangle_\delta = s\}$, and define the foliation $\{\mathcal{D}_s\}_{s\in(-1,1)}$ of $B \setminus \{v,-v\}$ so that for all s, $\mathcal{D}_s \subset B$ is the properly embedded disk which is totally geodesic with respect to g_t such that $\partial\mathcal{D}_s = \mathcal{C}_s$. Now let Σ be an properly embedded free boundary minimal disk in (B, g_t) which is invariant under rotation about v. In particular, $\partial\Sigma$ is equal to $\mathcal{C}_s$ for some $s \in (-1,1)$, so that, by the geometric maximum principal, $\Sigma = \mathcal{D}_s$. Since Σ meets ∂B orthogonally along $\partial\Sigma$, $s = 0$ and $\Sigma = \mathcal{D}_0$. The result follows. q.e.d.

Proposition 6.4. *There exists $\delta > 0$ such that, for all $t \in (-\delta,\delta)$ and for every vector $v \in \mathbb{R}^3$, there exists a unique (unoriented) properly embedded free boundary minimal surface in (B, g_t) which is diffeomorphic to the annulus $\mathbb{S}^1 \times [0,1]$ and invariant under rotation about v.*

Proof. We define $F :]0,\infty[\times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2$ by:

$$F(a,b,s) = (s, a^{-1}\cosh(as+b)).$$

Observe that F is a submersion into $\mathbb{R}^2$. Furthermore, for all (a,b), the image of $F(a,b,\cdot)$ is a catenoid whose surface of revolution is a properly embedded minimal surface. Let $\mathbb{S}^1$ be the unit circle in $\mathbb{R}^2$ and observe that $X := F^{-1}(\mathbb{S}^1)$ is a smooth hypersurface. Furthermore, the curve $F(r_0,0,\cdot)$ meets $\mathbb{S}^1$ orthogonally at the points $(r_0,0,\pm r_0 t_0)$. In particular, $X := F^{-1}(\mathbb{S}^1)$ is transverse to the line $\{(r_0,0,s) \mid s\in\mathbb{R}\}$ at these points. There, therefore, exists a neighbourhood Ω of $(r_0,0)$ in $]0,\infty[\times\mathbb{R}$ and smooth functions $G_\pm : \Omega\to\mathbb{R}$ such that $G_\pm(r_0,0)=\pm r_0 t$ and the graph of $G_\pm$ locally parametrises X.

Now define $\Theta : X\to\mathbb{R}$ by

$$\Theta(a,b,s) = \left\langle J_0F(a,b,s), \frac{\partial}{\partial r}F(a,b,r)|_{r=s}\right\rangle,$$

where J_0 is the standard complex structure over $\mathbb{R}^2$. The function Θ essentially measures the angle that the curve $F(a,b,\cdot)$ makes with $\mathbb{S}^1$ at the point $F(a,b,s)$. Now define $\Theta_\pm : \Omega\to\mathbb{R}$ by $\Theta_\pm(a,b) = \Theta(a,b,G_\pm(a,b))$. Observe that $\Theta_\pm(r_0,0)=0$. Furthermore, $\partial_a\Theta_-(r_0,0)$ and $\partial_a\Theta_+(r_0,0)$ are both non-zero with the same sign, but $\partial_b\Theta_-(r_0,0)$ and $\partial_b\Theta_+(r_0,0)$ are both non-zero with opposite signs. In particular, $\nabla\Theta_\pm(r_0,0)\neq 0$ and $\nabla\Theta_-(r_0,0)\neq\nabla\Theta_+(r_0,0)$, so that, upon reducing Ω if necessary, $\Theta_\pm^{-1}(\{0\})$ define smooth embedded curves in Ω which intersect transversally at the unique point $(r_0,0)$.

