J. DIFFERENTIAL GEOMETRY
31 (1990) 417-472

THE HYPERBOLIC METRIC AND THE GEOMETRY OF THE UNIVERSAL CURVE

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0. Introduction

0.1. A basic deformation of a compact Riemann surface is *pinching a nontrivial loop.* Understanding the limiting case, where the loop *degenerates to a point*, is actually tantamount to understanding the Deligne-Mumford stable curve compactification $\overline{\mathscr{M}_g}$ of the classical moduli space of Riemann surfaces. Degenerating families of Riemann surfaces are readily given by cut and paste constructions in the complex analytic category or by hyperbolic geometry, following Fenchel and Nielsen. A basic question is to relate the two approaches; to find the expansion for the hyperbolic metric in terms of the complex parametrization. The motivation is two-fold, we would like to be able to analyze the degeneration of an invariant of hyperbolic metrics by simply writing out its expansion in complex coordinates, and we would like to use the hyperbolic metric in the study of the analytic geometry of $\overline{\mathscr{M}_g}$.

The following example will play a central role in our investigation. Consider in $\mathbb{C}^3$ the smooth germ of a variety $V = \{(z,w,t) | zw = t, |z|, |w|, |t| < 1\}$ with projection $\Pi\colon V \to D$, $\Pi((z,w,t)) = t$, to the unit disc. The projection is almost a fibration: the t_0, fiber $t_0 \neq 0$, is an annulus $\{|t_0| < |z| < 1,\ w = t_0/z\}$, while the 0-fiber is two transverse discs $\{|z| < 1\} + \{|w| < 1\}$, intersecting only at the origin. $V \to D$ is a degenerating family of annuli. Each fiber of $V_0 = V - \{\text{origin}\}$ has a complete hyperbolic metric: the t_0-fiber,

$$ds_{t_0}^2 = \left(\frac{\pi}{\log|t_0|} \csc \frac{\pi \log|z|}{\log|t_0|} \left|\frac{dz}{z}\right|\right)^2$$

and the 0-fiber,

$$ds_0^2 = \left(\frac{|d\zeta|}{|\zeta| \log|\zeta|}\right)^2, \qquad \zeta = z, w.$$

Received February 19, 1988 and, in revised form, September 11, 1988. The author was partially supported by the National Science Foundation.

A simple observation is that the kernel of the differential $d\Pi\colon TV_0 \to TD$ defines a line bundle on V_0, the vertical bundle of the fibration. Each $v \in \operatorname{Ker} d\Pi$ is tangent to a fiber of Π and thus has a well-defined hyperbolic length $(ds_t^2(v))^{1/2}$. The line bundle extends over V and the system $\{ds_t^2\}$ defines a continuous metric, *the hyperbolic metric* $\langle\ ,\ \rangle_{\mathrm{hyp}}$ on the extension. In particular for the complex parameters (z, t)

$$ds_t^2 = ds_0^2\left(1 + \tfrac{1}{3}\Theta^2 + \tfrac{1}{15}\Theta^4 + \cdots\right) \tag{0.1}$$

with $\Theta = \pi \log|z|/\log|t|$; $\langle\ ,\ \rangle_{\mathrm{hyp}}$ is real analytic in the quantities $(1/\log|z|)$ and $(\pi\log|z|/\log|t|)$. We would like to find the analog of the expansion for a family with generic fiber a compact Riemann surface. This is a question on the Uniformization Theorem. It will turn out that the behavior for the compact fiber case is actually quite close to that of the example.

A degenerating family $\Pi\colon M \to B$ of compact Riemann surfaces is a proper holomorphic map Π of smooth complex manifolds with generic fiber a compact genus g Riemann surface [8], [12], [13]. A fiber R may have a finite number of nodes, points where the local model for the fibration *is* the above germ V. By hypothesis each component of $R-\{\text{nodes}\}$ will have negative Euler characteristic, i.e. R is a stable curve [10], [26]. Each fiber of $M_0 = M-\{\text{nodes}\}$ has a hyperbolic metric and $\operatorname{Ker} d\Pi$ defines a line bundle, extending to M, the vertical line bundle of the fibration (the dual of the relative dualizing sheaf). The system of hyperbolic metrics defines the *hyperbolic metric* $\langle\ ,\ \rangle_{\mathrm{hyp}}$ on the vertical line bundle. We are interested in the geometry of this metric and and in particular the analog of expansion (0.1).

But first we would like to review the plumbing construction for giving a degenerating family of Riemann surfaces. Let R be a surface with nodes and $R_0 = R-\{\text{nodes}\}$. R_0 has a pair of punctures a_j, b_j in place of the jth node of R, $j = 1, \cdots, m$. At the punctures take local coordinates $z_j, z_j(a_j) = 0$ and $w_j, w_j(b_j) = 0$. A family is constructed as follows, given $t = (t_1, \cdots, t_m)$, $t_j \neq 0$, remove the discs $|z_j| < |t_j|$ and $|w_j| < |t_j|$, t_j small, from R_0 to obtain an open surface R_t^*. Now form an identification space $R_t = R_t^*/\sim$ by identifying p to q if p and q lie in the domain of the coordinates z_j, w_j and $z_j(p)w_j(q) = t_j$. The union $\bigcup_{t\ \text{small}} R_t$ is a family of surfaces over the product of the punctured t_j-polydiscs. In fact the family extends to $\{R_t\}$, a degenerating family over the t-polydisc PD. By the jth collar in the fiber R_t we shall mean the image of the coordinate annulus $\{|t_j|/c < |z_j| < c, w_j = t_j/z_j\}$. An important observation is that the description of the collars in $\{R_t\}$ coincides with that of the example V. R_t has identifications $z_jw_j = t_j$, and the example has defining equation

$zw = t$; the fibers of $\{R_t\}$ above $\{t_k = 0\} \cap PD$ have nodes *in place of* the kth collar.

Our main result is an expansion in t for the hyperbolic metric of R_t. To describe the expansion for the *hyperbolic metric* we start by choosing z_j and w_j such that the R_0 hyperbolic metric has the local expression $(|d\zeta|/|\zeta|\log|\zeta|)^2$, $\zeta = z_j$, $\zeta = w_j$, ζ small, at the punctures. As the first approximation to the R_t hyperbolic metric, take the t_j-fiber metric of the example

$$\left(\frac{\pi}{\log|t_j|}\csc\frac{\pi\log|z_j|}{\log|t_j|}\left|\frac{dz_j}{z_j}\right|\right)^2$$

in the jth collar of R_t, the R_0 hyperbolic metric away from the collars, and interpolate between the choices at the collar boundaries. The result is a *grafted metric* dg_t^2 for R_t. The grafted curvature is -1 except at the collar boundaries, where the interpolation leads to a deviation of magnitude $(\log|t|)^{-2}$. By the maximum principle the R_t hyperbolic metric and the grafted metric differ by the same magnitude. Now to proceed with the expansion, the prescribed curvature equation can be solved to write the R_t hyperbolic metric in terms of the grafted metric and a compensating factor. The result is an expansion for the R_t hyperbolic metric ds_t^2 (see Expansion 4.2)

$$\begin{aligned} ds_t^2 = dg_t^2 \Bigg(1 - \frac{1}{3}\sum_{j=1}^{m}\left(\frac{\pi}{\log|t_j|}\right)^2 (D-2)^{-1}(\Lambda(z_j) + \Lambda(w_j)) \\ + O\left(\sum_{j=1}^{m}\frac{1}{(\log|t_j|)^4}\right)\Bigg), \end{aligned} \tag{0.2}$$

where D is the ds_t^2 Laplace Beltrami operator, D has nonpositive spectrum, $(D-2)^{-1}$ exists and is uniformly bounded in C^∞ norm, Λ is an indicator function for the collar boundary, $\Lambda(\zeta) = (a^4\eta_a)_a$ for $a = \log|\zeta|$, and η is a unit step function with step at $|\zeta| = c$, the collar boundary. Two immediate applications of (0.2) are an expansion for the length of the core geodesic in a collar (Example 4.3) and that the *hyperbolic metric* is continuous on the vertical line bundle for a degenerating family.

The general example of a degenerating family is the *universal curve* $\overline{\mathscr{C}_g}$ over $\overline{\mathscr{M}_g}$, the stable curve compactification of the classical *universal curve* $\mathscr{C}_g$. As above the tangents to the fibers define the vertical line bundle on $\overline{\mathscr{C}_g}$ and the *hyperbolic metric* is a continuous metric on the bundle. It is natural to consider the curvature 2-form Ω_{hyp} of the hyperbolic metric. The universal curve $\mathscr{C}_g$ can be considered as the moduli space of pairs

(Riemann surface, point). A Beltrami differential defines a deformation of a surface and thus also of a pair. In particular if R is smooth, $p \in R$ and ν a smooth Beltrami differential, we find the expansion (Lemma 5.7)

$$\Omega_{\mathrm{hyp}}(\nu,\nu)(p) = -2i(D-2)^{-1}(|\nu|^2)(p) + O(\|\nu\|_2\|K(\partial)\nu\|_2) \tag{0.3}$$

for the curvature in the ν-direction, where D is again the hyperbolic Laplacian, $\|\ \|_2$ is the C^2 norm on tensors, $K(\partial)$ is the covariant form of ∂, and the implied constant is absolute. An application is for ν a harmonic tensor with respect to the hyperbolic metric: for this case $K(\partial)\nu = 0$ and $(D-2)^{-1}|\nu|^2$ is everywhere negative by the maximum principle. Thus the curvature of the vertical line bundle restricted to $\mathscr{C}_g$ is strictly negative. For degenerating surfaces, it is not difficult to find Beltrami differentials corresponding to geometric intuition with $K(\partial)\nu$ small. Consequently formula (0.3) can also be used to analyze the limiting behavior of the curvature at the compactification divisor of $\overline{\mathscr{C}_g}$. There are a number of consequences.

The hyperbolic metric on $(\operatorname{Ker} d\Pi)$ is *good* (see Theorems 1.4 and 5.8), i.e., the nth power of the Chern form $(c_{1,\mathrm{hyp}})^n$, $c_{1,\mathrm{hyp}} = \frac{i}{2\pi}\Omega_{\mathrm{hyp}}$ computed on $\mathscr{C}_g$, defines by integration a current $[(c_{1,\mathrm{hyp}})^n]$ on $\overline{\mathscr{C}_g}$. $[(c_{1,\mathrm{hyp}})^n]$ is a closed (n,n) current that represents the nth power of the Chern class of $(\operatorname{Ker} d\Pi)$ in the rational-cohomology of $\overline{\mathscr{C}_g}$. In brief the hyperbolic metric on the restriction of $(\operatorname{Ker} d\Pi)$ to $\mathscr{C}_g$ already detects the extension of the bundle to $\overline{\mathscr{C}_g}$. As an application Ω_{hyp} can be used to compute the integral

$$\kappa_1 = \int_{\mathrm{fibre}} c_1((\operatorname{Ker} d\Pi)) \wedge c_1((\operatorname{Ker} d\Pi))$$

of the Chern class of $(\operatorname{Ker} d\Pi)$ along the fibers of $\Pi\colon \overline{\mathscr{C}_g} \to \overline{\mathscr{M}_g}$. The integral was previously evaluated for $\Pi\colon \mathscr{C}_g \to \mathscr{M}_g$ and $c_{1,\mathrm{hyp}}$ [37]. The result was simply $\omega_{\mathrm{wp}}/2\pi^2$, the Weil-Petersson Kähler form. Now since the hyperbolic metric is good the calculation is valid for $\overline{\mathscr{M}_g}$, in particular the current $[\omega_{\mathrm{wp}}/2\pi^2]$ represents the class κ_1 in the rational-cohomology of $\overline{\mathscr{M}_g}$. This observation, that the Kähler form is the pushdown of the square of the curvature, also provides for a Weil-Petersson Kähler potential in the form of a fiber integral. By our estimates it can be shown that the potential is continuous on $\overline{\mathscr{M}_g}$ and that the Kähler form is bounded below by a smooth positive $(1,1)$ form. From a result of Richberg [29] it follows that ω_{wp} is the limit in the sense of currents on $\overline{\mathscr{M}_g}$ of smooth Kähler forms in its cohomology class. Indeed, the current $[\omega_{\mathrm{wp}}/2\pi^2]$ is positive and rational, a result first found as a consequence of the sequence [33]–[36]. That the class κ_1 is positive is actually an earlier result of Mumford [26].

It is also interesting to consider the sign of the hyperbolic curvature on $\overline{\mathscr{C}_g}$. Arakelov has already shown that the vertical line bundle is negative on cycles transverse to the compactification locus $\overline{\mathscr{C}_g} - \mathscr{C}_g \subset \overline{\mathscr{C}_g}$ [3]. It is noted in [28] that the line bundle is nonpositive in general and that the bundle has degree zero restrictions to special cycles. In comparison we find that the hyperbolic curvature is essentially given by the Greens function $G(x,y)$ for $(D-2)^{-1}$. Specifically given a Beltrami differential ν, with $K(\partial)\nu$ small and ν supported near y, then by (0.3) the hyperbolic curvature $c_{1,\mathrm{hyp}}$ at x in the ν-direction is comparable to $G(x,y)$. In particular for a smooth fiber R of $\mathscr{C}_g$, $G(x,y)$ and $c_{1,\mathrm{hyp}}$ are everywhere negative (a consequence of the maximum principle for $(D-2)$). By contrast the Greens function $G_0(x,y)$ for a surface R with nodes is the direct sum in the sense of operators of the Greens functions of the components of $R_0 = R-\{\text{nodes}\}$, each with hyperbolic metric, [18], [19]. In particular for x and y in distinct components and ν supported near y, then ν represents a null curvature direction at x. Our complete result is as follows, if p is a nonnodal point of R, a fiber with nodes, and $R(p)$ the component of $R_0 = R-\{\text{nodes}\}$ containing p, then the negative curvature directions at p are precisely the deformations of R inducing nontrivial deformations of $R(p)-\{p\}$ (including opening the nodes on $R(p)$). At a node the negative directions are precisely the vectors with a nonzero component tangent to the fiber R. The null curvature directions at a node are spanned by the Beltrami differentials with compact support in R_0 and the tangents for opening any remaining nodes. As an application we obtain a geometric description of the cycles on $\overline{\mathscr{C}_g}$ with negative vertical line bundle. A point of $\overline{\mathscr{C}_g}$ is a pair (stable curve, point); a cycle is a parametrized pair. A cycle is negative precisely when there is *apparent motion* of the point on its algebraic component(s) in the stable curve: if either the point moves on the component or if the component itself is deformed. A node is a special location: if the point has the constant value *node* the vertical line bundle will have degree zero. And finally we note that the hyperbolic curvature is formally similar to the Weil-Petersson Riemann tensor [31], [38]. Our approach should apply to this case as well.

The paper is organized as follows. §1 contains a detailed discussion of the example $V = \{zw = t\}$ and a review of the definition of *good metrics*. As a model case we check that the hyperbolic metric for V is *good*. §2 is expository; a self-contained description of the local structure of $\overline{\mathscr{M}_g}$ and $\overline{\mathscr{C}_g}$ is presented. The description is a hybrid: deformations of a noded surface are given as combinations of classical deformations, supported away from the nodes, and plumbing constructions at the nodes.

§3 contains a discussion of grafting, constructing metrics by interpolating between hyperbolic metrics on overlapping subdomains. By standard estimates we find that the geometric difference between the grafted and hyperbolic metrics is bounded in C^k by the *thickness* of the overlaps and the C^0 difference on the overlaps of the subdomain metrics. The focus of the section is subsection four with three explicit grafting constructions and the accompanying estimates. In §4 we present the expansion for the hyperbolic metric and consider an application to geodesic length functions. §5 is devoted to the expansion for the derivatives of the hyperbolic metric; the immediate goal is to analyze the curvature form of the vertical line bundle. We start with an alternate organization for calculating perturbations of a metric. Computing perturbations of solutions of the Beltrami equation on the universal cover is replaced with a two-stage calculation. The first stage is completely local (no potential theory), and for the second stage the result is given in terms of the operator $(D-2)^{-1}$. As an instance of the first stage, we find the exact perturbations of the Laplacian and the Gauss curvature. The main application is the calculation in the second subsection of the hyperbolic curvature. The formula is simplest for the case of harmonic Beltrami differentials ($K(\partial)\nu = 0$), and still manageable for the case $K(\partial)\nu$ small. In the third subsection Beltrami differentials, corresponding to geometric intuition, with $K(\partial)\nu$ small are described. A feature of the current approach is that harmonic differentials, which are hard to write down, and the projection onto harmonic differentials, which also degenerates, are both avoided. In the fourth subsection the method is used to show that the hyperbolic metric is *good* and in the fifth subsection the curvature nulls are analyzed. And finally, standard potential theoretic estimates as well as a discussion of the operator $(D-2)^{-1}$ are given in the appendices.

The author would like to thank the University of Helsinki, Finland, University of Joensuu, Finland, Louis Pasteur University, France, and Institut des Hautes Etudes Scientifiques for their hospitality and support.

0.2. As an aid to the reader we now provide statements of the main results. The results will not be given in their complete generality.

The first item is the grafting construction (§3.3). Let $\{U, V\}$ be an open cover of a compact Riemann surface R, and η a smooth approximate characteristic function of V, i.e. η is 0 on $U-V$ and 1 on $V-U$. Given ds_1^2 a metric on U and ds_2^2 a metric on V, both compatible with the R-conformal structure, define

$$ds^2_{\text{graft}} = (ds_1^2)^{1-\eta}(ds_2^2)^{\eta},$$

the grafting of ds_1^2 and ds_2^2 relative to η. Note: the geometric interpolation will simplify the computation of the grafted curvature.