Now let $\tilde F : (-\delta,\delta)\times(0,\infty)\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}^2$ be such that $\tilde F(0,\cdot,\cdot,\cdot)=F$ and, for all (t,a,b), the surface of revolution of the curve $\tilde F(t,a,b,\cdot)$ is minimal with respect to the metric g_t. By transversality, upon reducing Ω and δ if necessary, we may suppose that, for all $t\in(-\delta,\delta)$, there exists a unique point $(a(t),b(t))\in\Omega$ such that the curve $\tilde F(t,a(t),b(t),\cdot)$ intersects $\mathbb{S}^1$ orthogonally with respect to the metric g_t. In particular, the surface of revolution of the curve $\tilde F(a(t),b(t),t,\cdot)$ about v is a properly embedded free boundary minimal annulus with respect to this metric, thus proving existence. Uniqueness also follows by uniqueness of the catenoid in the zero-curvature case together with a straightforward compactness argument. This completes the proof. q.e.d.

We henceforth refer to the embeddings constructed in Propositions 6.3 and 6.4, respectively, as the *critical disk* and the *critical catenoid* of the metric g_t with axis v.

6.2. Non-degenerate families of disks. Let e_1,e_2,e_3 be the canonical basis of $\mathbb{R}^3$. We parametrise the critical disk of the Euclidean metric by:

$$e_{\mathrm{disk}}(x,y) = (x,y,0).$$

Let $\mathrm{J}_{\mathrm{disk}} := (\mathrm{J}^h_{\mathrm{disk}},\mathrm{J}^\theta_{\mathrm{disk}})$ be the Jacobi operator of e_{disk} with respect to this metric. Lemma 2.5 and Proposition 2.6 immediately yield:

Proposition 6.5. *For all $\varphi \in C^\infty(D)$:*

$$\mathrm{J}^h_{\mathrm{disk}}\varphi = -\Delta\varphi,$$

where Δ is the standard Laplacian of $\mathbb{R}^2$, and:

$$\mathrm{J}^\theta_{\mathrm{disk}}\varphi = \varphi \circ \epsilon - \partial_\nu\varphi,$$

where $\epsilon : \partial D \to D$ is the canonical embedding, and ∂_ν is the partial derivative in the outward-pointing, normal direction over ∂D.

Proposition 6.6. $\mathrm{Ker}(\mathrm{J}_{\mathrm{disk}})$ *is 2-dimensional.*

Proof. Choose $\varphi \in \mathrm{Ker}(\mathrm{J}_{\mathrm{disk}})$. Since φ is harmonic, it is the real part of a holomorphic function, $\Phi : \overline{D} \to \mathbb{C}$. That is:

$$\varphi(z) = \mathrm{Re}(\Phi(z)) = \mathrm{Re}\left(\sum_{n=0}^{\infty} a_n z^n\right).$$

Observe that Φ is smooth over $\overline{D}$, and applying the Robin condition and the Cauchy–Riemann equations, therefore, yields:

$$\mathrm{Re}\left(\sum_{n=0}^{\infty}(1-n)a_n e^{in\theta}\right) = 0,$$

so that $a_n = 0$ for all $n \neq 1$, and $\varphi(z) = \mathrm{Re}(a_1 z) = \alpha x + \beta y$, where $a_1 = \alpha - i\beta$. It follows that $\mathrm{Ker}(\mathrm{J}_{\mathrm{disk}})$ is 2-dimensional, as desired. q.e.d.

Proposition 6.7. *If $\Sigma = D$ is the disk, then there exists $\delta > 0$ such that for all $t \in (-\delta, \delta)$, the family of embeddings $[e] \in \mathcal{Z}(\{g_t\})$ which are invariant under rotation about some unit vector in $\mathbb{R}^3$ constitutes a non-degenerate family diffeomorphic to $\mathbb{S}^2$.*

Proof. We define $\mathcal{I}_t : \mathbb{S}^2 \to \mathcal{Z}(\{g_t\})$ such that, for all $v \in \mathbb{S}^2$, $\mathcal{I}_t(v)$ is the critical disk of the metric g_t with axis v, oriented such that its normal coincides with v. We see that $\mathcal{I}_t$ is a smooth embedding. By Proposition 6.3, $\mathcal{I}_t(\mathbb{S}^2)$ accounts for all free boundary minimal embedded disks in $\mathcal{Z}(\{g_t\})$ which are invariant under rotation. By Proposition 6.6, when $t = 0$, the nullity of the Jacobi operator of $\mathcal{I}(v)$ with respect to the metric g_0 is equal to 2 for all $v \in \mathbb{S}^2$. By upper-semicontinuity, there exists $\delta > 0$ such that for all $|t| < \delta$ and for all $v \in \mathbb{S}^2$, the nullity of the Jacobi operator of $\mathcal{I}_t(v)$ with respect to the metric g_t is at most 2, and it follows by Proposition 5.1 that $\mathcal{I}_t(\mathbb{S}^2)$ is a non-degenerate family, as desired. q.e.d.

6.3. Non-degenerate families of catenoids. Let t_0 be as in Proposition 6.1. We parametrise the critical catenoid with axis e_3 by the map $e_{\mathrm{cat}} : [-t_0, t_0] \times \mathbb{S}^1 \to \mathbb{R}^3$ given by:

$$e_{\mathrm{cat}}(t,\theta) = (r_0^{-1}\cosh(t)\cos(\theta), r_0^{-1}\cosh(t)\sin(\theta), r_0^{-1}t).$$

Let $\mathrm{J}_{\mathrm{cat}} = (\mathrm{J}^h_{\mathrm{cat}}, \mathrm{J}^\theta_{\mathrm{cat}})$ be the Jacobi operator of e_{cat} with respect to the Euclidean metric.

Proposition 6.8. *For all $\varphi \in C^\infty([-t_0, t_0] \times \mathbb{S}^1)$ and for all $(t,\theta) \in \mathbb{R} \times \mathbb{S}^1$:*

$$(\mathrm{J}^h_{\mathrm{cat}}\varphi)(t,\theta) = -\frac{2r_0^2}{\cosh^4(t)}\varphi(t,\theta) - \frac{r_0^2}{\cosh^2(t)}(\Delta\varphi)(t,\theta),$$

where Δ is the standard Laplacian of $\mathbb{R} \times \mathbb{S}^1$, and, for all $\theta \in \mathbb{S}^1$:

$$(\mathrm{J}^\theta_{\mathrm{cat}}\varphi)(\pm t_0, \theta) = \varphi(\pm t_0, \theta) \mp t_0(\partial_t\varphi)(\pm t_0, \theta).$$

Proof. Observe that the parametrisation e_{cat} is conformal and that, for all $(t,\theta) \in \mathbb{R} \times \mathbb{S}^1$, $(e^*_{\mathrm{cat}} g_0)(t,\theta) = r_0^{-2}\cosh^2(t)(dt^2 + d\theta^2)$. Thus if Δ_{cat} denotes the Laplacian operator of the metric $e^*_{\mathrm{cat}}\delta$, then:

$$\Delta_{\mathrm{cat}} = \frac{r_0^2}{\cosh^2(t)}\Delta.$$

Let I be an interval, and let $f : I \rightarrow]0, \infty[$ be a smooth, positive function. We recall that the principle directions of the surface of revolution of f are those parallel and normal to the direction of revolution. Moreover, the principle curvature in the direction of revolution (with respect to the outward-pointing normal) is equal to $1/(f\sqrt{1 + (f')^2})$. When this surface is minimal, the other principle curvature is then equal to $-1/(f\sqrt{1 + (f')^2})$. Thus, if A denotes the shape operator of e_{cat}, then:

$$\|A\|^2 = \frac{2r_0^2}{\cosh^4(t)},$$

and so, by Lemma 2.5:

$$(\mathrm{J}^h_{\mathrm{cat}}\varphi)(t,\theta) = -\frac{2r_0^2}{\cosh^4(t)}\varphi(t,\theta) - \frac{r_0^2}{\cosh^2(t)}(\Delta\varphi)(t,\theta),$$

as desired. Finally, by Proposition 2.6, bearing in mind that the shape operator of the unit sphere in $\mathbb{R}^3$ coincides with Id:

$$(\mathrm{J}^\theta\varphi)(\pm t_0, \theta) = \varphi(\pm t_0, \theta) \mp t_0(\partial_t\varphi)(\pm t_0, \theta),$$

and this completes the proof. q.e.d.