Of special interest is the case ds_1^2 and ds_2^2 each hyperbolic (typically not complete). We would like to estimate the difference between a grafted and the hyperbolic metric for R. To this end let $\| \ \|_{k,B}$ denote the C^k norm for functions on the open set $B \subset R$; the norm is defined relative to the R hyperbolic metric (§3.2). Now for the estimate (§3.3): fix $\varepsilon > 0$, $1 > \delta > 0$ and an integer $k \geq 0$; assume that the grafting data is chosen such that an ε-neighborhood of $\operatorname{supp}(d\eta)$ is contained in $U \cap V$, and $\|ds_1^2/ds_2^2 - 1\|_{0,U\cap V} < \delta$. With these conditions and for R compact there exists a constant $c = c(\varepsilon, \delta, k, \|\eta\|_{k+1})$ such that

$$\|ds^2_{\text{hyp}}/ds^2_{\text{graft}} - 1\|_{k,R} \leq c\|ds_1^2/ds_2^2 - 1\|_{0,U\cap V}.$$

In practice $U \cap V$ will be a cylinder and the C^k norm of η can be bounded in terms of the injectivity radius of $\operatorname{supp}(d\eta) \subset U \cap V$, i.e. the thickness of the overlap. Thus the difference of the grafted metric and the hyperbolic metric is bounded in terms of the thickness of the overlap and the jump in the component metrics on the overlap.

We will now describe how there is a clear choice of component metrics for the case of pinching. Consider a surface R with a single node n and set $R_0 = R - \{n\}$. R_0 has two punctures a and b in place of the node. Let z be a local coordinate at a and w a local coordinate at b, chosen such that the z and w local expression for the R_0 complete hyperbolic metric is simply $(|d\zeta|/(|\zeta|\log|\zeta|))^2$. Our method is actually valid (see §4) for z, w arbitrary local coordinates. Nevertheless the construction is simplest for the above case; the key point is that a neighborhood of the node n in R is isometric to a neighborhood of the origin in the 0-fiber of the $zw = t$ example. Let $c_1, c_2 > 0$ be chosen such that the annulus $\{c_1 < |\zeta| < c_2\}$ is contained in the image of the z and w coordinate charts. Pick η a function of $|\zeta|$ such that $\eta = 0$ for $|\zeta| < c_1$ and $\eta = 1$ for $|\zeta| > c_2$.

Given t, $|t| < c_1^2$ we wish to plumb R_0 to obtain R_t and at the same time describe a grafted metric for R_t. The plumbing: remove $\{|z| < |t|\}$ and $\{|w| < |t|\}$ from R_0 to obtain R^*; now identify $p \in \operatorname{Domain}(z) \cap R^*$ with $q \in \operatorname{Domain}(w) \cap R^*$ if $z(p)w(q) = t$. The result is a compact surface R_t with z, w coordinates on the pinching collar, i.e. *pinching collar*$=\{|t|/c_2 < |z| < c_2\}$. The next step is to specify the component metrics for the grafting. For the pinching collar take

$$ds_1^2 = \left(\frac{\pi}{\log|t|}\csc\frac{\pi\log|z|}{\log|t|}\frac{|dz|}{|z|}\right)^2$$

(the metric has a symmetry $w = t/z$; thus if z were replaced with w we would have the same metric ds_1^2 for the collar). For ds_2^2 take the original R_0 hyperbolic metric restricted to $R_t - \{|t|/c_1 < |z| < c_1\}$ (the identification to form $R_t = R^*/\sim$ only involved points of $\{|t|/c_1 < |z|, |w| < c_1\} \subset R^*$, thus $R_t - \{|t|/c_1 < |z| < c_1\}$ can be considered as a subdomain of R_0). The intersection (pinching collar) $\cap$ $(R_t - \{|t|/c_1 < |z| < c_1\})$ has two components, each a band; graft ds_1^2 to ds_2^2 relative to η to obtain dg_t^2 ($\eta = \eta(|z|)$ in the z-band and $\eta = \eta(|w|)$ in the w-band).

Now from Expansion 4.2 if ds^2_{hyp} is the R_t hyperbolic metric and D the associated Laplacian then there exists a $\delta = \delta(c_1, c_2)$ such that for $|t| < \delta$

$$ds^2_{\text{hyp}} = dg_t^2 \left(1 - \frac{1}{3}\left(\frac{\pi}{\log|t|}\right)^2 (D-2)^{-1}(\Lambda(z) + \Lambda(w)) + O\left(\frac{1}{(\log|t|)^2}\right)\right),$$

where $\Lambda(\zeta) = \Lambda(|\zeta|) = (x^4\eta_x)_x$ for $x = \log|\zeta|$, and the O-term is for C^∞ functions on R_t with the constant bounded solely in terms of c_1 and c_2.

Remarks. $(D-2)^{-1}$ exists and is bounded in C^∞, constants independent of R (see appendix A.4). Our technique applies for the case of opening several nodes as well as to include deformations supported away from the collars.

If $l(t)$ is the length of the closed geodesic in the collar core for the R_t hyperbolic metric then from the expansion we find (Example 4.3)

$$l(t) = \frac{2\pi^2}{\log(1/|t|)} + O\left(\frac{1}{(\log|t|)^4}\right)$$

for $|t| < \delta$ and the constant depends only on c_1 and c_2.

Now we turn to §5. There are two main results: the hyperbolic metric is good for the vertical line bundle, and the classification of the curvature nulls for the metric. We have already seen from the pinching expansion that the metric is not smooth in t at $t = 0$. We expect the connection 1-form and curvature 2-form to have a *pole* at $t = 0$. Recall for a line bundle with metric $\langle\ ,\ \rangle$ and σ a nonvanishing local section, that the connection form is simply $\Theta = \partial \log\langle\sigma, \sigma\rangle$ and the curvature is $\Omega = \bar{\partial}\Theta$. If a metric is smooth, except on a subvariety, then there are two ways in which one can try to interpret the differentiation: in the sense of distributions, or to consider the current defined by integration, against the expression given on the complement of the subvariety. For a good metric the two notions agree. Mumford's *good* criterion is a growth condition for the connection and curvature forms.

The final application is the curvature nulls. The hyperbolic curvature is to be considered as a measure on discs embedded in the local manifold covers of $\overline{\mathscr{C}_g}$ (note: $\overline{\mathscr{C}_g}$ is a V-manifold). We analyze the growth rate of the curvature for each possible embedded disc (§5.6). A consequence is the following classification.

Case 1: the curvature is negative definite on $\mathscr{C}_g$.

Case 2: p a nonnodal point of a noded fiber R of $\overline{\mathscr{C}_g}$ and $R(p)$ the algebraic component of R containing p. For S a component of R denote by $B(S)$ the deformations of R given by Beltrami differentials compactly supported on $S-\{\text{nodes}\}$. The tangent space $T_p\overline{\mathscr{C}_g}$ is a direct sum of five subspaces: $T_pR \oplus B(R(p)) \oplus$ plumbing the nodes on $R(p) \oplus B(R - R(p)) \oplus$ plumbing the remaining nodes. A tangent vector at p is negative for the hyperbolic curvature if and only if one of the first three components is nonzero.

Case 3: p a node of a noded fiber R of $\overline{\mathscr{C}_g}$. Local coordinates for a neighborhood of p are given as a combination of the plumbing at p, plumbing any remaining nodes, and classical deformations of $R-\{\text{nodes}\}$. Let z, w be the variables for the $zw = t$ plumbing at p and $TV \subset T_p\overline{\mathscr{C}_g}$ the subspace spanned by the z and w tangents. A tangent vector at p is negative for the hyperbolic curvature if and only if its TV component is nonzero.

The operator $(D-2)^{-1}$ plays an important role in the pinching expansion. We give two estimates for the Greens function $G_s(z, z_0)$, the kernel for the integral operator $(D - s(s-1))^{-1}$ on $L^2(R)$. Let $\delta(z, z_0)$ denote the hyperbolic distance between points of R.

Lemma A.4.1. *Given $\delta_0 > 0$ and $s > 1$ there exists a constant c_0 such that*

$$0 < -G_s(z, z_0) < c_0 e^{(1-s)\delta(z,z_0)}$$

for $\delta(z, z_0) > 1$ and provided the injectivity radius at z or z_0 is at least δ_0.

Remark. There is no hypothesis for R and the constant is R-independent.

There is a lower estimate with the same order. A constant $c_1 > 0$ is given; a *short geodesic* γ is a closed geodesic on R with length $l(\gamma) < c_1$. The *collar* about γ is $C(\gamma) = \{z | \delta(z, \gamma) < \log(1/l(\gamma))\}$; this is essentially the same as a pinching collar. For an integer $g > 1$ there exists a constant $c(g)$ such that if R is a compact surface of genus g, z lies on a short geodesic, and z_0 is in the adjoining component of $R - \bigcup_{\gamma \text{ short}} C(\gamma)$, then $-G_s(z, z_0) \geq c(g)e^{(1-s)\delta(z,z_0)}$.

1. The model case: $zw = t$

1.1. The family $V = \{zw = t\} \cap$ unit-polydisc, over the t-unit disc D, is the standard example of degeneration. Since this is also the model for the general case, we start in §1.2 by reviewing the geometry of V. In particular, V is smooth and $\Pi\colon V \to D$ is a submersion on V_0, $V_0 = V - \{\text{origin}\}$. The tangents to the fibers of the (almost) fibration $\Pi\colon V_0 \to D$ define a line bundle $(\operatorname{Ker} d\Pi)$ on V. Each fiber of V_0 has a complete hyperbolic metric and thus we obtain a hyperbolic metric on $\operatorname{Ker} d\Pi$ (each $v \in \operatorname{Ker} d\Pi$ is a tangent vector to some fiber of V_0). The metric is continuous but not smooth.

Mumford gave a C^2 growth condition for such metrics; the curvature form, considered as a current, of a *good metric* represents the Chern class of the line bundle. We recall Mumford's discussion in §1.3 and check in §1.4 that the hyperbolic metric for the example satisfies the criterion.

1.2. By $zw = t$ we shall mean the specific variety $V = \{(z, w, t) | zw = t,\ |z|, |w|, |t| < 1\}$. The defining function $zw - t$ has differential $zdw + wdz - dt$. There are several simple consequences: $zw = t$ is a smooth variety, (z, w) are global coordinates, while (z, t) and (w, t) are not. Consider the projection $\Pi\colon V \to D$, onto the t-unit disc. Π is a submersion, except at $(z, w) = (0, 0)$; thus we can consider $V \xrightarrow{\Pi} D$ as a family of Riemann surfaces. The t-fiber, $t \neq 0$, is the hyperbola $zw =$ constant, or equivalently the annulus $\{|t| < |z| < 1, w = t/z\} = \{|t| < |w| < 1, z = t/w\}$. The 0-fiber is the intersection of the unit ball with the union of the coordinate axes in $\mathbf{C}^2$, on removing the origin the union is $\{0 < |z| < 1\} \cup \{0 < |w| < 1\}$.

In as much as Π is a submersion on $V_0 = V - \{\text{origin}\}$ the kernel $\operatorname{Ker} d\Pi$, the vertical line bundle $(\operatorname{Ker} d\Pi)$, defines a line bundle over V_0. A tangent vector v to V, $v = a\frac{\partial}{\partial z} + b\frac{\partial}{\partial z}$, is in the kernel of $d\Pi$ provided $v(t) = v(zw) = aw + bz = 0$, thus $\sigma = z\frac{\partial}{\partial z} - w\frac{\partial}{\partial w}$ is a section of $(\operatorname{Ker} d\Pi)$, nonvanishing on V_0. Since the origin has codimension 2 in V, $(\operatorname{Ker} d\Pi)$ has a unique extension to a line bundle over V and σ extends to a nonvanishing section (we shall use the same notation for the extensions).

Each fiber of $V_0 \to D$ has a complete hyperbolic metric

$$\begin{aligned}
&t \neq 0, \qquad \text{fiber} = \{|t| < |z| < 1\},\\
&ds_t^2 = \left(\frac{\pi}{\log|t|}\csc\frac{\pi\log|z|}{\log|t|}\left|\frac{dz}{z}\right|\right)^2,\\
&t = 0, \qquad \text{fiber} = \{0 < |z| < 1\} \cup \{0 < |w| < 1\},\\
&ds_0^2 = \left(\frac{|d\zeta|}{|\zeta|\log|\zeta|}\right)^2 \quad \text{for } \zeta = z, w.
\end{aligned} \tag{1.1}$$

A vector v of $(\operatorname{Ker} d\Pi)$ over V_0 is tangent to a particular t-fiber of the family $V_0 \to D$ and thus we can consider the hyperbolic length $ds_t^2(v)$. The system $\{ds_t^2\}$ defines the hyperbolic metric $\langle\ ,\ \rangle$ on the line bundle $(\operatorname{Ker} d\Pi)$. Consider the squared-length of σ in a neighborhood of a point p of V_0 with $z \neq 0$ ((z,t) are local coordinates at p), $\sigma = z\frac{\partial}{\partial z}$ relative to (z,t) and

$$\langle\sigma,\sigma\rangle = \left(\frac{\pi}{\log|t|}\csc\frac{\pi\log|z|}{\log|t|}\right)^2 \quad \text{for } t \neq 0.$$

More generally, observe that

$$\frac{\pi}{\log|t|}\csc\frac{\pi\log|z|}{\log|t|}$$

depends real analytically on the parameter $\varepsilon = 1/\log|t|$.

We summarize: $\langle\ ,\ \rangle$ is a continuous metric, degenerate only at the origin-fiber for the line bundle $(\operatorname{Ker} d\Pi)$ over V. Since $\Theta\csc\Theta$, $-\pi < \Theta < \pi$, is an even real analytic function of Θ we see that $\langle\ ,\ \rangle$ is real analytic in the two quantities $(\pi/\log|t|)^2$ and $(\pi\log|z|/\log|t|)^2$. Indeed

$$ds_t^2 = \left(\frac{|dz|}{|z|\log|z|}\right)^2 (\Theta\csc\Theta)^2 \quad \text{for } \Theta = \frac{\pi\log|z|}{\log|t|}$$

$$= ds_0^2\left(1 + \frac{1}{3}\Theta^2 + \frac{1}{15}\Theta^4 + \cdots\right).$$

We would like to make two remarks about the first perturbation, the Θ^2 term. In effect consider the universal cover of $\{0 < |z| < 1\}$ by substituting $\zeta = -i\log z$. We obtain the upper half plane model: ds_0^2 becomes $(|d\zeta|/|\operatorname{Im}\zeta)^2$ and Θ^2 becomes $(\pi\operatorname{Im}\zeta/\log|t|)^2$. The hyperbolic Laplacian is $\Delta = (2\operatorname{Im}\zeta)^2\partial^2/\partial\zeta\delta\overline{\zeta}$; then $\Delta\Theta^2 = 2\Theta^2$, Θ^2 is a 2-eigenfunction (see 4.2 for an explanation). A fundamental domain for the covering is $\{0 < \operatorname{Re}\zeta < 2\pi\}$, a lift of a neighborhood of the cusp is $N = \{0 < \operatorname{Re}\zeta < 2\pi, \operatorname{Im}\zeta > c > 0\}$, the function Θ^2 is in $L^p(N)$ (relative to the hyperbolic area element) only for $p < \frac{1}{2}$. We will see that the $L^{(1/2)-\varepsilon}$ estimate also holds for the general case (§4).

1.3. Given a metric, an obvious issue is to consider its curvature. For a smooth metric on a holomorphic vector bundle, the characteristic classes of the bundle can be expressed in terms of curvature. Mumford has given a criterion to generalize the result to metrics smooth, except on a subvariety. We recall, and at times directly quote, Mumford's discussion of good metrics for a vector bundle [27].

Let $\overline{X}$ be a smooth variety with $\overline{X} - X$ a divisor of $\overline{X}$ with normal crossings. Consider polycylinders $D^r \subset \overline{X}$, $r = \dim \overline{X}$, D the unit disc, such that $D^r \cap \overline{X} - X =$ {union of coordinate hyperplanes $z_1 = 0, \cdots, z_k = 0$}; hence $D^r \cap X = (D^*)^k \times D^{r-k}$, D^* the punctured disc. On each D^* take the hyperbolic metric $ds^2 = (|dz|/(|z|\log|z|))^2$ and on each D the Euclidean metric; we have the product metric on $(D^*)^k \times D^{r-k}$, denoted by ω.

Definition 1.1. A complex-valued C^∞ p-form η on X is said to have Poincaré growth on $\overline{X} - X$, provided there is a set of polycylinders $U_\alpha \subset \overline{X}$ covering $\overline{X} - X$ such that for each $U_\alpha \cap X$ there is a constant C_α and

$$|\eta(v_1, \cdots, v_p)|^2 \leq C_\alpha \omega_{u_\alpha}(v_1, v_1) \cdots \omega_{u_\alpha}(v_p, v_p)$$

for all choices of p tangent vectors and $|z_j| \leq \frac{1}{2}$, all j.

The property is independent of covering and the first consequence is that a p-form η with Poincaré growth defines, by integration against C^∞ $(r-p)$-forms, a p-current $[\eta]$ on $\overline{X}$.

Definition 1.2. A complex-valued p-form η on X is *good* on $\overline{X}$ if both η and $d\eta$ have Poincaré growth.

The set of all good forms is a differential graded algebra. In particular, the wedge of good forms is a good form and also $d([\eta]) = [d\eta]$. Next let $\overline{E}$ be a holomorphic rank n vector bundle over $\overline{X}$, let E be the restriction of $\overline{E}$ to X and let h be a Hermitian metric on E.

Definition 1.3. A Hermitian metric h on E is *good* on $\overline{X}$ if for all $x \in \overline{X} - X$, and all local frames $e_1, \cdots, e_n$ of $\overline{E}$ on a neighborhood U of x and for $(\overline{X} - X) \cap U$ given as above by $\prod_{j=1}^k z_j = 0$, then for $h_{ij} = h(e_i, e_j)$,

(i) $|h_{ij}|$, $(\det h)^{-1} \leq C(\sum_{j=1}^k \log|z_j|)^{2n}$ for some $C > 0, n \geq 1$,

(ii) the 1-forms $(h^{-1}\partial h)_{ij}$ are *good* on $\overline{X} \cap U$.

A key point is that given (E, h), there is at most one extension $\overline{E}$ of E to $\overline{X}$ for which h is good. In this way a good metric determines the bundle extension. The main result is the following:

Theorem 1.4. *If $\overline{E}$ is a vector bundle on $\overline{X}$, $\overline{X}$ compact, and h is a good Hermitian metric on $E = \overline{E}|_X$, then the Chern forms $c_p(E, h)$ are good on $\overline{X}$, and the current $[c_p(E, h)]$ represents the cohomology class $c_p(\overline{E}) \in H^{2p}(\overline{X}, \mathbb{C})$.*

1.4. As an example and also a precursor of the general case, we now check that the hyperbolic metric on the vertical line bundle is good. The variety $V = \{zw = t\}$ plays the role of $\overline{X}$, the 0-fiber of $\Pi\colon V \to D$ the role of $\overline{X} - X$ and of course $(\operatorname{Ker} d\Pi)$ is a vector bundle over V with frame σ. The Poincaré comparison form is

$$\omega = \left(\frac{|dz|}{|z|\log|z|}\right)^2 + \left(\frac{|dw|}{|w|\log|w|}\right)^2.$$

We take (a, b), $a = \log z$, $b = \log w$, as local coordinates and $\alpha = \log|z|$, $\beta = \log|w|$. Start with the ω-norms of the a and b tangents:

$$\frac{\partial}{\partial a} = z\frac{\partial}{\partial z}, \quad \frac{\partial}{\partial b} = w\frac{\partial}{\partial w}, \quad \omega\left(\frac{\partial}{\partial a}, \frac{\partial}{\partial a}\right) = \alpha^{-2}, \quad \omega\left(\frac{\partial}{\partial b}, \frac{\partial}{\partial b}\right) = \beta^{-2}.$$

The squared-length of the defining section σ is

$$\langle\sigma, \sigma\rangle = \left(\frac{\pi}{\alpha+\beta}\csc\frac{\pi\alpha}{\alpha+\beta}\right)^2.$$

We shall estimate $\langle\sigma,\sigma\rangle$, $\partial\log\langle\sigma,\sigma\rangle$ and $\overline{\partial}\partial\log\langle\sigma,\sigma\rangle$. Substituting $\Theta = \alpha/(\alpha+\beta)$ then $\langle\sigma,\sigma\rangle = (\alpha^{-1}A(\Theta))^2$ for $A(\Theta) = \pi\Theta\csc\pi\Theta$; for $|\Theta| < 1$, $A(\Theta)$ is nonvanishing, real analytic. By symmetry it is enough to treat $0 < \Theta < c < 1$; the 0th-order estimate $\langle\sigma,\sigma\rangle \le c\alpha^{-2}$ is immediate. For the higher order estimates write $\frac{1}{2}\log\langle\sigma,\sigma\rangle = -\log\alpha + \log A(\Theta)$ and treat the two terms separately. Now $(-\log\alpha)_a = -(2\alpha)^{-1}$, $(\log\alpha)_{a\bar{a}} = (2\alpha)^{-2}$, and the b-derivatives are 0; all are bounded by the ω-norms. For the derivatives of A if we let u, v be generic tangents then

$$u\log A(\Theta) = \frac{A'(\Theta)}{A(\Theta)}\Theta_u = B(\Theta)\Theta_u,$$
$$vu\log A(\Theta) = B'(\Theta)\Theta_v\Theta_u + B(\Theta)\Theta_{vu}.$$

Since A and B are bounded for $|\Theta| < c < 1$, it only remains to check the derivatives of $\Theta = \alpha/(\alpha+\beta)$:

$$\Theta_a = \frac{\beta}{2(\alpha+\beta)^2}, \quad \Theta_b = \frac{-\alpha}{2(\alpha+\beta)^2}, \quad \Theta_{a\bar{a}} = \frac{-\beta}{2(\alpha+\beta)^3},$$
$$\Theta_{a\bar{b}} = \frac{\alpha-\beta}{4(\alpha+\beta)^3}, \qquad \Theta_{b\bar{b}} = \frac{\alpha}{2(\alpha+\beta)^3},$$

and recall that $\omega(\partial/\partial a, \partial/\partial a) = \alpha^{-2}$, $\omega(\partial/\partial b, \partial/\partial b) = \beta^{-2}$. The restriction $\Theta < c$ implies $\beta/\alpha > c_0 > 0$; the estimates follow for $|z| \to 0$, $\alpha, \beta \to -\infty$. We record the result with the following:

Lemma 1.5. *The hyperbolic metric on the vertical line bundle over $zw = t$ is good.*

2. The description of $\overline{\mathscr{M}_g}$ and $\overline{\mathscr{C}_g}$ near the noded surfaces

2.1. Our purpose in this section is expository, to go over the construction of the local deformation space, as well as the associated family, for a Riemann surface with nodes (a stable curve). The definition of a Riemann surface with nodes and the deformation theory of a finitely punctured surface are briefly reviewed in §§2.2 and 2.3. In the final subsection the local description of $\overline{\mathscr{M}_g}$, the moduli space of stable curves, and $\overline{\mathscr{C}_g}$, the *universal curve* over $\overline{\mathscr{M}_g}$, are given in detail. For references the reader should consult [8], [10], [12], [13], [16], [22], [24], [26], [28].

2.2. A *Riemann surface with nodes* R is a connected complex space, such that every point has a neighborhood isomorphic to either the unit disc, $\{|z| < 1\}$ in $\mathbb{C}$, or the germ of the intersection of the coordinate axes in $\mathbb{C}^2$, $\{(z,w)|zw = 0, |z|, |w| < 1\}$. Examples are given by the t-fibers of $zw = t$. The special points of R are *nodes*; $R-\{\text{nodes}\}$ is a union $\bigcup_{j=1}^n R_j$ of nonsingular Riemann surfaces, called the *parts* of R. Provided R is compact, each R_j can be described as a compact surface minus a finite number of points; we assume that the Euler characteristic of each R_j is negative. There are two immediate consequences: R is a *stable curve* in the sense of Mumford et al. ([10], [22], [26], [28]) and each R_j carries a complete hyperbolic metric.

On removing a node $p \in R$ we obtain two punctures a and b of $R-\{p\}$. We will associate the punctures a and b with the node p, and refer to a and b as being paired to form p. We saw in §1.2 the description of a section σ of the vertical line bundle, near a node p: if $U = \{|z| < 1, |w| < 1, zw = 0\}$ is the neighborhood of p, then on $0 < |z| < 1$, $\sigma = z\frac{\partial}{\partial z}$ and on $0 < |w| < 1$, $\sigma = -w\frac{\partial}{\partial w}$. The reciprocal section $\sigma^{-1} = \frac{dz}{z}$, $\sigma^{-1} = -\frac{dw}{w}$ of the reciprocal line bundle is the prototype of a *regular* 1-*differential* (a section of the relative dualizing sheaf). A *regular q-differential* on R is the assignment of a meromorphic q-differential Θ_j for each part R_j of R such that: (i) each Θ_* has poles only at the punctures of R_*, their order is at most q, and (ii) if a is paired to b then $\operatorname{Res}_a \Theta_* = (-1)^q \operatorname{Res}_b \Theta_*$.

2.3. We briefly review the local deformation theory of a compact Riemann surface minus a finite number of points. At a puncture the hyperbolic metric singles out special coordinate charts (rs coordinates); we describe how these charts are affected by a deformation.

In the Kodaira-Spencer setup the infinitesimal deformation space of a compact complex manifold M is $\check{H}^1(M,\mathscr{O}(v.f.))$ ([25], [11]). For example, the infinitesimal deformation space of the pair (R,q)=(compact Riemann surface, point) is $\check{H}^1(R,\mathscr{O}(\kappa^{-1}\zeta_q^{-1}))$, κ the R-canonical bundle, ζ_q the point line bundle for q. By the Dolbeault isomorphism $\check{H}^1(R,\mathscr{O}(\kappa^{-1}\zeta_q^{-1}))$ $\cong H_{\bar\partial}^{0,1}(R,\mathscr{E}(\kappa^{-1}\zeta_q^{-1}))$, $\mathscr{E}(\kappa^{-1}\zeta_q^{-1})$ the sheaf of germs of smooth vector fields vanishing at q [25]. By definition a (0,1) form with values in smooth vectors fields is a *smooth Beltrami differential*, a $(-1,1)$ tensor. A Beltrami differential ν with support disjoint from q represents a trivial infinitesimal deformation of (R,q) exactly when there exists a smooth vector field F on

R, vanishing at q, such that $\overline{\partial} F = \nu$. Each class in $H^{0,1}_{\overline{\partial}}(R, \mathscr{E}(\kappa^{-1}\zeta_q^{-1}))$ has a representative with support disjoint from q.

Associated to a Beltrami differential of absolute value less than 1 is a finite deformation of R. Specifically given an atlas $\{U_\alpha, z_\alpha\}$ and ν, define new charts as follows: for $z = z_\alpha$, the local coordinate on $U_\alpha \subset R$ (z_α maps U_α into $\mathbb{C}$), and $\nu(z)$, the local expression for ν, let $w_\alpha = w(z)$ be a homeomorphism solution of the Beltrami equation $w_{\overline{z}} = \nu(z) w_z$ on $z_\alpha(U_\alpha)$. The new atlas $\{U_\alpha, w_\alpha \circ z_\alpha\}$ defines the new surface R_ν. In practice local holomorphic coordinates for the Teichmüller space of $R - \{p_1, \cdots, p_k\}$ are given as follows: choose $\nu_1, \cdots, \nu_n$ spanning the Dolbeault group $H^{0,1}_{\overline{\partial}}(R, \mathscr{E}((\kappa\zeta_{p_1}\cdots\zeta_{p_k})^{-1}))$; for $s = (s_1, \cdots, s_n)$ small, $\nu(s) = \sum_j s_j \nu_j$, satisfies $|\nu(s)| < 1$ and $R_s = R_{\nu(s)}$ is a Riemann surface. The assignment $s \to \{R_s\}$ is a local chart for Teichmüller space [1].

We shall emphasize the similarity for degeneration of hyperbolic metrics between the model case and the compact case. The first step is to pick appropriate local coordinates on the surface.

Definition 2.1. Let R be a surface with hyperbolic metric ds^2. A local coordinate z on $U \subset R$ is rs (rotationally symmetric for the hyperbolic metric) if either

(i) (simple closed geodesic case) z maps U to the annulus $|t|^{1/2}/c < |z| < |t|^{1/2}c$, $c, |t| > 0$, and the z-coordinate expression for ds^2 is

$$\left(\frac{\pi}{\log|t|} \csc \frac{\pi \log|z|}{\log|t|} \left|\frac{dz}{z}\right|\right)^2;$$

(ii) (puncture case) z maps U to the punctured disc $0 < |z| < c$, $c > 0$, and the z-coordinate expression for ds^2 is $(|dz|/(|z| \log|z|))^2$.

Each simple closed geodesic and each puncture has a family of rs coordinates parametrized by c and a rotation.

We wish to consider the effect of a deformation on the rs coordinates at a puncture. Let q be a puncture of R and ν, $|\nu| < 1$, a Beltrami differential, and let (U_0, z_0) be a chart about q, $z_0(q) = 0$, with $\operatorname{supp}(\nu) \cap U_0 = \emptyset$. By definition (U_0, z_0) will also be a chart about q for R_ν. If z_0 is an rs coordinate for R it will in general not be an rs coordinate for R_ν, $\nu \neq 0$.

To measure the deviation introduce the uniformization of R and R_ν by the upper half plane H. Conjugate both coverings such that the ideal point q is represented by ∞, and a loop once around q lifts to the deck transformation $z \to z+1$, $z \in H$. How lift the Beltrami differential ν to H and let $w(z)$ be the solution of $w_{\overline{z}} = \nu w_z$, that is a self-homeomorphism of H fixing $0, 1$ and ∞ [2]. For our choice of coverings w commutes with the translation, $w(z+1) = w(z) + 1$, thus for $u = e^{2\pi i z}$, respectively $v = e^{2\pi i w}$

(u an rs coordinate in the domain, v an rs coordinate in the range), the map $w(z)$ induces a map $u \to v$. Write $\hat{w}(u) = v$ for this map of punctured discs (H/translations); $\hat{w}$ represents the R_ν hyperbolic metric in the R-local-coordinate u. If $\mathrm{supp}(\nu) \subset \{\mathrm{Im}\, z \le c\}$ then $\hat{w}$ is holomorphic on $|u| < e^{-2\pi c}$. A simple construction shows that $\hat{w}$ on $|u| < e^{-2\pi c}$ can be an arbitrary holomorphic map; rs coordinates are not preserved by deformations. Nevertheless it is a standard result of Teichmüller theory that $\nu \in L^\infty(H) \to \hat{w} \in C^0(\text{disc})$ is a real analytic map of Banach spaces [2]. Thus $\hat{w}$ is close to the identity for ν small and there is an expansion for $\hat{w}$ in terms of ν. This is sufficient for our purposes.

2.4. We recall the description of the universal curve $\overline{\mathscr{C}_g}$ over the compactified moduli space ([8], [24], [28]). $\overline{\mathscr{C}_g}$ and $\overline{\mathscr{M}_g}$ are examples of complex V-manifolds. A complex V-manifold is a pair $(M, \mathscr{A})$, where M is a Hausdorff space and $\mathscr{A} = \{A_\alpha\}$ is an open covering: associated to each A_α is a triple $(\tilde{A}_\alpha, \Gamma_\alpha, \varphi_\alpha)$, where $\tilde{A}_\alpha$ is connected open in $\mathbb{C}^n$, $\Gamma_\alpha \subset \mathrm{GL}(n; \mathbb{C})$ is a finite group stabilizing $\tilde{A}_\alpha$ and $\varphi_\alpha \colon \tilde{A}_\alpha \to A_\alpha$ is Γ_α-invariant, inducing a homeomorphism of $\tilde{A}_\alpha/\Gamma_\alpha$ to A_α. There is a compatibility condition for triples when $A_\alpha \subset A_\beta$, and by hypothesis $\mathscr{A}$ is a basis for the topology of M [4], [27]. We are interested in questions on the local differential geometry of $\overline{\mathscr{C}_g}$ and $\overline{\mathscr{M}_g}$; it will be sufficient to study the *local manifold covers*, the $\tilde{A}_\alpha$.

We start with the plumbing construction for R a surface with a pair of punctures $\{a, b\}$. The data is (U, V, F, G, t): U and V are disjoint disc coordinate neighborhoods of the punctures, $a \in U$, $b \in V$, $F \colon U \to \mathbb{C}$, $G \colon V \to \mathbb{C}$ coordinate mappings, $F(a) = 0$, $G(b) = 0$ and t a sufficiently small complex number. The plan is to construct a degenerating family R_t, analogous to the model family $zw = t$. Pick a constant $c > 0$ such that $F(U)$ and $G(V)$ contain the disc $\{|\zeta| < c\}$. Assume $|t| < c^2$, remove from R the discs $\{|F| \le |t|/c\} \subset U$ and $\{|G| \le |t|/c\} \subset V$ to obtain an open surface R_t^*. Form an identification space R_t, $t \ne 0$, by identifying $p \in \{|t|/c < |F| < c\} \subset R_t^*$ with $q \in \{|t|/c < |G| < c\} \subset R_t^*$ if and only if $F(p)G(q) = t$. R_t is the *plumbing* for the prescribed data.

Remarks. F and G are holomorphic homeomorphisms, thus R_t is a Riemann surface. The constant c specifies the size of the (image of the annuli) $\{|t|/c < |F|, |G| < c\}$ in R_t; for $c > |t|^{1/2}$ the surface R_t is independent of c.

Now we describe the *plumbing family*, $\{R_t\}$, over the t-disc. Let $\gamma = c^2$, $D_\gamma = \{|t| < \gamma\}$, $M = R_\gamma^* \times D_\gamma$ and $N = \{(z, w, t) | zw = t,\ |z|, |w| < c$ and $|t| < c^2\}$. M and N are complex manifolds with holomorphic projection to D_γ. There are two holomorphic maps of (appropriate) subsets of M to

N:

$$(p,t) \xrightarrow{\hat{F}} (F(p), t/F(p), t) \quad \text{and} \quad (q,t) \xrightarrow{\hat{G}} (G(q), t/G(q), t).$$

The identification space $M \cup N/\{\hat{F} + \hat{G} \text{ equivalence}\}$ is a *degenerating family* $\{R_t\}$ with projection to D_γ (an analytic fiber space of Riemann surfaces in the sense of Kodaira [23]). The $t \neq 0$ fiber is R_t, constructed above; the 0-fiber is a noded Riemann surface. Finally for the sake of latter consideration, note that $N \to D_\gamma$ is a local model for $\{R_t\} \to D_\gamma$ in a neighborhood of the node.

2.4.M. Description of the local manifold covers of $\overline{\mathscr{M}_g}$. R is a Riemann surface with nodes $p_1, \cdots, p_m$ ($m = 0$ is allowed). First we discuss the data for constructing a local manifold cover of a neighborhood of $R \in \overline{\mathscr{M}_g}$. $R_0 = R - \{\text{nodes}\}$ is a union of Riemann surfaces with punctures. Fix $U_0 \subset R_0$, open, such that each component of R_0 intersects U_0 in a nonempty relatively compact set. Pick plumbing data $(U_j, V_j, F_j, G_j, t_j)$ for each of the j nodes of R; assume that the U_j, V_j are mutually disjoint, and that U_0 and $\bigcup_j (U_j \cup V_j)$ are disjoint. The deformation space of R_0, $\mathrm{Def}(R_0)$, is the product of the Teichmüller spaces of the components of R_0. Pick Beltrami differentials $\nu_1, \cdots, \nu_n$, supported in U_0, spanning the tangent space at R_0 of $\mathrm{Def}(R_0)$.