For any function $\varphi \in C^\infty([-t_0, t_0] \times \mathbb{S}^1)$, we consider the Fourier transform of φ in the θ direction. For all $(t,\theta) \in \mathbb{R} \times \mathbb{S}^1$, we write:

$$\varphi(t,\theta) = \sum_{n \in \mathbb{Z}} \varphi_n(t) e^{in\theta},$$

where, for all $n \in \mathbb{Z}$, φ_n is the n'th Fourier mode of φ. Since the Jacobi equation is linear and homogeneous, we readily obtain:

Proposition 6.9. *A function $\varphi \in C^\infty([-t_0,t_0]\times\mathbb{S}^1)$ is an element of* $\mathrm{Ker}(\mathrm{J}_{\mathrm{cat}})$ *if and only if, for all $n \in \mathbb{Z}$:*

$$\begin{aligned}\varphi_n'' + (\frac{2}{\cosh^2(t)} - n^2)\varphi_n &= 0,\\ \varphi_n(\pm t_0) \mp t_0\varphi_n'(\pm t_0) &= 0.\end{aligned} \tag{6.2}$$

Proposition 6.10. *In that case $n=0$, there exists no non-trivial solution $\varphi_0 \in C^\infty([-t_0,t_0])$ to* (6.2)*.*

REMARK 6.11. The functions constructed in the proof of this result are obtained by considering the normal perturbations of e_{cat} arising from dilatations and from translations in the e_3 direction.

Proof. The solution space to any second-order, linear ODE (ignoring boundary conditions) is 2-dimensional. By inspection, we verify that the solution space to (6.2) with $n=0$ is spanned by $u(t) := 1 - t\tanh t$ and $v(t) := \tanh t$, and we verify that no linear combination of these solutions satisfies the boundary conditions. It follows that there exists no non-trivial solution to (6.2) with $n=0$, as desired. q.e.d.

Proposition 6.12. *For each $n\in\mathbb{Z}$ such that $|n|\geqslant 2$, there exists no non-trivial solution $\varphi_n \in C^\infty([-t_0,t_0])$ to* (6.2)*.*

Proof. Choose $|n| \geqslant 2$ and define $f_n : [-t_0,t_0] \longrightarrow \mathbb{R}$ by $f_n(t) = -n^2 + 2/\cosh^2 t$. Observe that, since $|n|\geqslant 2$, $f_n(t)\leqslant -2$. We now argue by contradiction. Suppose there exists a non-trivial solution, φ_n to (6.2) with $|n|\geqslant 2$. Since (6.2) is linear, upon multiplying by -1 if necessary, we may assume that $\varphi_n(0)\geqslant 0$. Since (6.2) is even, upon replacing $\varphi_n(t)$ with $\varphi_n(-t)$ if necessary, we may assume that $\varphi_n'(0)\geqslant 0$. Since φ_n is non-trivial, $\varphi_n(0)$ and $\varphi_n'(0)$ cannot both be equal to 0. Also, if $\varphi_n > 0$ over an interval I, then $\varphi_n'' = -f_n\varphi_n \geqslant 2\varphi_n > 0$ over I, and so φ_n is strictly convex over I. We deduce that $\varphi_n(t), \varphi_n'(t) > 0$ for all $t\in(0,t_0]$, and we, therefore, define $\gamma :]0,t_0] \to \mathbb{R}$ by:

$$\gamma(t) = \frac{\varphi_n'(t)}{\varphi_n(t)}.$$

For all t, $\gamma'(t) = -f_n(t) - \gamma(t)^2 \geqslant 2 - \gamma(t)^2$. Moreover, since $\gamma(t) > 0$ for all $t>0$, $\mathrm{LimInf}_{t\to 0}\,\gamma(t)\geqslant 0$. Since $\beta(t) := \sqrt{2}\mathrm{tanh}(\sqrt{2}t)$ satisfies:

$$\beta'(t) = 2 - \beta(t)^2,$$

with initial condition $\beta(0)=0$, it follows that $\gamma(t)\geqslant\beta(t)$ for all $t\in[0,t_0]$. In particular, bearing in mind that $t_0>1$:

$$\gamma(t_0) \geqslant \beta(t_0) = \sqrt{2}\mathrm{tanh}(\sqrt{2}t_0) > \sqrt{2}\mathrm{tanh}(\sqrt{2}) > 1 > t_0^{-1}.$$

Since the boundary condition implies that $\gamma(t_0) = t_0^{-1}$, this is absurd, and there, therefore, exists no solution to (6.2) with $|n|\geqslant 2$ as desired. q.e.d.

Proposition 6.13. *The only non-trivial solutions to* (6.2) *with* $n = \pm 1$ *are given by:*

$$\varphi_{\pm 1}(t) = a\left(\sinh(t) + \frac{t}{\cosh(t)}\right),$$

for some $a \in \mathbb{C}$.

Remark 6.14. The functions constructed in the proof of this result are obtained by considering the normal perturbations of e_{cat} arising from rotations about the axes e_1 and e_2 and from translations in the e_1 and e_2 directions.

Proof. The solution space to any second-order ODE (ignoring boundary conditions) is 2-dimensional. By inspection, we verify that the solution space to (6.2) with $n = \pm 1$ is spanned by $u(t) := \sinh t + t/\cosh t$ and $v(t) := 1/\cosh t$ and that $au + bv$ satisfies the boundary condition if and only if $b = 0$. This completes the proof. q.e.d.

Proposition 6.15. $\mathrm{Ker}(J)$ *is* 2*-dimensional.*

Proof. Choose $\varphi \in \mathrm{Ker}(\mathrm{J}^h, \mathrm{J}^\theta)$ and for $n \in \mathbb{Z}$, let φ_n be the n'th Fourier mode of φ. By Propositions 6.10 and 6.12, $\varphi_n = 0$ for $n \neq \pm 1$, and, by Proposition 6.13:

$$\varphi_{\pm 1} = a\left(\sinh(t) + \frac{t}{\cosh(t)}\right),$$

for some $a \in \mathbb{C}$. It follows that:

$$\varphi = \left(\sinh(t) + \frac{t}{\cosh(t)}\right)(a\cos(\theta) + b\sin(\theta)),$$

for some $a, b \in \mathbb{R}$. Since the space of all such functions is 2-dimensional, this completes the proof. q.e.d.

Proposition 6.16. *If* $S = \mathbb{S}^1 \times [0,1]$ *is the annulus, then there exists* $\delta > 0$ *such that for all* $t \in (-\delta, \delta)$*, the family of embeddings* $[e] \in \mathcal{Z}(\{g_t\})$ *which are invariant under rotation about some vector constitutes a non-degenerate family diffeomorphic to two disjoint copies of* $\mathbb{RP}^2$.