Now, using the data, we parametrize the small deformations of R_0. Let $\nu(s) = \sum_{k=1}^n s_k \nu_n$, $s \in \mathbb{C}^n$, $|s|$ small, and as in §2.3 let $R_s = R_{0,\nu(s)}$. Since U_0 is disjoint from the U_j, V_j, then the F_j, G_j are also holomorphic coordinates on R_s. Given $t = (t_1, \cdots, t_m)$, small, for each j form the plumbing of R_s with data $(U_j, V_j, F_j, G_j, t_j)$ to obtain $R_{s,t}$. The tuple $(s,t) = (s_1, \cdots, s_n, t_1, \cdots, t_m)$ are local coordinates for a *local manifold cover* of $\overline{\mathscr{M}_g}$ in a neighborhood of R. The $R_{s,t}$ are precisely the *small* (stable curve) deformations of R.

2.4.C. Description of the local manifold covers of $\overline{\mathscr{C}_g}$. For the plumbing construction we described the family $\{R_t\}$ over the t-disc. Now we shall construct the family $\{R_{s,t}\}$ over the (s,t)-polydisc.

We start with the family $\{R_s\}$, fiber=union of Riemann surfaces with punctures, over the s-polydisc. Bers has given the global description of the family over Teichmüller space [7]. A local description is obtained as follows. Set $M = \bigcup_s R_s$, for small s, M is a complex manifold with holomorphic projection to D_s. As a smooth manifold M is the product $R_0 \times D_s$. A coordinate covering of M is obtained as follows: Let $\{(U_\alpha, z_\alpha)\}$ be an atlas for R_0, and for each α let $w_\alpha(z,s)$ be a homeomorphism-solution of $w_{\bar{z}} = \nu(s) w_z$ on $z_\alpha(U_\alpha) \subset \mathbb{C}$, normalized such that $w_\alpha(z,s)$ is holomorphic in s. Then $W_\alpha = (w_\alpha \circ z_\alpha, s)$ is a chart, mapping $U_\alpha \times D_s \subset$

$R_0 \times D_s = M$ to $\mathbb{C}^{n+1}$. The chart W_α is *holomorphic*: a function h on $U_\alpha \times D_s \subset M$ is holomorphic if and only if $h \circ W_\alpha^{-1}$ is holomorphic on $W_\alpha(U_\alpha \times D_s) \subset \mathbb{C}^{n+1}$. As a warning we remind the reader that $W_\alpha(z,s)$ is not holomorphic but rather quasiconformal in z; the distinction is simple, w_α is not holomorphic relative to the R_0-conformal structure but *is*, by definition, relative to the R_s-conformal structure.

To summarize: $\{(U_\alpha \times D_s, W_\alpha)\}$ is a holomorphic atlas for M and the natural projection $M \xrightarrow{\Pi} D_s$ is holomorphic with fiber R_s (a proof is sketched at the end of this section). This is the local description of the family. As an application consider the coordinate description of the vertical line bundle $(\operatorname{Ker} d\Pi)$ of the projection. On a W_α chart, $W_\alpha(U_\alpha \times D_s) \subset \mathbb{C}^{n+1}$, Π is the projection onto the last n coordinates; $\frac{\partial}{\partial w}$ is a holomorphic section of $\operatorname{Ker} d\Pi$.

Now we give the construction of the family $\{R_{s,t}\}$ over the $D_s \times D_t$ polydisc. Let $(U_j, V_j, F_j, G_j, t_j)$ be the data for plumbing the nodes of R_0. Start with the first node, $j = 1$, and the above family $M \xrightarrow{\Pi} D_s$: for each s form the plumbing family of the s-fiber R_s. The result is a holomorphic family $M_1 \xrightarrow{\Pi \times \Pi_1} D_s \times D_{t_1}$ with fiber R_{s,t_1}; proceed by induction on j. The result is a holomorphic family $\{R_{s,t}\} \xrightarrow{\Pi_0} D_s \times D_t$ with fiber $R_{s,t}$. This is the description which we shall use for the local manifold covers of $\overline{\mathscr{C}_g} \to \overline{\mathscr{M}_g}$.

Remark. We sketch an argument that (i) $\{(U_\alpha \times D_s, W_\alpha)\}$ is a holomorphic atlas for M and (ii) the fiber of $M \xrightarrow{\Pi} D_s$ is R_s. $\{(U_\alpha \times D_s, W_\alpha)\}$ is an atlas for a C^0-structure; it is enough to show that the compositions $W_\alpha \circ W_\beta^{-1}$ are holomorphic. The solutions $w_\alpha(z,s)$ are holomorphic in s, thus the $W_\alpha \circ W_\beta^{-1}$ are holomorphic in s. On $z_\alpha(U_\alpha \cap U_\beta)$, w_α and $w_\beta \circ z_\beta \circ z_\alpha^{-1}$ are solutions of the same Beltrami equation, $w_{\bar{z}} = \nu w_z$, it is then standard that $w_\alpha \circ (w_\beta \circ z_\beta \circ z_\alpha^{-1})^{-1}$ and thus $W_\alpha \circ W_\beta^{-1}$ are holomorphic in z. Hence $W_\alpha \circ W_\beta^{-1}$ is separately holomorphic in z and s; and therefore jointly holomorphic. To see the structure of a fiber, fix an s_0; then by definition R_{s_0} and the s_0-fiber of $M \to D_s$ are surfaces with the same holomorphic atlas $\{(U_\alpha, w_\alpha(z_\alpha, s_0))\}$.

3. Grafting hyperbolic metrics

3.1. The goal is to give good approximations to the hyperbolic metric of a compact surface R, particularly for the limiting case *degeneration*. Our method is to choose hyperbolic metrics for each element U_α of an open cover $\{U_\alpha\}$ of R, such that on the overlaps the *jumps are small*. An approximating metric is constructed by interpolating between the possible choices on the overlaps. Of course, the approach depends on having good estimates for the difference between the approximating and the actual hyperbolic metric. A maximum principle argument shows that the C^{k+1} difference is estimated by the C^k norm of the curvature of the approximating metric. A Schauder theory argument provides an estimate for the C^k norm of the jumps, and also of the approximating curvature, in terms of the C^0 norm of the jumps and the *thickness* of the overlap. In brief the C^{k+1} difference between the approximating and actual hyperbolic metric is estimated by the C^0 norm of the jumps and the thickness of the overlap.

In §3.2 we review the definition of invariant differentiation and the associated norms. We describe the basic construction in §3.3 and recall the two basic estimates, Lemma 3.5 and 3.6. Finally in the last subsection we give three instances of the construction in detail, and derive the estimates for the curvature. Lemma 3.5 and the three constructions are key ingredients of the considerations in later sections.

3.2. We start with a setup for invariant differentiation and define a C^k norm on the space of metrics, compatible with a fixed conformal structure.

Let R be an arbitrary Riemann surface, κ its canonical bundle, $S(p,q)$, $p,q \in \mathbf{Z}$, the space of smooth sections of $\kappa^{p/2} \otimes \overline{\kappa}^{q/2}$ and $S(r) = S(r,-r)$, smooth sections of $(\kappa \otimes \overline{\kappa}^{-1})^{r/2}$. A section of $S(r)$ will transform as follows: if (U_α, z_α), (U_β, z_β) are charts for R with f_α, f_β the local expressions for f, and $\gamma = z_\beta \circ z_\alpha^{-1}$ the change of coordinates, then $\gamma_r^*(f_\beta) = f_\alpha$, where

$$\gamma_r^*(f_\beta) = f_\beta(\gamma(z))(\gamma'(z)/\overline{\gamma'(z)})^{r/2}.$$

In particular, f has a well-defined absolute value. Fix a conformal metric ds^2 for R; then for z a generic local coordinate, $ds^2 = \rho^2(z)|dz|^2$. Metric (essentially the covariant) derivatives are defined as follows: $K_r = \rho^{r-1}\frac{\partial}{\partial z}\rho^{-r}$ and $L_r = \rho^{-r-1}\frac{\partial}{\partial \overline{z}}\rho^r$.

Basic Properties 3.1. For $f \in S(r)$ then $\overline{f} \in S(-r)$ and

(i) for $f \in S(r)$ and γ a change of coordinates, $K_r\gamma_r^* f = \gamma_{r+1}^* K_r f$, $L_r\gamma_r^* f = \gamma_{r-1}^* L_r f$, thus $K_r\colon S(r) \to S(r+1)$ and $L_r\colon S(r) \to S(r-1)$.

(ii) $K_r = \overline{L_{-r}}$.

(iii) For $f \in S(p)$ and $g \in S(r-p)$ then $fg \in S(r)$ and

$$K_r(fg) = gK_pf + fK_{r-p}g, \qquad L_r(fg) = gL_pf + fL_{r-p}g.$$

(iv) $D_r = 4L_{r+1}K_r + r(r+1)$ is the ∂-Laplacian on $S(r)$, in particular,

$$D_0 = 4\rho^{-2}\frac{\partial^2}{\partial z \partial \overline{z}}$$

is the standard Laplacian on functions.

(v) $L_{r+1}K_r = K_{r-1}L_r + \frac{r}{2}C$, C the ds^2 Gauss curvature.

(vi) For $C = -1$, $D_{r+1}K_r = K_rD_r$ and $D_rL_{r+1} = L_{r+1}D_{r+1}$.

Item (i) is verified for L_r as follows; the reader will check the remaining. For $f \in S(r)$ the product $(ds^2)^{r/2}f$ defines a section of κ^r, a holomorphic line bundle. The exterior differential $\overline{\partial}$ is well defined for sections of κ^r; thus $\overline{\partial}((ds^2)^{r/2}f) \in S(2r, 2)$ and $(ds^2)^{(-r-1)/2}\overline{\partial}((ds^2)^{r/2}f) \in S(r-1)$, as desired.

We write $|P|$ for the number of factors in a product P of operators L_* and K_*. A C^k norm is defined on $S(r)$ by

$$\|f\|_0 = \sup_R |f|, \qquad \|f\|_k = \sum_{\substack{P\\ |P|\leq k}} \|Pf\|_0.$$

Given a subset $B \subset R$, we can localize the norm $\| \ \|_{k,B}$ by restricting the supremum to B. The norm $\| \ \|_k$ involves k-derivatives of the metric. To see this we compare two metrics for R; their ratio ds_1^2/ds_2^2 is a nonvanishing function.

Definition 3.2. The metric ds_1^2 is ε C^k-close to ds_2^2 provided the norm $\|\log ds_1^2/ds_2^2\|_k$, relative to ds_2^2, is at most ε.

Lemma 3.3. *Let $\| \ \|_{k,j}$ be the C^k norm defined by ds_j^2, $j = 1, 2$. There exists an increasing function $c(\varepsilon)$ such that if ds_1^2 is ε C^k-close to ds_2^2, then $\|f\|_{k,1} \leq c(\varepsilon)\|f\|_{k,2}$ for $f \in S(r)$.*

Proof. If $K_{r,j}$ and $L_{r,j}$ are the ds_j^2 derivatives and $\Phi = (ds_1^2/ds_2^2)^{1/2}$, then $K_{r,1} = \Phi^{r-1}K_{r,2}\Phi^{-r}$ and $L_{r,1} = \Phi^{-r-1}L_{r,w}\Phi^r$. Thus a product of p operators relative to ds_1^2 can be written as a product of p operators relative to ds_2^2 and multiplications by $\Phi^{\pm 1}$. The conclusion follows.

Corollary 3.4. *There exists a constant $c(k, \varepsilon_0)$ such that if ds_1^2 is ε C^k-close to ds_2^2, $\varepsilon < \varepsilon_0$, then ds_2^2 is $c(k, \varepsilon_0)\varepsilon$ C^k-close to ds_1^2.*

3.3. Now we shall give the grafting construction as well as the basic estimates. The estimates are based on the maximum principle; the arguments are simplest if R is compact. And so for the construction R is a general surface and for the estimates R is compact.

Let $\{U, V\}$ be an open cover of R and η_0 a smooth function on $U \cap V$ chosen such that for

$$\eta = \begin{cases} 0, & \text{on } U - V, \\ \eta_0, & U \cap V, \\ 1, & V - U, \end{cases}$$

η will be smooth on all of R ($U \cap V$ need not be connected). Obviously η is a smooth approximation of the characteristic function of V. Given ds_1^2 a metric on U and ds_2^2 a metric on V (neither need be complete) then define

$$ds^2_{\text{graft}} = (ds_1^2)^{1-\eta}(ds_2^2)^{\eta}$$

to be the *grafting* of ds_1^2 and ds_2^2 relative to η. On the overlap ds^2_{graft} interpolates between the two possible choices of metric. We shall refer to ds_1^2 and ds_2^2 as the component metrics of the grafting. We check that the definition is coordinate independent: if κ is the R canonical bundle then a metric ds^2 for R is a section of $|\kappa|^2$, an $\mathbb{R}^+$-bundle with defining cocycle taking values in the multiplicative group of positive reals. Thus for $a \in \mathbb{R}$, $(ds^2)^a$ is a section of $|\kappa|^{2a}$, and for example ds^2_{graft} is a metric.

Of particular interest to us is the case of the ds_j^2 hyperbolic. Just as a piecewise linear function on $\mathbb{R}^n$ with small C^1 jumps is close to a linear function, we shall see below that for a compact surface a piecewise hyperbolic metric with small C^1 jumps is even C^k close to the constant curvature -1 metric. As the general case consider graftings relative to any finite open cover $\{U_\alpha\}$ of R, provided that the sets only intersect in pairs. We write η for the collection of interpolation functions on the $U_\alpha \cap U_\beta$, supp(graft) for the union of the overlaps, and finally $\Phi = ds_\alpha^2/ds_\beta^2$ for the collection of ratios.

Lemma 3.5. [17]. *Let ds^2 and K be a smooth metric and its Gauss curvature for a compact surface R. If ds^2_{hyp} is the R hyperbolic metric and $\Psi = ds^2_{\text{hyp}}/ds^2$, then*

$$\|\Psi - 1\|_0 \leq \|K + 1\|_0,$$

and given an integer $k \geq 0$ and ε_0, $0 < \varepsilon_0 < 1$, there exists $c = c(k, \varepsilon_0)$ such that for $\|\Psi - 1\|_0 < \varepsilon_0$

$$\|\Psi - 1\|_{k+1} < c\|K + 1\|_k.$$

Proof. In §4.2 we recall that $f = \log \Psi$ satisfies the equation $Df - K = e^{2f}$, where D and K are for the smooth metric. The first inequality is a simple application of the maximum principle Estimate A.2 (see appendix). For the second inequality first note that $\|\Psi - 1\|_0 < \varepsilon_0$ provides a C^0 bound for f, and that a C^p bound for f leads to a C^p bound for $\Psi = e^{2f}$. The inequality follows by induction using Estimate A.3 (see appendix); the proof is complete.

Remarks. If K is somewhere positive then the first inequality provides a lower bound, but no upper bound. We will only be interested in the case $K_{\text{graft}} \approx -1$. It is an important feature that the estimate is by local data, the curvature. If the component metrics for a grafting are hyperbolic, all that has to be checked is the curvature on the overlap.

For the sake of reference we compute K_{graft}. Let U_α, $\alpha = 1, 2$, be overlapping open sets with hyperbolic metrics ds_α^2 and interpolation function η ($= 1$ on $U_2 - U_1$). If z is a conformal coordinate on $U_\alpha \cap U_\beta$, then $ds_\alpha^2 = \rho_\alpha^2 |dz|^2$ and

$$\Delta \log \rho_\alpha = \rho_\alpha^2, \qquad \Delta = \frac{\partial^2}{\partial z \partial \overline{z}}.$$

For $\Psi = \frac{1}{2}\log(ds_2^2/ds_1^2)$, the grafted metric is $ds_{\text{graft}}^2 = (\rho_1 e^{\eta\Psi})^2 |dz|^2$ with curvature

$$K_{\text{graft}} = -(\rho_1 e^{\eta\Psi})^{-2} \Delta \log(\rho_1 e^{\eta\Psi}) = -e^{-2\eta\Psi} D \log(\rho_1 e^{\eta\Psi})$$

for $D = 4L_1K_0$ the ds_1^2 Laplacian. We find, on substituting $\Delta \log \rho_\alpha = \rho_\alpha^2$, the expansion

$$K_{\text{graft}} = -e^{-2\eta\Psi}(1 + 4\Psi L_1 K_0 \eta + 8 \operatorname{Re} L_0 \eta K_0 \Psi + \eta(e^{2\Psi} - 1)) \tag{3.1}$$

relative to the ds_1^2 derivatives. Note that $K_{\text{graft}} = -1$ on the complement of $\operatorname{supp}(d\eta)$.

From the expansion for K_{graft} we see that for its C^k norm it is enough to estimate the C^{k+1} norm of $\Psi = \frac{1}{2}\log ds_2^2/ds_1^2$ and the C^{k+2} norm of η on the overlap. In fact, by the estimate below it is enough to estimate the C^0 norm of Ψ provided the overlap is *uniformly thick*. We shall use this criterion for the constructions: C^0-close hyperbolic metrics on uniformly thick overlaps.

Lemma 3.6. [17]. *Given an integer $k > 0$ and $c_0, \varepsilon_0, \varepsilon_1 > 0$, $0 < \varepsilon_1 < \varepsilon_0$, there exists a constant $c = c(k, c_0, \varepsilon_0, \varepsilon_1)$ such that if hyperbolic metrics ds_1^2 and ds_2^2 are at least c_0 C^0-close on the ds_2^2 ball $B(p, \varepsilon_0)$, then*

$$\|\Psi\|_{k,B(p,\varepsilon_1)} \leq c\|\Psi\|_{0,B(p,\varepsilon_0)}$$

for $\Psi = \frac{1}{2}\log(ds_1^2/ds_2^2)$.

Note. It is not necessary that $B(p, \varepsilon_0)$ be embedded.

Proof. Again, consult §4.2 to find $D\Psi + 1 = e^{2\Psi}$ for D the ds_2^2 Laplacian. By hypothesis $\|\Psi\|_0 < c_0$ and again a C^p bound on Ψ provides a C^p bound on $e^{2\Psi}$. The proof is by induction using Estimate A.3; the proof is complete.

3.4. Start with a surface R with a pair of punctures. We construct a family R_t by plumbing, and a grafted metric by combining the model metric in the plumbing collar and the R metric in the complement. There are four possible situations: the plumbing functions F and G *and* the R metric may or may not be compatible (have the same rotational symmetry in the collar) with the model metric. The possibilities are covered by the *primary grafting*, the *model grafting* and the *compound grafting*.

3.4.PG. The primary grafting. The first step is to describe the data. Let U, V be disjoint disc coordinate neighborhoods of a pair of punctures $p \in U$, $q \in V$. Let ds^2_R be a hyperbolic metric on R which is *complete near* p *and* q. Consider coordinate mappings: u and $F(u)$ on U (maps of U into $\mathbb{C}$), both vanishing at p as well as v and $G(v)$ on V, both vanishing at q. Set $\lambda = (F'(0))^{-1}$ and $\mu = (G'(0))^{-1}$ and introduce the normalized functions $f = \lambda F$, $g = \mu G$; the reader should keep in mind that $f(u) \approx u$ and $g(v) \approx v$. Set $\tau = \lambda\mu t$ and note that the plumbing for the $FG = t$ identification coincides with the plumbing for the $fg = \lambda\mu FG = \tau$ identification.

Set for the entire discussion positive constants A, $A < 1$, and δ, δ small; we allow a finite number of modifications to A and δ. Assume, decreasing A if necessary, that $A < |f| < 2A$ defines a relatively compact annulus in U and $A < |g| < 2A$ a relatively compact annulus in V. The inner boundaries of the annuli are approximately $|u| = A$ and $|v| = A$; with this in mind we define η smooth on U and on V such that is vanishes on $|u| < (1-\delta)A$ and $|v| < (1-\delta)A$ and is identically 1 on $|u| > (1+\delta)A$ and $|v| > (1+\delta)A$.

Now we start the construction of the plumbed surface R_t; fix $t \neq 0$ sufficiently small ($\tau = \lambda\mu t$). Consider f as a map to the z-plane and g as a map to the w-plane. Use the *charts* f and g to pull back the plumbing as well as the metric for the model case. The plumbing: remove the discs $|f| < |\tau|$ and $|g| < |\tau|$ from R and define R_t by identifying $|\tau| < |f| < 2A$ to $|\tau| < |g| < 2A$ by the rule $u \equiv v$ if and only if $f(u)g(v) = \tau$ (see §2.4). The metrics: first consider three annuli on R_t, contained in the identification locus,

$$\text{(I)} \qquad \{A < |f| < 2A\} = \{|\tau|/(2A) < |g| < |\tau|/A\},$$

$$\text{(II)} \qquad \{|\tau|/A < |f| < A\} = \{|\tau|/A < |g| < A\},$$

$$\text{(III)} \qquad \{|\tau|/2A < |f| < |\tau|/A\} = \{A < |g| < 2A\}.$$

Enlarge the annuli to obtain $\mathrm{I}_\delta, \mathrm{II}_\delta$ and III_δ by replacing A with $e^{-2\delta}A$ in the definition of I and III and A with $e^{2\delta}A$ in the definition of II; $\{\mathrm{I}_\delta, \mathrm{II}_\delta, \mathrm{III}_\delta\}$ is an open cover of the t-collar in R_t. As hyperbolic metric on I_δ and III_δ we choose ds^2_R and on II_δ we take the pullback by f of the τ-fiber model metric (1.1), specifically

$$f^* ds^2_\tau = \left(\frac{\pi}{\log|\tau|} \csc \frac{\pi \log|f|}{\log|\tau|} \frac{|df|}{|f|}\right)^2 .$$

The metrics ds^2_R on I_δ, III_δ and $f^* ds^2_\tau$ on II_δ are grafted relative to η to obtain ds^2_{graft}, the *primary grafting*. The data for the construction is the tuple $(ds^2_R, U, V, F, G, \eta, t)$; henceforth we shall specify the construction by the data.

Remark. The t-fiber model metric has a symmetry $w = t/z$. Since $fg = \tau$, it follows that $f^*ds^2_\tau = g^*ds^2_\tau$ (simply replace the letter f by the letter g). Thus the $\mathrm{I}_\delta \cap \mathrm{II}_\delta$ and $\mathrm{II}_\delta \cap \mathrm{III}_\delta$ graftings are formally the same. Therefore it is enough to analyze (or estimate) the former.

Introduce norms $|||f|||$ and $|||g|||$ to measure the deviation of $f(u)$ and $g(v)$ from the identity map. We take $|||\ |||$ as the C^0 deviation of f^{-1} and g^{-1} on the disc of radius $2A$; f, g are holomorphic, a C^0 estimate automatically bounds the derivatives.

Lemma 3.7. *With the above notation there exists a positive constant c_k, depending only on $|||f|||, |||g|||, \eta, A, \delta$ and an integer $k \geq 0$, such that for $\Psi = \frac{1}{2}\log(ds^2_R/f^*ds^2_\tau)$, $|\tau| < e^{-2\delta}A$, then*

$$\|K_{\text{graft}} + 1\|_k \leq C_k \|\Psi\|_{0,\text{overlap}}.$$

Proof. As already noted using the K_{graft} expansion (3.1) and Lemma 3.6, the norm $\|K_{\text{graft}}+1\|_k$ is estimated by the C^0 norm of Ψ, the C^{k+2} norm of η and the injectivity radius of $\operatorname{supp}(d\eta)$, all considered on the overlap. The overlap is $e^{-2\delta}A < |f| < e^{2\delta}A$. On the overlap, $f^*ds^2_\tau$ is C^∞-close to $f^*ds^2_0$ (this is the model case); it is enough to estimate the norm of η and the injectivity radius of $\operatorname{supp}(d\eta)$ in terms of $f^*ds^2_0$. Simply take η as a function of f, independent of τ, its C^k norm is bounded. Now if η is chosen with support of $d\eta$ in $(1-\delta)A < |f| < (1+\delta)A$ then certainly the injectivity radius of $\operatorname{supp}(d\eta)$ in the overlap is bounded below. We have estimated the norm and support of η; the curvature is bounded by the remaining quantity $\|\Psi\|_{0,\text{overlap}}$. The argument is complete.

3.4.MG. The model grafting. We will now describe the grafting of the fiber metrics for the model case. The $\mathrm{I}_\delta \cap \mathrm{II}_\delta$ and the $\mathrm{II}_\delta \cap \mathrm{III}_\delta$ grafting will be formally the same (see the above remark); it is enough to consider the $\mathrm{I}_\delta \cap \mathrm{II}_\delta$ overlap. The data for the *model grafting* is: $U = \{|u| < 1\}$, $ds^2_u = (|du|/(|u|\log|u|))^2$, $F(u) = u$, η is a function of $\log|u|$ with $\operatorname{supp}(\eta') \subset (\log A - \delta, \log A + \delta)$ and t is small. As an example and also since K_{graft} determines the hyperbolic metric (§4.1), we will start by computing the first perturbation of K_{graft} at $t = 0$. The estimate for the norm of $K_{\text{graft}} + 1$ is given afterwards.

Let $\Psi = \frac{1}{2}\log ds^2_t/ds^2_0$. Then the ratio and K_{graft} are coordinate independent, and we may change variables. Let

$$\zeta = \log u = a + ib \quad \text{and} \quad \Theta = \frac{\pi \log|u|}{\log|t|} = \frac{\pi a}{\log|t|};$$

thus $ds_0^2 = (|d\zeta|/a)^2$ and $\Psi = -\log(\sin\Theta/\Theta) = \frac{1}{6}\Theta^2 + O(\Theta^4)$. The functions Ψ and η are invariant under rotations of the annulus, thus they are functions of the single variable $a = \log|u|$ and *in effect* $K_0 = \frac{1}{2}a\frac{\partial}{\partial a}$, $L_0 = \frac{1}{2}a\frac{\partial}{\partial a}$. Now by (3.1) the curvature is simply

$$K_{\text{graft}} = -e^{-2\eta\Psi}(1 + \Psi a^2\eta_{aa} + 2a^2\eta_a\Psi_a + \eta(e^{2\Psi} - 1)).$$

We expand, simplify, and write ε for $(\pi/\log|t|)$ and $\equiv$ for calculations modulo ε^4. Then

$$\begin{aligned}\Theta &= \varepsilon a, \qquad \Psi \equiv \tfrac{1}{6}\varepsilon^2 a^2,\\ K_{\text{graft}} &\equiv -(1 - \tfrac{1}{3}\eta\varepsilon^2a^2)(1 + \tfrac{1}{6}\varepsilon^2a^4\eta_{aa} + \tfrac{2}{3}\varepsilon^2a^3\eta_a + \tfrac{1}{3}\varepsilon^2a^2\eta)\\ &\equiv -1 - \tfrac{1}{6}\varepsilon^2(a^4\eta_{aa} + 4a^3\eta_a) \equiv -1 - \tfrac{1}{6}\varepsilon^2((a^4\eta_a)_a).\end{aligned}$$

The a-term will appear in the expansions of §§4 and 5; for the sake of later reference we have the following.

Definition 3.8. The *model curvature perturbation* is

$$\Lambda(u) = (a^4\eta_a)_a \quad \text{for } a = \log|u|, u \in \mathbb{C}.$$

Remarks. As set up, $\eta = 1$ on $\mathrm{II}_\delta - \mathrm{I}_\delta$, and of course η_a is an approximate Dirac delta for the overlap. The implicit O-terms are uniform in a, given the uniform convergence of the t-fiber model metrics on $\log|u|$ compacta. In fact for the same reason, given an integer $k \geq 0$ there exists a constant $c_k > 0$, depending only on η, such that $\|K_{\text{graft}}+1\|_k \leq c_k(\log|t|)^{-2}$.

3.4.CG. The compound grafting. This is the general case, i.e., the R-metric, the model metric and the plumbing are not compatible. Our solution is to introduce, as the correction, an additional grafting in the center of the collar. For the new grafting $K_{\text{graft}} = -1 + O(|t|^{(1/2)-2\delta})$, an exponentially small term compared to $1/[\log(1/|t|)]$.

As data for the *compound grafting* we have: $U = \{|u| < 1\}$, $ds_U^2 = (|du|/(|u|\log|u|))^2$, $V = \{|v| < 1\}$, $ds_V^2 = (|dv|/(|v|\log|v|))^2$, F, G, and t small. As before $f = \lambda F$, $g = \mu G$ and $\tau = \lambda\mu t$. We subdivide II_δ by setting

$$(\mathrm{II}_1) \qquad \{|\tau|^{1/2} < |f| < A\} = \{|\tau|/A < |g| < |\tau|^{1/2}\},$$

$$(\mathrm{II}_2) \qquad \{|\tau|/A < |f| < |\tau|^{1/2}\} = \{|\tau|^{1/2} < |g| < A\}.$$

Enlarge the annuli to obtain $\mathrm{II}_{1,\delta}$ and $\mathrm{II}_{2,\delta}$ by replacing A with $e^{2\delta}A$ and $|\tau|^{1/2}$ with $|\tau|^{(1/2)+2\delta}$ for an inner boundary and with $|\tau|^{(1/2)-2\delta}$ for an outer boundary.

The $\mathrm{I}_\delta \cap \mathrm{II}_{1,\delta}$ and $\mathrm{II}_{2,\delta} \cap \mathrm{III}_\delta$ graftings are straightforward: take

$$\left(\frac{\pi}{\log|\tau|}\csc\frac{\pi\log|\zeta|}{\log|\tau|}\frac{|d\zeta|}{|\zeta|}\right)^2$$

for $u = \zeta$ as the metric ds_1^2 on $\mathrm{II}_{1,\delta}$ and for $v = \zeta$ as the metric ds_2^2 on $\mathrm{II}_{2,\delta}$ and $\eta = \eta(\log|\zeta|)$. In both cases these are precisely *model graftings*; we already have the desired estimates.

All that remains is to discuss the grafting on $OV = \mathrm{II}_{1,\delta} \cap \mathrm{II}_{2,\delta}$. The estimates are simplest if we introduce a third (comparison) metric ds^2_{aux}. The first step is a change of variables, $u = e^{\zeta \log|\tau|}$, $\zeta = a + ib$, $0 < a < 1$,

$$ds_1^2 = \{\pi \csc \pi a |d\zeta|)^2,$$
$$OV = \{\tfrac{1}{2} - 2\delta < a < \tfrac{1}{2} + 2\delta\},$$

and now we define

$$ds^2_{\mathrm{aux}} = \left(\pi \csc\left(\frac{\pi \log|f|}{\log|\tau|}\right)\left|\frac{uf'(u)}{f(u)}\right| |d\zeta|\right)^2 .$$

We must give a C^0 estimate on OV for

$$\Psi = \tfrac{1}{2}\log ds_1^2/ds^2_{\mathrm{aux}} = \log(\sin\Theta/\sin\pi a) + \log|f(u)/(uf'(u))|,$$

where $\Theta = \pi\log|f|/\log|\tau|$ and $a = \log|u|/\log|\tau|$. Observe that since $f'(0) = 1$ then

$$\Theta = \frac{\pi\log|f|}{\log|\tau|} = \pi\frac{\log|u|}{\log|\tau|} + \pi\frac{\log|f(u)/u|}{\log|\tau|} = \pi a + O\left(\frac{|u|}{\log|\tau|}\right);$$

in consequence the sine term for Ψ is $O(|u|/\log|\tau|)$ for $\zeta \in OV$, constant depending on $|||f|||$. Now the second term for Ψ is $\tfrac{1}{2}\operatorname{Re}\log(f(u)/(uf'(u))) = \operatorname{Re}(cu + \cdots)$, which is clearly $O(|u|)$, constant depending on $|||f|||$. In conclusion on OV since $|\tau|^{(1/2)+2\delta} < |u| < |\tau|^{(1/2)-2\delta}$ then Ψ is $O(|\tau|^{(1/2)-2\delta})$, constant depending on $|||f|||$.

The auxiliary metric was intentionally chosen so that the change of variables $g = \tau/f$ provides the corresponding expression with f replaced by g (see remark in 3.4.PG). And so we have the same estimate for $\log ds_2^2/ds^2_{\mathrm{aux}}$ and by the triangle inequality the same estimate for $\log ds_1^2/ds_2^2$ on OV.

The grafting will be done in the $\zeta = a+ib$ variable with an interpolation function $\eta = \eta(a)$, where η' has support $\{\tfrac{1}{2} - \delta < a < \tfrac{1}{2} + \delta\}$; note that the C^k norm of η is t-independent. By Lemma 3.6 ds^2_{aux} is C^k-close to ds_1^2 on $\{\tfrac{1}{2} - \tfrac{3}{2}\delta < a < \tfrac{1}{2} + \tfrac{3}{2}\delta\}$ and by a second application ds_2^2 is C^k-close to ds^2_{aux} on $\operatorname{supp}(\eta')$. Given the expansion (3.1) for K_{graft} we have the following.

Lemma 3.9. *With the above notation, there exists a positive constant c_k, depending only on $|||f|||, |||g|||, \eta, \delta$ and an integer $k \geq 0$ such that*

$$\|K_{\mathrm{graft}} + 1\|_{k,OV} \leq C_k|t|^{(1/2)-2\delta}.$$

Example. *The geometry of the compound metric in the collar.* We shall estimate the length of the core geodesic. First note that for $\alpha = 1, 2$,

ds_α^2 and ds_{graft}^2 coincide on $\text{II}_\alpha - OV$ and by the above $\log ds_{\text{graft}}^2/ds_\alpha^2$ is $O(|t|^{(1/2)-2\delta})$ on OV. Now the shortest closed curve for ds_1^2 is $|u| = \frac{1}{2}\log|\tau|$ of length $2\pi^2/\log 1/|\tau|$; by our estimates the same curve has comparable ds_{graft}^2 length. Thus the shortest ds_{graft}^2 closed curve γ has length $l_\gamma \to 0$. This initial estimate forces γ to be contained entirely in OV. On OV, ds_{graft}^2 and ds_1^2 are comparable, lengths are comparable by the same factor:

$$l_\gamma = \frac{2\pi^2}{\log 1/|\tau|}(1 + O(|\tau|^{1/2)-2\delta})).$$

4. The pinching expansion

4.1. We give the expansion for the hyperbolic metrics of a degenerating family of Riemann surfaces $\{R_{s,t}\}$. The expansion is similar to that for the model metric $ds_t^2 = ds_0^2(1 + \frac{1}{3}\Theta^2 + \cdots)$. The first term will be the grafting of the model metric and the $R_{s,0}$ metric. The second term will correspond to solving for the perturbation of the grafted metric from a constant curvature -1 metric. The remainder will be $O((1/\log|t|)^4)$ in the C^∞ norm for functions on $R_{s,t}$. A special feature of our approach is that no restriction is placed on the form of the plumbing functions.

§4.2 contains basic information on the prescribed curvature equation. The pinching expansion is presented in §4.3. The section ends with an example, the geodesic length function of a pinched geodesic.

4.2. Given a metric ds^2 and a smooth function f, then $e^{2f}ds^2$ is a conformal metric. An invariant of $e^{2f}ds^2$ is its Gauss curvature $K(e^{2f}ds^2)$. It is interesting to invert the process: given K_0, find a function f_0 such that $e^{2f_0}ds^2$ has curvature K_0 ([6], [20], [21]). We would like to review a few results on this question.