Proof. We define $\mathcal{I}_{t,+} : \mathbb{S}^2 \to \mathcal{Z}(\{g_t\})$ such that, for all $v \in \mathbb{S}^2$, $\mathcal{I}_{t,+}(v)$ is the extremal catenoid of the metric g_t with axis v, oriented such that its normal points towards the axis of rotation. We define $\mathcal{I}_{t,-} : \mathbb{S}^2 \to \mathcal{Z}(\{g_t\})$ such that for all $v \in \mathbb{S}^2$, $\mathcal{I}_{t,-}(v) = \mathcal{I}_{t,+}(v)$ with the reverse orientation. We see that $\mathcal{I}_{t,\pm}$ quotients down to a smooth embedding of $\mathbb{RP}^2$ into $\mathcal{E}$. By Proposition 6.4, $\mathcal{I}_{t,\pm}(\mathbb{RP}^2)$ accounts for all free boundary minimal embeddings in $\mathcal{Z}(\{g_t\})$ which are invariant under rotation. By Proposition 6.15, when $t = 0$, the nullity of the Jacobi operator of $\mathcal{I}_{0,\pm}(v)$ with respect to the metric g_0 is equal to 2 for all $v \in \mathbb{RP}^2$. By upper-semicontinuity, there exists $\delta > 0$ such that for all $|t| < \delta$ and for all $v \in \mathbb{S}^2$, the nullity of the Jacobi operator of $\mathcal{I}_{t,\pm}(v)$

with respect to the metric g_t is at most 2, and it follows by Proposition 5.1 that $\mathcal{I}_{t,\pm}(\mathbb{RP}^2)$ is a non-degenerate family, as desired. q.e.d.

6.4. Calculating the degree. Let Σ be a compact surface with boundary. Let δ be a positive real number chosen as in Proposition 6.7 if Σ is diffeomorphic to the disk, D; as in Proposition 6.16 if Σ is diffeomorphic to the annulus, $\mathbb{S}^1 \times [0,1]$; and equal to 1 otherwise. We have (cf. [**26**]):

Proposition 6.17. *For all $t \in (-\delta, \delta)$, there exists $N \in \mathbb{N}$ such that if $S \subseteq B$ is an embedded surface in B which is diffeomorphic to Σ and free boundary minimal with respect to g_t, then either:*

(1) *S is invariant by rotation about some unit vector v; or*
(2) *for all unit vectors $v \in \mathbb{S}^2$, and for all $k \geqslant N$, $R_{v,2\pi/k}(S) \neq S$.*

Proof. Suppose the contrary. There exists a sequence $(k_m)_{m\in\mathbb{N}}$ in $\mathbb{N}$ converging to ∞, a sequence $(v_m)_{m\in\mathbb{N}}$ of unit vectors in $\mathbb{R}^3$ and a sequence $(S_m)_{m\in\mathbb{N}}$ of embedded surfaces in B diffeomorphic to Σ such that for all m, S_m is free boundary minimal with respect to g_t, is not invariant under rotation about any vector, but satisfies $R_{v_m,2\pi/k_m}(S_m) = S_m$. Upon extracting a subsequence, we may suppose that $(v_m)_{m\in\mathbb{N}}$ converges to a unit vector v_∞ in $\mathbb{R}^3$, say. By Theorem 2.3, upon extracting a further subsequence, we may suppose that $(S_m)_{m\in\mathbb{N}}$ converges to an embedded submanifold S_∞ which is also diffeomorphic to Σ and free boundary minimal with respect to g_t. We claim that S_∞ is invariant under rotation about v_∞. Indeed, choose $\theta \in \mathbb{R}$. Since $(k_m)_{m\in\mathbb{N}}$ converges to ∞, there exists a sequence $(l_m)_{m\in\mathbb{N}} \in \mathbb{Z}$ such that $(2\pi l_m/k_m)_{m\in\mathbb{N}}$ converges to θ. However, for all m:

$$R_{v_m,2\pi l_m/k_m}(S_m) = (R_{v_m,2\pi/k_m})^{l_m}(S_m) = S_m,$$

and taking limits yields $R_{v_\infty,\theta}(S_\infty) = S_\infty$, so that S_∞ is invariant under rotation about v_∞, as asserted. If Σ is diffeomorphic to the disk, D, then by Proposition 6.3, S_∞ is the critical disk of the metric g_t with axis v. However, by Proposition 6.7, the family of critical disks of the metric g_t is non-degenerate, and, by Theorem 5.12, is, therefore, isolated in $\mathcal{Z}(\{g_t\})$. Thus, for sufficiently large m, S_m is also a critical disk of g_t, and is, therefore, invariant under rotation about some vector, which is absurd. If Σ is diffeomorphic to the annulus, $\mathbb{S}^1 \times [0,1]$, then we likewise obtain a contradiction using Propositions 6.4 and 6.16 and Theorem 5.12. Finally, if Σ is neither diffeomorphic to the disk, nor to the annulus, then, in particular, S_∞ cannot be a surface of revolution. This is absurd, and the result follows. q.e.d.