First the equation; let R be a compact surface with conformal metrics, ds_0^2 with curvature K_0 and ds^2 with curvature K. If z is a generic conformal coordinate we write $ds_0^2 = (\lambda(z)|dz|)^2$, $ds^2 = (\rho(z)|dz|)^2$ and $ds^2 = e^{2f}ds_0^2$. The ds^2 curvature equation is $-\rho^{-2}\Delta\log\rho = K$ for Δ the z-Euclidean Laplacian or equivalently $-e^{-2f}\lambda^{-2}\Delta\log(e^f\lambda) = K$ or

$$D_0 f - K_0 = -e^{2f}K \tag{4.1}$$

for D_0 the ds_0^2 Laplacian. Conversely given a putative curvature function $K < 0$ and provided $K_0 < 0$ then the equation (4.1) has a unique solution h, the metric $e^{2h}ds_0^2$ has curvature K ([6], [21]).

A special case is the *curvature correction equation* to find the function f such that $e^{2f}ds_0^2$ will have constant curvature -1,

$$D_0 f - K_0 = e^{2f}. \tag{4.2}$$

We shall approximate f by the solution of

$$D_0 f_1 - K_1 = 1 + 2f_1. \tag{4.3}$$

The difference will be estimated by the following.

Lemma 4.1. *Let R be a compact surface. Given ε, $0 < \varepsilon < 1$, and an integer $k \geq 0$ there exists a constant $c = c(k, \varepsilon)$ such that for K_0, K_1 satisfying $\|K_0 + 1\|_0$, $\|K_1 + 1\|_0 < \varepsilon$, the solutions of* (4.2) *and* (4.3) *satisfy*

$$\|e^{2f} - (1 + 2f_1)\|_{k+1} \leq c(\|K_1 - K_0\|_k + \|K_0 + 1\|_k^2).$$

Proof. First consider the C^0 estimates for f, f_1 and $(f - f_1)$. By Estimate A.2 (see appendix) applied to (4.3) and (4.2), $\|2f_1\|_0 \leq \|K_1 + 1\|_0 < \varepsilon$ (the C^0 estimate for f_1) and $\|e^{2f} - 1\|_0 \leq \|K_0 + 1\|_0 < \varepsilon$. Two consequences of the second inequality are $\|f\|_0 < c(\varepsilon)$ (the C^0 estimate for f) and that there exists a constant C such that for $E = e^{2f} - 1 - 2f$, $\|E\|_0 \leq C\|K_0 + a\|_0^2$. To estimate $(f - f_1)$ consider the difference of (4.2) and (4.3), $D_0(f - f_1) + K_1 - K_0 = e^{2f} - 1 - 2f_1$. Set $F = e^{2f} - 1 - 2f_1$ and apply Estimate A.2 to conclude $K_1 - K_0 \geq F$ at max of $(f - f_1)$, and $K_1 - K_0 \leq F$ at min of $(f - f_1)$. Now to give the C^0 estimate for $(f - f_1)$, $2f - 2f_1 = F - E$ and thus $2\|f - f_1\|_0 \leq \|F\|_0 + \|E\|_0 \leq \|K_1 - K_0\|_0 + C\|K_0 + 1\|_0^2$, the desired initial estimate.

Now to consider the C^1 estimates: apply A.3 to the difference of (4.2) and (4.3) to obtain the estimate for $(f - f_1)$; apply A.3 to (4.2) to obtain the estimate for f and thus for E. The C^1 estimate for $F = 2(f - f_1) + E$ now follows. To continue, proceed by finite induction using Estimate A.3 to obtain the C^p estimate for $(f - f_1)$ and F. The proof is complete.

4.3. Consider a degenerating family $\{R_{s,t}\}$, as described in §2.4.C. We present our expansion for the hyperbolic metric of $R_{s,t}$; the initial term is the compound grafted metric and the second term is the correction for the model curvature perturbation. The specific features are: we do not expand in the s-variables (we assume the R_s hyperbolic metric is given), nevertheless the O-term is s-independent for small s, and there is no special hypothesis for the plumbing functions F and G. In Example 5.5, §5.2, we describe the first term for the expansion in the s-variables, the resulting O-term is $O((1/\log|t|)^4 + |s|^2)$.

Return to 2.4.M and 2.4.C to review the description of a degenerating family. R is a Riemann surface with nodes $p_1, \cdots, p_m$, $R_0 = R -$

{nodes}. Choose $U_0 \subset R_0$ to support the smooth Beltrami differentials $\nu_1, \cdots, \nu_n$, $\nu(s) = \sum_{k=1}^n s_k \nu_k$, $s \in \mathbf{C}^n$, small, and choose plumbing data $(U_j, V_j, F_j, G_j, t_j)$ for each of the j nodes. The deformation $R_s = R_{\nu(s)}$ of R_0 is defined in terms of the Beltrami equation, and $R_{s,t}$ is obtained by plumbing R_s with the above data.

Now to review the description of the grafted metric (§3.4) for $R_{s,t}$. Let u_j, v_j be the rs coordinates (Definition 2.1) for the jth pair of punctures of R_s. For the remainder of the section we consider F_j, resp. G_j, as a function of u_j, resp. v_j; normalize the rs coordinates (by rotation) so that F_j, G_j have positive derivative at the origin. As before, set $\lambda_j = \lambda_j(s) = (F_j'(0))^{-1}$, $f_j = \lambda_j F_j$, $\mu_j = \mu_j(s) = (G_j'(0))^{-1}$, $g_j = \mu_j G_j$ and $t_j = \lambda_j \mu_j \tau_j$. The data for the compound grafting is $(ds^2_{R_s}, U_j, V_j, F_j, G_j$, choice of $\eta, t_j)$ and we write $dg^2_{s,t}$ for the grafted metric on $R_{s,t}$. Recall that the compound grafting (3.4.CG) consists of a model grafting at the collar boundaries and a compensating grafting at the collar core. Keeping our notation, we state the main results of the section.

Expansion 4.2. *Let ds^2_{hyp} be the $R_{s,t}$ hyperbolic metric, D the associated Laplacian and $dg^2_{s,t}$ the $R_{s,t}$ grafted metric. There exists a $\delta_0 > 0$ such that for $|t|, |s| < \delta_0$,*

$$ds^2_{\text{hyp}} = dg^2_{s,t}\left(1 - \frac{\pi^2}{3}\sum_{j=1}^m \left(\frac{1}{\log|\tau_j|}\right)^2 (D-2)^{-1}(\Lambda(u_j) + \Lambda(v_j)) + O\left(\sum_{j=1}^m \left(\frac{1}{\log|\tau_j|}\right)^4\right)\right),$$

where the O-term is for the C^∞ norm (§3.2) on functions on $R_{s,t}$ and the constant is bounded solely in terms of: δ_0, the norms (see 3.4.PG) $|||f_j|||, |||g_j|||$ relative to the R_0 rs coordinates, and the choice of interpolation function η.

Remarks. Λ is the model curvature perturbation of §3.4.MG. Replacing $(1/\log|\tau|)^2$ with $(1/|\log|t|)^2$ will produce a term of order $(1/\log|t|)^3$.

Proof. The argument consists of combining the estimates of 3.4.MG, 3.4.CG, Lemmas 3.9 and 4.1. We start and consider again the curvature $K_{s,t}$ for the grafted metric in the jth plumbing collar. By the considerations of 3.4.MG and 3.4.CG the curvature in the jth collar is

$$K_{s,t} = -1 - \frac{1}{6}\left(\frac{\pi}{\log|\tau_j|}\right)^2 (\Lambda(u_j) + \Lambda(v_j)) + O\left(\left(\frac{1}{\log|\tau_j|}\right)^4\right),$$

where the O-term is for the $dg^2_{s,t}$ C^∞ norm and the constant is bounded in terms of δ_0, choice of η and $|||f_j|||$, $|||g_j|||$. Next we consider the solution of equation (4.3) for the first perturbation of curvature. By Lemma 4.1 with $K_0 = K_{s,t}$ and

$$K_1 = -1 - \frac{1}{6}\sum_j \left(\frac{\pi}{\log|\tau_j|}\right)^2 (\Lambda(u_j) + \Lambda(v_j))$$

we find that the resulting metric is C^∞-close to the actual $R_{s,t}$ hyperbolic metric with magnitude $\sum_j(\pi/\log|\tau_j|)^4$. The constant depends on δ_0, choice of η and $|||f_j|||$, $|||g_j|||$. This is almost the desired conclusion, *except* for the s-dependence: the C^∞ norms are relative to $dg^2_{s,t}$, the norms $|||f_j|||$, $|||g_j|||$ are relative to the R_s rs coordinates, and for inverting the curvature perturbation the $dg^2_{s,t}$ Laplacian has been used.

We will check these items in order. By Lemma 3.5 and the estimates for $K_{s,t}$ we have that the metrics ds^2_{hyp} and $dg^2_{s,t}$ are $O(\sum_j(\pi/\log|\tau_j|)^2)$ C^∞-close and thus by Lemma 3.3 and Corollary 3.4 the two norms can be interchanged. The s-dependence of the rs coordinates was discussed in §2.3. The consequence, $|||f_j|||$, $|||g_j|||$ vary continuously in s, and thus are bounded by $|s| < \delta_0$ and the norms of f_j and g_j relative to the R_0 rs coordinates. Finally to compare Laplacians set $e^{2f_{s,t}} = ds^2_{\text{hyp}}/dg^2_{s,t}$, thus $D_{s,t} = e^{2f_{s,t}}D_{\text{hyp}}$ and note that there is a Neumann series expansion for $(D_{s,t} - 2)^{-1}$ in terms of $(D_{\text{hyp}} - 2)^{-1}$. In particular $(D_{s,t} - 2)^{-1} = (D_{\text{hyp}} - 2)^{-1} + O(\sum_j(\pi/\log|\tau_j|)^2)$, where the O-term is for the operator norm on C^∞ functions. We can interchange Greens operators. The proof is complete.

Remarks. We could write down the next term; it seems premature to do this, the expansion already involves $(D-2)^{-1}$ whose t-expansion we do not completely understand. In applications it may only be necessary to have an L^∞ approximation of metrics; in such cases the grafted metric could simply be replaced by the component metrics for the grafting. In the following example we will see that the curvature correction term is actually one order of magnitude smaller in the collar core.

Example 4.3. *Geodesic length functions.* If l_j is the length of the ds^2_{hyp} geodesic γ_j in the jth collar of $R_{s,t}$, then for $|s|, |t| < \delta_0$

$$l_j = \frac{2\pi^2}{\log 1/|\tau_j|} + O\left(\frac{1}{(\log|\tau_j|)^2}\sum_{k=1}^m \frac{1}{(\log|\tau_k|)^2}\right),$$

where the constant depends on δ_0, $|||f_j|||$, $|||g_j|||$ and choice of η. The proof is based on the Example of 3.4.CG and inequality (A.4.4) of appendix A.4. Specifically since $ds^2_{\text{hyp}}/dg^2_{s,t} - 1$ has order $\sum 1/(\log|\tau|)^2$, the length $l_{s,t,j}$ of the $dg^2_{s,t}$ geodesic in the jth collar satisfies

$$\left|\frac{l_j}{l_{j,s,t}} - 1\right| \le C \sum \frac{1}{(\log|\tau|)^2}.$$

The initial estimate forces the $dg^2_{s,t}$ geodesic to lie in a collar $\mathscr{C}(\gamma_j)$ of fixed width about γ_j. Inequality (A.4.4) provides that on $\mathscr{C}(\gamma_j)$ the magnitude of $(D-2)^{-1}\sum(\Lambda(u)+\Lambda(v))$ is $O(1/\log 1/|\tau_j|)$. Thus in consequence, on $\mathscr{C}(\gamma_j)$,

$$ds^2_{\text{hyp}} = dg^2_{s,t}\left(1 + O\left(\frac{1}{\log(1/|\tau_j|)} \sum_{k=1}^{m} \frac{1}{(\log|\tau_k|)^2}\right)\right)$$

and since comparable metrics have comparable geodesic lengths,

$$\left|\frac{l_j}{l_{j,s,t}} - 1\right| \le \frac{c}{\log(1/|\tau_j|)} \sum_{k=1}^{m} \frac{1}{(\log|\tau_k|)^2}.$$

Substituting

$$l_{s,t,j} = \frac{2\pi^2}{\log(1/|\tau_j|)}(1 + O(|\tau_j|^{(1/2)-2\delta}))$$

from the example of 3.4.CG, gives the desired estimate.

5. The hyperbolic curvature

5.1. The focus of this section is the curvature 2-form for the hyperbolic metric on the vertical line bundle. In §5.2 we describe a procedure for calculating the perturbations of a metric. Formulas are given for the first two deformations of the Laplacian and Gauss curvature. As an application we consider the s-dependence (see §4.3) of a degenerating metric. A second application is the calculation in §5.3 of the connection 1-form and curvature 2-form for the hyperbolic metric on the vertical line bundle. In §5.4 we describe Beltrami differentials ν with $K_{-2}\nu$ small. The idea is to start with a quadratic differential holomorphic on a subdomain and multiply by a cutoff function. The corresponding Beltrami differential will be smooth and have small K_{-2}-derivative. These special Beltrami differentials are used in §5.5 to show that the hyperbolic metric is good. In the final subsection we analyze the limiting behavior of the curvature. A consequence is the classification of the null curvature directions for the vertical line bundle.

5.2. There are several procedures for calculating the perturbations of a hyperbolic metric ([1], [2], [15], [31], [32], [37]). We now describe a slightly different organization for the calculation. In the first stage perturbations of all orders are computed in terms of derivatives of the Beltrami differential and the original metric. The calculation does not involve the map, inducing the deformation, or any potential theory. The second stage involves perturbations of the prescribed curvature equation. The procedure is valid for arbitrary smooth Beltrami differentials and arbitrary metrics. For hyperbolic metrics perturbations are given solely in terms of: the metric, derivatives of the Beltrami differential and the operator $(D-2)^{-1}$, for D the hyperbolic Laplacian.

We start by recalling the description of a Hermitian form. Let $V_{\mathbb{R}}$ be an $\mathbb{R}$-vector space with almost complex structure $\mathbf{J}$, $\mathbf{J}^2 = -1$. Set $V_{\mathbb{C}} = V_{\mathbb{R}} \otimes \mathbb{C}$ and $V_{\mathbb{C}} = V^{(1,0)} \oplus V^{(0,1)}$ the $\pm i$ $\mathbf{J}$-eigenspace splitting. We write V for the $\mathbb{C}$-vector space $V^{(1,0)}$ and recall the canonical geometric isomorphism $v \in V \to \operatorname{geom}(v) = v + \overline{v} \in V_{\mathbb{R}}$. For h a Hermitian form on V there is the associated metric k on $V_{\mathbb{R}}$,

$$k(\operatorname{geom}(v), \operatorname{geom}(w)) = 2\operatorname{Re} h(v, w)$$

and Kähler form

$$\omega(\operatorname{geom}(v), \operatorname{geom}(w)) = k(\mathbf{J}\operatorname{geom}(v), \operatorname{geom}(w)) = -2\operatorname{Im} h(v, w).$$

Any two of the three, $k, w, \mathbf{J}$, determines the third.

As an example, let $V_{\mathbb{R}} = \mathbb{C}$, $\mathbf{J}$ be multiplication by i and $z = x + iy$ the coordinate. A Hermitian form is given by $h = \frac{a}{2}\, dz \otimes d\overline{z}$, for some $a > 0$, the metric by $a(dx^2 + dy^2)$ and the Kähler form by $-2\operatorname{Im} h = a(dx \otimes dy - dy \otimes dx) = a(dx \wedge dy) = \frac{ia}{2} dz \wedge d\overline{z}$. Observe that given $\mathbf{J}$, passing from ω to k is coordinate independent, $k(\ ,\ ) = -\omega(\mathbf{J},\ )$. In the complex coordinate z the net effect is to replace $dx \wedge dy$ with $dx^2 + dy^2$. Also note that associated to an area form for $V_{\mathbb{R}}$ is a $(1,1)$ form for V. For instance, given a diffeomorphism of Riemann surfaces, the pullback of an area form is necessarily a Kähler form. We shall use these observations in the following paragraphs.

Definition 5.1. Let $f\colon R \to S$ be a diffeomorphism of Riemann surfaces, k a metric for S, and ω the Kähler form. The $\mathbf{J}$-*pullback metric* for R is

$$k_f = -(f^*\omega)(\mathbf{J}_R,\ )$$

for $\mathbf{J}_R$ the R complex structure.

To continue with the example, let R with generic coordinate z, and S with generic coordinate w, be Riemann surfaces and $w(z)$ the map f in coordinates. If $k = (\alpha(w)|dw|)^2$, then

$$f^*\omega = \alpha(w(z))^2(|w_z|^2 - |w_{\overline{z}}|^2)\tfrac{i}{2} dz \wedge d\overline{z}$$

and the J-pullback metric is

$$k_f = \alpha(w(z))^2(|w_z|^2 - |w_{\overline{z}}|^2)|dz|^2.$$

A simple property of the J-pullback metric is that the metric distortion of a quasiconformal map is independent of the particular solution of the Beltrami equation.

Lemma 5.2. *Let R be a Riemann surface with metric k and Kähler form ω. If ν is a smooth Beltrami differential, and w_1, w_2 homeomorphism-solutions of the Beltrami equation $w_{\overline{z}} = \nu w_z$, then $w_1^*(k_{w_1^{-1}}) = w_2^*(k_{w_2^{-1}})$.*

Proof. It is a standard calculation that the $\overline{w}$-derivative of $w_2 \circ w_1^{-1}$ vanishes identically, and thus $w_2 = g \circ w_1$, g holomorphic. Next, consider the composition $R \xrightarrow{w_1} S \xrightarrow{g} T$ as a mapping of Riemann surfaces, S with metric $k_{w_1^{-1}}$ and T with metric $k_{(g\circ w_1)^{-1}}$. The map g is holomorphic, and thus $g_* \circ \mathbf{J}_S = \mathbf{J}_T \circ g_*$ for $\mathbf{J}$ the appropriate complex structure. Now starting with the pullback metric (pullback by g of a symmetric 2-tensor)

$$\begin{aligned} g^* k_{(g\circ w_1)^{-1}} &= -((g\circ w_1)^{-1})^*\omega(\mathbf{J}_T g_*, g_*) = -((g\circ w_1)^{-1})^*\omega(g_*\mathbf{J}_S, g_*) \\ &= -((w_1)^{-1})^*\omega(\mathbf{J}_T,\ ) = k_{w_1^{-1}}, \end{aligned}$$

since $(g^{-1})_* \circ g_* = \mathrm{id}$, and finally applying w_1^* gives the desired conclusion.

The purpose for the lemma is suggested by the following observation. A differential invariant of a quasiconformal map, that is, independent of the particular solution of the Beltrami equation, is necessarily expressible in terms of the Beltrami differential and its derivatives. To see this let ν be the Beltrami differential, and $w(z)$ a particular solution of $w_{\overline{z}} = \nu w_z$. First note that the equation can be used to express $\overline{z}$-derivatives of w in terms of z-derivatives of ν and w. Thus the invariant can be given as an expression in the z-derivatives of w and $z, \overline{z}$ derivatives of ν. The second observation is that the first k, $k \geq 0$, z-derivatives at a fixed point z_0 of a homeomorphism-solution of the Beltrami equation can be arbitrarily specified. This is a simple consequence of a basic fact: if $w(z)$ is a solution and g is holomorphic, then $g \circ w$ is also a solution. To summarize: the invariant at z_0 is independent of the quantities $w, w_z, w_{zz}, \cdots$ and thus can be expressed solely in terms of the derivatives of ν.

We shall carry out this procedure for the pullback Laplacian and curvature. It will be more convenient to use the inverse map, consider $f: R \to S$ a diffeomorphism of surfaces, k a metric on R and $k_{f^{-1}}$ the J-pullback metric on S. Let ν be the Beltrami differential of f, and $A = (1 - |\nu|^2)^{-1}$. For the complex local coordinates z on R, w on S, let $w(z)$ be the map, and

$(\alpha(z)|dz|)^2$ the metric on R, and define the operators

$$\partial(\nu) = \left(\frac{\partial}{\partial z} - \overline{\nu}\frac{\partial}{\partial \overline{z}}\right), \qquad \overline{\partial(\nu)} = \left(\frac{\partial}{\partial \overline{z}} - \nu\frac{\partial}{\partial z}\right).$$

Let C be the curvature of $k_{f^{-1}}$, $C_* = C \circ f$, and D the Laplacian, D_* its pullback by f, i.e., for h a function on S, $(Dh) \circ f = D_*(h \circ f)$.

Lemma 5.3. *With notation as above,*

$$D_* = 4\alpha^{-2}\left((1+|\nu|^2)A\frac{\partial^2}{\partial z \partial \overline{z}} + 2\operatorname{Re}\left((\overline{\partial(\nu)}A)\frac{\partial}{\partial z} - A\frac{\partial}{\partial z}\left(\nu\frac{\partial}{\partial z}\right)\right)\right),$$

$$C_* = -\frac{1}{2}D_* \log(\alpha^2 A) + 4\alpha^{-2}\operatorname{Re}(-\overline{\partial(\nu)}(\overline{\nu}\nu_z A) + \overline{\nu}(\nu_z)^2 A + \nu_{zz}).$$

Proof. We write $w(z)$ for f, $z(w)$ for the inverse map, and $J = (|w_z|^2 - |w_{\overline{z}}|^2)^{-1}$, and recall the elementary formulas

$$z_w = J\overline{w_z}, \qquad z_{\overline{w}} = -Jw_{\overline{z}},$$

$$\frac{\partial}{\partial w} = J\overline{w_z}\partial(\nu), \qquad \frac{\partial}{\partial \overline{w}} = Jw_z\overline{\partial(\nu)},$$

$$z_w z_{\overline{w}} = -AJ\nu, \qquad |z_w|^2 \pm |z_{\overline{w}}|^2 = AJ(1 \pm |\nu|^2),$$

$$\frac{\partial^2}{\partial w \partial \overline{w}} = (|z_w|^2 + |z_{\overline{w}}|^2)\frac{\partial^2}{\partial z \partial \overline{z}} + 2\operatorname{Re}\left(z_{w\overline{w}}\frac{\partial}{\partial z} + z_w z_{\overline{w}}\frac{\partial^2}{\partial z^2}\right).$$

We start by evaluating $z_{w\overline{w}}$,

$$z_w = J\overline{w_z} = \frac{A}{w_z},$$

$$\overline{\partial(\nu)}\left(\frac{A}{w_z}\right) = \frac{-A}{(w_z)^2}(w_{z\overline{z}} - \nu w_{zz}) + \frac{1}{w_z}\overline{\partial(\nu)}A.$$

Now $w_{\overline{z}} = \nu w_z$ and thus $w_{z\overline{z}} = \nu_z w_z + \nu w_{zz}$. On substituting we obtain

$$-\frac{A}{w_z}\nu_z + \frac{1}{w_z}\overline{\partial(\nu)}A$$

and multiply by Jw_z for the result

$$z_{w\overline{w}} = J(-A\nu_z + \overline{\partial(\nu)}A).$$

The metric $k_{f^{-1}}$ is $\alpha^2(|z_w|^2 \pm |z_{\overline{w}}|^2)|dw|^2 = \alpha^2 J|dw|^2$, the Laplacian $4\alpha^{-2}J^{-1}\partial^2/\partial w\partial\overline{w}$ and the formula for D_* now follows.

To start, the curvature is

$$C = -\tfrac{1}{2}D\log\alpha^2 J = -\tfrac{1}{2}D\log\alpha^2 A + \tfrac{1}{2}D\log|w_z|^2.$$

The first term, after composing with f, is the first term of the result. The second term is

$$2\alpha^{-2}J^{-1}\frac{\partial^2}{\partial w\partial\overline{w}}\log|w_z|^2 = 4\operatorname{Re}\alpha^{-2}J^{-1}\frac{\partial^2}{\partial w\partial\overline{w}}\log w_z.$$

We start with

$$\begin{aligned}\frac{\partial}{\partial w}\log w_z &= J\frac{\overline{w_z}}{w_z}(w_{zz}-\overline{\nu}w_{z\overline{z}}) = \frac{A}{(w_z)^2}(w_{zz}-\overline{\nu}w_{z\overline{z}})\\ &= \frac{w_{zz}}{(w_z)^2} - \overline{\nu}\nu_z\left(\frac{A}{w_z}\right),\end{aligned}$$

since $w_{z\overline{z}} = \nu_z w_z + \nu w_{zz}$. Recalling the evaluation of $(A/w_z)_{\overline{w}}$ above, we substitute and find for the second term

$$\begin{aligned}\left(\overline{\nu}\nu\frac{A}{w_z}\right)_{\overline{w}} &= AJ\overline{\partial(\nu)}(\overline{\nu}\nu_z) + \overline{\nu}\nu_z J(-A\nu_z + \overline{\partial(\nu)}A)\\ &= -AJ\overline{\nu}(\nu_z)^2 + J\overline{\partial(\nu)}(A\overline{\nu}\nu_z).\end{aligned}$$

Finally we consider

$$\begin{aligned}\left(\frac{w_{zz}}{(w_z)^2}\right)_{\overline{w}} &= Jw_z\overline{\partial(\nu)}\left(\frac{w_{zz}}{(w_z)^2}\right)\\ &= \frac{J}{(w_z)^2}(-2w_{zz}(w_{z\overline{z}}-\nu w_{zz}) + w_z(w_{zz\overline{z}}-\nu w_{zzz})) = J\nu_{zz},\end{aligned}$$

where for the last step we have substituted $w_{z\overline{z}} - \nu w_{zz} = \nu_z w_z$, $w_{zz\overline{z}} - \nu w_{zzz} = \nu_{zz}w_z + 2\nu_z w_{zz}$. On collecting terms we have the desired formula, and hence the proof is complete.

Example 5.4. *The first two perturbations of D and C.* As above, $f\colon R \to S$ is a diffeomorphism of surfaces, k the metric on R and $k_{f^{-1}}$ the **J**-pullback metric on S. We are interested in the ν-expansions of D_* and C_*. Write $D_{(n)}$, $C_{(n)}$ and so on, for the term of homogeneity n in ν, for example $A_{(2)} = ((1-|\nu|^2)^{-1})_{(2)} = |\nu|^2$. Now by inspection from Lemma 5.3,

$$\begin{aligned}D_{(1)} &= -8\alpha^{-2}\operatorname{Re}\frac{\partial}{\partial z}\nu\frac{\partial}{\partial z},\\ C_{(1)} &= 4\alpha^{-2}\operatorname{Re}\left(\frac{\partial}{\partial z}\nu\frac{\partial}{\partial z}\log\alpha^2 + \nu_{zz}\right),\\ D_{(2)} &= 2|\nu|^2 D_{(0)} + 8\alpha^{-2}\operatorname{Re}|\nu|^2_{\overline{z}}\frac{\partial}{\partial z},\\ C_{(2)} &= -\frac{1}{2}D_{(2)}\log\alpha^2 - \frac{1}{2}D_{(0)}(\log A)_{(2)} - 4\alpha^{-2}\operatorname{Re}(\overline{\nu}\nu_z)_{\overline{z}}.\end{aligned}$$

The expression for $C_{(2)}$ is simplified upon substituting the expansions for $D_{(2)}$ and A,

$$C_{(2)} = 2|\nu|^2 C_{(0)} - \tfrac{1}{2}D_{(0)}|\nu|^2 - 4\alpha^{-2}\operatorname{Re}(2|\nu|^2_{\overline{z}}(\log\alpha)_z + (\overline{\nu}\nu_z)_{\overline{z}}).$$

The formula can be stated in an intrinsic form using the invariant derivatives of §3.2, $K_r = \alpha^{r-1}\frac{\partial}{\partial z}\alpha^{-r}$, and $L_r = \alpha^{-r-1}\frac{\partial}{\partial \overline{z}}\alpha^{r}$. The reader can check the following formulas:

$$\begin{aligned}
&K_{-2}\nu = (-2(\alpha^{-1})_z\nu + \alpha^{-1}\nu_z),\\
&K_{-1}(\nu K_0 h) = \alpha^{-2}(\nu h_z)_z \quad \text{for an arbitrary function } h,\\
&K_{-1}K_{-2}\nu = \alpha^{-2}((\nu(\log\alpha^2)_z)_z + \nu_{zz}),\\
&L_{-1}K_{-2}\nu = -(\alpha^{-2})_{z\bar z}\nu - (\alpha^{-2})_z\nu_{\bar z} + (\alpha^{-2})_{\bar z}\nu_z + \alpha^{-2}\nu_{z\bar z}.
\end{aligned}$$

The second and third formulas simplify the $D_{(1)}$ and $C_{(1)}$ expressions,

$$D_{(1)} = -8\operatorname{Re} K_{-1}\nu K_0, \qquad C_{(1)} = 4\operatorname{Re} K_{-1}K_{-2}\nu,$$

and $D_{(2)}$ is easy,

$$D_{(2)} = 2|\nu|^2 D_{(0)} + 8\operatorname{Re}(L_0|\nu|^2)K_0.$$

Finally the reader can check that

$$\begin{aligned}
&\operatorname{Re}(\bar\nu L_{-1}K_{-2}\nu + |K_{-2}\nu|^2 + \tfrac{1}{2}C_{(0)}|\nu|^2)\\
&\quad = \alpha^{-2}\operatorname{Re}(2|\nu|^2_{\bar z}(\log\alpha)_z + (\bar\nu\nu_z)_{\bar z}),
\end{aligned}$$

and on substituting back into the $C_{(2)}$ expansion,

$$C_{(2)} = -\tfrac{1}{2}D_{(0)}|\nu|^2 - 4\operatorname{Re}(L_1(\bar\nu K_{-2}\nu)).$$

As a remark we note that if ν is a harmonic tensor, i.e., $\nu = \alpha^{-2}\bar\varphi$, φ a holomorphic quadratic differential, then $K_{-2}\nu = 0$ and the formulas are simpler.

Example 5.5. *The s-t-expansion for a degenerating family.* As an application we find the first s-term for the expansion of the hyperbolic metric of the degenerating family $\{R_{s,t}\}$. Return to the discussion of §4.3. Start with the hyperbolic metric on the initial cusped surface R_0. For each cusped surface $R_s = R_{\nu(s)}$ take the **J**-pullback metric of the R_0 metric. Use *this new metric* on R_s to do the grafting and obtain $dg^2_{s,t}$. The **J**-pullback metric is hyperbolic near the cusps, and thus the construction and estimates are given by the previous discussion. Away from the cusps the **J**-pullback metric has curvature $-1 + O(x)$. If we solve the curvature correction equation for the first s-term as well as the first grafting term then by Lemma 4.1 the resulting metric on $R_{s,t}$ differs from the hyperbolic metric by magnitude $O((1/\log|t|)^4 + |s|^2)$. Writing $dg^2_{s,t}$ for the grafted metric, we have the initial s-t-expansion for the $R_{s,t}$ hyperbolic metric,

$$ds^2_{\text{hyp}} = dg^2_{s,t}\left(1 + (D-2)^{-1}\left[-\frac{\pi^2}{3}\sum_{j=1}^{m}\left(\frac{1}{\log|\tau_j|}\right)^2(\Lambda(u_j)+\Lambda(v_j)) + 4\sum_{k=1}^{n} s_k \operatorname{Re} K_{-1}K_{-2}\nu_k\right] + O\left(\sum_{j=1}^{m}\frac{1}{(\log|\tau_j|)^4} + \sum_{k=1}^{n}|s_k|^2\right)\right),$$

where K_{-1}, K_{-2} are relative to the R_0 hyperbolic metric.

In general our plan for the second stage of computing the perturbation of a metric follows the example. Specifically let $h\colon R \to S$ be a deformation of Riemann surfaces, R now with hyperbolic metric k. Take $k_{h^{-1}}$ as the initial approximation to the hyperbolic metric on S. On S solve the curvature correction equation $Df - C = e^{2f}$, where D and C are for the known metric $k_{h^{-1}}$. The metric $e^{2f}k_{h^{-1}}$ on S has constant curvature -1. In fact by introducing the pullback Laplacian and curvature the correction equation can also be given on R, as an equation for $\hat{f} = f \circ h$,

$$D_*\hat{f} - C_* = e^{2\hat{f}}. \tag{5.1}$$

The special feature of the setup is that if the deformation of R to S is specified by a Beltrami differential ν, then (5.1) does not explicitly involve the map h. By Lemma 5.3, D_* and C_* can be expanded in ν. Thus if ν depends on a parameter and we wish to solve to order n, we can start with (5.1) and formally expand in the parameter to obtain the successive perturbations. In particular, for generic parameters a and b the first two perturbations are $D_a\hat{f} + D\hat{f}_a - C_a = 2e^{2\hat{f}}\hat{f}_a$ and $D_{ab}\hat{f} + D_a\hat{f}_b + D_b\hat{f}_a + D\hat{f}_{ab} - C_{ab} = 4e^{2\hat{f}}\hat{f}_a\hat{f}_b + 2e^{2\hat{f}}\hat{f}_{ab}$. Now for $a = b = 0$, then $C = -1$, $\hat{f} = 0$ and

$$\begin{aligned}\hat{f}_a &= (D-2)^{-1}C_a,\\ \hat{f}_{ab} &= (D-2)^{-1}(C_{ab} - D_a\hat{f}_b - D_b\hat{f}_a + 4\hat{f}_a\hat{f}_b),\end{aligned} \tag{5.2}$$

where D is now the hyperbolic Laplacian on R, $(D-2)^{-1}$ is bounded in C^∞ norm by A.3.1 and C_a, C_{ab}, D_a and D_b are given by Example 5.4. We shall use (5.2) in the following sections.

Remarks. It is not hard to see that the general perturbation $\hat{f}_{a\cdots b}$ is also given in the form $(D-2)^{-1}(C_{a\cdots b}+$ lower order terms). Finally we note that the method can be extended to the case of R with a finite number of punctures. The current approach is based on the solution of the curvature equation (4.2), maximum principle arguments and Lemma 5.3. The new hypothesis would be compact support for the Beltrami differentials. Thus the **J**-pullback metric would be conformal and have curvature -1 in a neighborhood of the cusps. Existence of solutions of (4.2) is not difficult for this situation. The maximum principle is extended to the puncture case by first proving that all solutions for the curvature type equations necessarily vanish at the punctures. This is done in A.4 for $(D-2)^{-1}F$, F with compact support. And finally, Lemma 5.3 has already been given without compactness or curvature assumptions.

5.3. As an application of the formalism of the preceding section, we shall derive formulas for the connection 1-form and curvature 2-form of the hyperbolic metric on the vertical line bundle. The formulas are simpler for harmonic Beltrami differentials [37], but unfortunately these are the most difficult to write down. One way to resolve the dilemma is as follows. First write the connection and curvature in the form: principal term + correction term bounded by $\|K_{-2}\nu\|_k$. Then in the next subsection for certain infinitesimal deformations, such as varying t for $zw = t$, we describe a corresponding Beltrami differential ν with $\|K_{-2}\nu\|_k$ small. The result is that the connection and curvature are given as a sum, a principal term, whose order of magnitude and negativity is clear and a smaller error term.