Theorem 6.18.

$$\mathrm{Deg}(\Pi) = \begin{cases} \pm 2 & \text{if } \Sigma \text{ is diffeomorphic to } D; \\ \pm 2 & \text{if } \Sigma \text{ is diffeomorphic to } \mathbb{S}^1 \times [0,1]; \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

REMARK 6.19. We recall that the degree theory constructed in this paper has been designed to count *oriented* surfaces. In the present case, this means that every free boundary minimal surface will be counted twice, once for each orientation, so that the degree will always be even.

Proof. Let $Z_0 \subseteq \mathcal{Z}(\{g_t\})$ be the set of embeddings which are free boundary minimal with respect to g_t and invariant under rotation about some vector. By Propositions 6.7 and 6.16, Z_0 constitutes a non-degenerate family. By Theorem 5.12, there exists a neighbourhood Ω of Z_0 in $\mathcal{E}$ such that:

$$\mathcal{Z}(\{g_t\}) \cap \overline{\Omega} = Z_0.$$

Upon reducing Ω if necessary, we may suppose, furthermore, that it is invariant under the action of SO(3). Now let N be as in Proposition 6.17, let $p \geqslant N$ be prime, let v be a unit vector in $\mathbb{R}^3$, and let $G \subseteq \mathrm{SO}(3)$ be the subgroup generated by $R_{v,2\pi/p}$. We calculate the contribution to the degree from embeddings in $\overline{\Omega}^c$ by repeating the proof of Theorem 4.2 in a G-invariant manner. Thus, pick $[e] \in \mathcal{Z}(\{g_t\}\,|\overline{\Omega}^c)$. By definition, $R_{v,2\pi/p} \circ e(\Sigma) \neq e(\Sigma)$, and, since p is prime, $R_{v,2\pi k/p} \circ e(\Sigma) \neq e(\Sigma)$ for all $1 \leqslant k < p$. Since e is minimal, there exists an open, dense subset V of Σ such that $R_{v,2\pi k/p} \circ e(V) \cap e(V) = \emptyset$ for all $1 \leqslant k < p$. Choose $q \in V$ and let U be a neighbourhood of $e(q)$ in B such that for all $1 \leqslant k < p$:

$$R_{v,2\pi k/p}(U) \cap U = \emptyset, \text{ and } R_{v,2\pi k/p}(U) \cap e(\Sigma) = \emptyset.$$

Now let $X := \{x_0\}$ be the manifold consisting of a single point, and denote $g_{x_0} := g_t$. Let $f_1, ..., f_m$ be a basis of $\mathrm{Ker}(\mathrm{J}_{g_{x_0},e})$, and let $\varphi_1, ..., \varphi_m$ be as in Proposition 2.10 with U as above. For $1 \leqslant k \leqslant m$, define $\tilde{\varphi}_k$ by:

$$\tilde{\varphi}_k = \sum_{l=1}^{p} \varphi_k \circ R_{v,2\pi l/p},$$

and observe that $\tilde{\varphi}_k$ is G-invariant. Following the proof of Theorem 4.2 with $\tilde{\varphi}_k$ instead of φ_k, we now obtain an extension $\tilde{X}$ of X such that $\partial_\omega \mathcal{Z}(\tilde{X}|\Omega) = \partial_\omega \mathcal{Z}(\tilde{X}|\overline{\Omega}^c) = \emptyset$ and, for all $x \in \tilde{X}$, g_x is invariant under the action of G, and $\mathrm{P}_{x,e} + \mathrm{J}_{x,e}$ defines a surjective map from $T_x\tilde{X} \times C^\infty(\Sigma)$ into $C^\infty(\Sigma) \times C^\infty(\partial\Sigma)$ for all $[e] \in \mathcal{Z}(\{x\}\,|\overline{\Omega}^c)$. In particular, by Theorem 4.5, $\mathcal{Z}(X|\overline{\Omega}^c)$ is a smooth $\mathrm{Dim}(\tilde{X})$-dimensional manifold and $\Pi(\partial(\mathcal{Z}(X|\overline{\Omega}^c))) \subseteq \partial X$.

Now, let $x \in \tilde{X}$ be a regular value of the restriction of Π to $\mathcal{Z}(\tilde{X}|\overline{\Omega}^c)$. Since g_x and Ω^c are both invariant under the action of G, $\mathcal{Z}(\{x\}\,|\overline{\Omega}^c)$ decomposes into disjoint orbits of G. By Proposition 6.17 together with a compactness argument, none of these orbits is trivial, so that, by primality, they all have order p. It follows that:

$$\mathrm{Deg}(\Pi|\overline{\Omega}^c) = \sum_{[e] \in \mathcal{Z}(\{x\}|\overline{\Omega}^c)} \mathrm{Sig}(\mathrm{J}_{x,e}) = 0 \bmod p.$$

Finally, by Theorem 5.12, extending $\tilde{X}$ further if necessary:

$$\mathrm{Deg}(\Pi|\Omega) = (-1)^{\mathrm{Ind}(Z_0)}\chi(Z_0),$$

and combining these relations yields:

$$\mathrm{Deg}(\Pi) = (-1)^{\mathrm{Ind}(Z_0)}\chi(Z_0) \text{ mod } p.$$

Since $p > 0$ is arbitrary, we have:

$$\mathrm{Deg}(\Pi) = (-1)^{\mathrm{Ind}(Z_0)}\chi(Z_0),$$

and the result now follows by Propositions 6.7 and 6.16. q.e.d.

6.5. Proof of Theorem 1.3. We now complete the proof of Theorem 1.3. For $s \in \mathbb{R}$, denote $g_s := e^{-2sf}g$, and let Rc^s be the Ricci-curvature tensor of this metric. Then:

$$\frac{\partial}{\partial_s}\bigg|_{s=0} \mathrm{Rc}^s = (n-2)\mathrm{Hess} f + \Delta f g > 0.$$

Thus, for sufficiently small, positive s, g_s has positive Ricci curvature and f is still strictly convex with respect to g_s. We now use Theorem 6.18 to prove existence. Indeed, let t_m be any sequence of positive numbers converging to 0. Fix m and let $X = \{g_{t_m}\}$ be the manifold consisting of a single point. By Theorem 4.11, there exists an extension $\tilde{X}$ of X such that $\mathcal{Z}(\tilde{X})$ has the structure of a smooth $\mathrm{Dim}(\tilde{X})$-dimensional manifold and the canonical projection $\Pi : \mathcal{Z}(\tilde{X})$ has a well-defined integer valued degree. By Theorem 6.18, $\mathrm{Deg}(\Pi) = \pm 2$. In particular, for any regular value x of Π in $\tilde{X}$, there exists an embedding $e_m : \mathbb{S}^1 \times [0,1] \to B$ which is free boundary minimal with respect to g_x. Moreover, by Sard's Theorem, $g_m := g_x$ may be chosen so that $(g_m)_{m\in\mathbb{N}}$ also converges to g. It now follows by Theorem 2.3 that there exists an embedded submanifold $\Sigma_\infty \subseteq B$ towards which $(\Sigma_m)_{m\in\mathbb{N}}$ converges. In particular, Σ_∞ is diffeomorphic to $\mathbb{S}^1 \times [0,1]$ and is free boundary minimal with respect to g, as desired.

Remark 6.20. Observe that Theorem 6.18 and the same argument as above also recovers the result [**12**] of Grüter and Jost.

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