The connection is a 1-form and the curvature a $(1,1)$-form; each is determined by its restriction to complex lines. All we need consider are 1-parameter families of smooth surfaces; such a family is specified by an initial smooth surface R and a Beltrami differential ν. As in §2.4.C, let $\{(\mathscr{O}_\beta, z_\beta)\}$ be an atlas for R and $w_\beta(z,s)$ a homeomorphism-solution of $w_{\bar z} = s\nu w_z$ on $z_\beta(\mathscr{O}_\beta)$, depending holomorphically on s with $w_z = 1$ for $s = 0$. Then, dropping the subscript from w_β and z_β, $(w(z,s),s)$ is the coordinate on $\mathscr{O}_\beta \times D_s \subset \{R_{s\nu}\}$, where $\{R_{s\nu}\}$ is the family, D_s (the base) is a small disc in the s-plane, and $(w(z,s),s)$ is holomorphic in s with s-tangent field $w_s\frac{\partial}{\partial w} + \frac{\partial}{\partial s}$. Recall that $\frac{\partial}{\partial w}$ is a holomorphic section over $\mathscr{O}_\beta \times D_s$ of the vertical line bundle and that by 5.2 the hyperbolic length of $\frac{\partial}{\partial w}$ is $\langle \frac{\partial}{\partial w}, \frac{\partial}{\partial w}\rangle = e^{2f}\alpha^2\,\mathrm{Jac}$, where $(\alpha(z)|dz|)^2$ is the initial R-hyperbolic metric, f is the solution of the curvature correction equation (5.1) for the **J**-pullback metric, and $\mathrm{Jac} = (|w_z|^2 - |w_{\overline{w}}|^2)^{-1}$ for $w = w(z,s)$. In particular for

$$h = \log\left\langle \frac{\partial}{\partial w}, \frac{\partial}{\partial w}\right\rangle = 2f + 2\log\alpha - \log(|w_z|^2(1-|s\nu|^2)),$$

then $\Theta = \partial h$ is the connection 1-form, and $\Omega = \overline{\partial}\partial h$ is the curvature 2-form for the hyperbolic metric.

Lemma 5.6. *With the above notation, for $s = 0$ and for $\frac{\partial}{\partial\sigma} = w_s\frac{\partial}{\partial w} + \frac{\partial}{\partial s}$, the connection 1-form is given by*

$$\Theta\left(\frac{\partial}{\partial z}\right) = 2(\log\alpha)_z, \qquad \Theta\left(\frac{\partial}{\partial\sigma}\right) = 2f_s - w_{zs},$$

and the curvature 2-form by

$$\Omega\left(\frac{\overline{\partial}}{\partial z}, \frac{\partial}{\partial z}\right) = \frac{1}{2}\alpha^2, \quad \Omega\left(\frac{\overline{\partial}}{\partial z}, \frac{\partial}{\partial\sigma}\right) = 2f_{s\overline{z}} - \alpha K_{-2}\nu,$$

$$\Omega\left(\frac{\overline{\partial}}{\partial\sigma}, \frac{\partial}{\partial\sigma}\right) = 2f_{s\overline{s}} + \nu\overline{\nu}.$$

Proof. Start with $(w(z,s),s)$ as coordinate on $\mathscr{O}_\alpha \times D_s$ and

$$\begin{aligned} h = h(w(z,s),s) &= \log\left\langle\frac{\partial}{\partial w}, \frac{\partial}{\partial w}\right\rangle \\ &= 2f + 2\log\alpha(z) - \log(|w_z|^2(1-|s\nu|^2)). \end{aligned}$$

In particular for $s = 0$ and $h = 2\log\alpha$, the first equation is immediate, and the third equation is a consequence of the curvature -1 equation. For the second equation we simply note that $\Theta(\frac{\partial}{\partial\sigma}) = \frac{\partial h}{\partial\sigma}$ and that $w(z,s)$ and $s\nu$ are holomorphic in s. The result is

$$\Theta\left(\frac{\partial}{\partial\sigma}\right) = 2f_s - (\log w_z)_s + \frac{\nu\overline{s\nu}}{1-|s\nu|^2},$$

and setting $s = 0$ gives the second equation. Again, since $w(z,s)$ is holomorphic in s, we see for the last equation that

$$\Omega\left(\frac{\overline{\partial}}{\partial\sigma}, \frac{\partial}{\partial\sigma}\right) = \frac{\partial^2 h}{\partial\overline{\sigma}\partial\sigma} = \left(\Theta\left(\frac{\partial}{\partial\sigma}\right)\right)_{\overline{s}} = 2f_{s\overline{s}} + \nu\overline{\nu}.$$

Now $w(z,s)$ is not holomorphic in z, and so for the fourth line we must first solve for $\Theta(\frac{\partial}{\partial\sigma})$, $\Theta(\frac{\partial}{\partial\sigma}) = \frac{\partial h}{\partial\sigma} - w_s\Theta(\frac{\partial}{\partial w})$. Substituting for $\frac{\partial h}{\partial\sigma}$, setting $s = 0$, using that $\Theta(\frac{\partial}{\partial w}) = 2(\log\alpha)_z$, and $w_z = 1$, we have the formula

$$\Theta\left(\frac{\partial}{\partial\sigma}\right) = 2f_s - w_{zs} - 2w_s(\log\alpha)_z.$$

The final step is the $\overline{z}$-derivative,

$$\begin{aligned} \Omega\left(\frac{\overline{\partial}}{\partial z}, \frac{\partial}{\partial\sigma}\right) = \left(\Theta\left(\frac{\partial}{\partial\sigma}\right)\right)_{\overline{z}} &= 2f_{s\overline{z}} - \nu_z - 2(\log\alpha)_z - 2w_s(\log\alpha)_{z\overline{z}} \\ &= 2f_{s\overline{z}} - \alpha K_{-2}\nu - \frac{1}{2}\alpha^2 w_s. \end{aligned}$$

Recalling that $\frac{\partial}{\partial\sigma} = w_s\frac{\partial}{\partial w} + \frac{\partial}{\partial s}$ the fourth equation now follows, and hence the calculation is complete.

We now write the connection and curvature in the form: principal term + correction term bounded by $\|K_{-2}\nu\|_k$.

Lemma 5.7. *With the above notation, given an integer $k \geq 1$*

$$\Theta\left(\frac{\partial}{\partial\sigma}\right) = -w_{zs} + O(\|K_{-2}\nu\|_k),$$

$$\Omega\left(\frac{\overline{\partial}}{\partial z}, \frac{\partial}{\partial\sigma}\right) = 4((D-2)^{-1}K_{-1}K_{-2}\nu)_{\overline{z}} - \alpha K_{-2}\nu,$$

$$\Omega\left(\frac{\overline{\partial}}{\partial z}, \frac{\partial}{\partial\sigma}\right) = O\left(\left\langle\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\right\rangle^{1/2}\|K_{-2}\nu\|_{k+1}\right),$$

$$\Omega\left(\frac{\overline{\partial}}{\partial\sigma}, \frac{\partial}{\partial\sigma}\right) = -2(D-2)^{-1}(|\nu|^2 + 2\operatorname{Re}(2L_1\overline{\nu}K_{-2}\nu + D_{\overline{s}}f_s) - 4f_sf_{\overline{s}}),$$

$$\begin{aligned}\Omega\left(\frac{\overline{\partial}}{\partial\sigma}, \frac{\partial}{\partial\sigma}\right) &= -2(D-2)^{-1}|\nu|^2 + O(\|\nu\|_{k+1}\|K_{-2}\nu\|_{k+2})\\ &= O(\|\nu\|^2_{k+3}),\end{aligned}$$

where the remainder terms are bounded in C^k norm over the surface, and the constants depend only on the integer k.

Proof. The proof is essentially by inspection. Combining formulas (5.2) with those of Example 5.3, we have

$$\begin{gathered}f_s = (D-2)^{-1}2K_{-1}K_{-2}\nu, \qquad D_s = -4K_{-1}\nu K_0,\\ C_{s\overline{s}} = -\tfrac{1}{2}D|\nu|^2 - 4\operatorname{Re}(L_1\overline{\nu}K_{-2}\nu),\\ f_{s\overline{s}} = (D-2)^{-1}(C_{s\overline{s}} - 2\operatorname{Re}(D_{\overline{s}}f_s) + 4f_sf_{\overline{s}}).\end{gathered}$$

The formula for line two of the lemma is immediate. Now to estimate f_s, use Lemma A.4.2, $\|(D-2)^{-1}F\|_{k+1} \leq c_k\|F\|_k$ and thus $\|f_s\|_k \leq c_k\|K_{-1}K_{-2}\nu\|_{k-1} \leq c_k\|K_{-2}\nu\|_k$. The estimates in lines one and three follow from the formulas of Lemma 5.6. The last three lines remain to be considered. The principal term of $f_{s\overline{s}}$ is the $-\frac{1}{2}D|\nu|^2$, contributed by $C_{s\overline{s}}$, the others will be remainder terms. Thus $\Omega = 2(D-2)^{-1}(-\frac{1}{2}D)|\nu|^2 + |\nu|^2 +$ remainder, and substituting $D = (D-2)+2$ gives the principal term as well as the formula. The remainder estimates are clear, given A.4.2, except possibly for the term $D_{\overline{s}}f_s = -4\overline{K_{-1}\nu K_0}f_s = -4\overline{K_{-2}\nu}L_0f_s - 4\overline{\nu}L_{-1}L_0f_s$, which is also bounded since $\|f_s\|_k \leq c_k\|K_{-2}\nu\|_k$. Hence the argument is complete.

5.4. It is essential to have a description of the tangent bundle of $\overline{\mathscr{M}_g}$, suited for calculation, in a neighborhood of a noded surface. Ideally one would want to use harmonic Beltrami differentials, constructed from the hyperbolic metric and holomorphic quadratic differentials. Unfortunately this is not an elementary approach, the distortion of the hyperbolic metric in a collar is infinite, and the construction of quadratic differentials involves residues of $(3g-2)$-meromorphic forms along the fibers of $\Pi\colon \mathscr{C}_g \to \overline{\mathscr{M}_g}$ [24]. As an alternative we now describe differentials ν with $K_{-2}\nu$ small, such that for varying t of $uv = t$, ν is supported in the collar, and for the remaining deformations the support is disjoint from the collars. Essentially we have localized the situation to the model case and the case, deformations of a punctured surface.

5.4.T. We start with a Beltrami differential describing the deformation, varying t for the family $uv = t$. The desired estimates are a matter of inspection given the differential. The idea is to consider the differential $\lambda^{-2}\chi\overline{u}^{-2}$ on $\{|t| < |u| < 1\}$, where $ds^2_{\text{hyp}} = (\lambda(u)|du|)^2$, and χ is the characteristic function of a collar.

The discussion is simpler if we use a vertical strip, the universal cover of the annulus. Fix a value t_0 of t, and define

$$z = \frac{\log u}{\log|t_0|}, \quad w = \frac{\log v}{\log|t_0|}, \quad 1+\varepsilon = \frac{\log t}{\log|t_0|}$$

(this is a new use of the letters z and w). The identification $uv = t$ becomes $z + w = 1+\varepsilon$, and $t = |t_0|$ corresponds to $\varepsilon = 0$. The t_0-fiber of the model case $uv = t$ now has universal cover $\{0 < \operatorname{Re} z < 1\}$ with hyperbolic metric $ds^2_{\text{strip}} = (\pi \csc \pi x|dz|)^2$, $z = x+iy$ and deck translations $z \to z + 2\pi i n/\log|t_0|$, $n \in \mathbb{Z}$. Choose $\varphi(x)$, $0 \le x \le 1$, such that $\varphi(0) = 0$, $\varphi(1) = 1$ and φ' has compact support in $(0,1)$. Define a mapping $f(z) = z + \varepsilon\varphi(x)$ of the width 1 strip to the width $1+\operatorname{Re}\varepsilon$ strip. Let $z_\varepsilon, w_\varepsilon$, $z_\varepsilon + w_\varepsilon = 1+\varepsilon$, be the coordinates on the width $1+\operatorname{Re}\varepsilon$ strip $\{0 < \operatorname{Re} z_\varepsilon, \operatorname{Re} w_\varepsilon < 1+\operatorname{Re}\varepsilon\}$. Now if we express the map f in z_* coordinates, $z_\varepsilon = f(z_0)$, then for $\operatorname{Re} z_0$ small, $z_\varepsilon = z_0$ and for w_* coordinates, $1+\varepsilon - w_\varepsilon = f(1-w_0)$. Thus for $\operatorname{Re} w_0$ small, $f(1-w_0) = 1 - w_0 + \varepsilon$, hence $w_\varepsilon = w_0$. Thus the map f, expressed in z_* coordinates, is the identity on the far left side of the strip and, expressed in w_* coordinates, is the identity on the far right side of the strip. It follows that the map f and the plumbing induce the same deformation of the strip.

The infinitesimal deformation of the strip is given by the Beltrami differential $\nu = \frac{\partial}{\partial\varepsilon}(f_{\overline{z}}/f_z)_{\varepsilon=0} = \frac{1}{2}\varphi_x$. We are interested in a special form for φ_x, specifically $\frac{1}{2}\varphi_x = c\alpha^{-2}\chi$, for $ds^2_{\text{strip}} = (\alpha(z)|dz|)^2$ and χ the approximate characteristic function of a subinterval in $(0,1)$: χ is smooth, equal

to 1 on
$$\left(-\frac{2\delta}{\log|t_0|}, 1+\frac{2\delta}{\log|t_0|}\right),$$
and vanishing on the complement of
$$\left(-\frac{\delta}{\log|t_0|}, 1+\frac{\delta}{\log|t_0|}\right),$$
$\delta > 0$, δ small. We make a *special assumption* that for the u, v coordinates $\chi(|u|)$, $e^{-2\delta} < |u| < 1$, and $\chi(|v|)$, $e^{-2\delta} < |v| < 1$, $uv = t_0$ are independent of t_0. We must also normalize the constant c such that $\nu = c\alpha^{-2}\chi$ represents the unit ε deformation. The condition is that $\varphi(1) = 1$ and thus $1 = \int_0^1 \varphi_x\, dx = 2c\int_0^1 \alpha^{-2}\chi\, dx$. To find c, recall that $\alpha^{-2} = (\frac{1}{\pi}\sin \pi x)^2$ and the description of χ; thus $c = \pi^2 + O((\log 1/|t_0|)^{-3})$ or for our purposes c is a constant. We would also like to find the multiple of ν that represents the t-tangent for $uv = t$. This is a matter of scaling, for $a = \log b/\log|t_0|$, then
$$\frac{\partial}{\partial b} = \frac{1}{b\log|t_0|}\frac{\partial}{\partial a},$$
and in particular
$$\frac{\partial}{\partial t} = \frac{1}{t_0\log|t_0|}\frac{\partial}{\partial \varepsilon}.$$
To summarize: φ is bounded independent of t_0 and $\frac{\partial}{\partial t}$ is represented by
$$\hat{\nu} = \frac{1}{t_0\log|t_0|}\nu = \frac{\pi^2}{t_0\log|t_0|}\alpha^{-2}\chi.$$
We require two estimates for the Beltrami differential $\hat{\nu}$. The first is for the absolute value $|\nu|^2$ on a fixed width band in the collar $B = \{\gamma < \operatorname{Re} z < 1-\gamma\}$, $\gamma > 0$, γ small. The quantities α and χ are t_0 independent on B and thus $|\hat{\nu}|^2 \approx c/|t_0\log|t_0||^2$ on B. The second estimate is for $K_{-2}\hat{\nu}$. Taking u as the coordinate, $z = \log u/\log|t_0|$, we find
$$\begin{aligned}\hat{\nu} &= \frac{c}{t_0(\log|t_0|)^3}\left(\frac{\pi}{\log|t_0|}\csc\left(\frac{\pi\log|u|}{\log|t_0|}\right)\frac{1}{|u|}\right)^{-2}\chi\overline{u}^{-2}\\ &= \frac{c}{t_0(\log|t_0|)^3}\lambda^{-2}\chi\overline{u}^{-2}\end{aligned}$$
for $ds^2_{\text{hyp}} = (\lambda(u)|du|)^2$ the metric on the u-annulus. Now $K_{-2} = \lambda^{-3}\frac{\partial}{\partial u}\lambda^2$ and on $\operatorname{supp}(\chi_u) = \{e^{-2\delta} < |u|, |v| < e^{-\delta}\}$, the t_0-fiber metric converges C^∞ uniformly to the 0-fiber model metric. Since $(\chi\overline{u}^{-2})_u$ is t_0 independent it follows that $K_{-2}\nu = \lambda^{-3}(\chi\overline{u}^{-2})_u$ is bounded in C^k.

To summarize: given $k \geq 0$, there exists a constant c_k such that $\|K_{-2}\hat{\nu}\|_k \leq c_k|t_0(\log|t_0|)^3|$. We also require two simple estimates for the derivatives of $f = z + \varepsilon\varphi$: $f_\varepsilon = \varphi$ and $f_{z\varepsilon} = \frac{1}{2}\varphi_x$ are bounded independent of t_0.

5.4.S. We want a Beltrami differential representing a deformation of a surface R_0 with punctures. Start with φ a holomorphic quadratic differential on R_0 and χ a smooth function, vanishing in a neighborhood of the punctures and identically 1 outside a larger neighborhood. The idea is to consider the Beltrami differential $\nu = (ds^2_{\text{hyp}})^{-1}\chi\overline{\varphi}$. Provided φ has at most simple poles at the punctures, then $K_{-2}\nu$ tends to zero in C^∞ norm as the support of χ increases, and thus we can choose the support to bound $K_{-2}\nu$. A second feature of the choice is the following. For a plumbing family R_t, constructed from R_0, there are inclusion maps of the open surfaces $R_t^* = R_0 - \{t\text{-discs at punctures}\}$ to R_0 and thus $\chi\varphi$ also defines a quadratic differential on R_t^* and on R_t. Since by Expansion 4.2 the R_t hyperbolic metric on $\operatorname{supp}(\chi)$ converges C^∞ uniformly to the R_0 metric, our estimates for $K_{-2}\nu$ will apply in this case as well.

Consider a puncture of R_0 with rs coordinate u and pass to the universal cover by setting $\zeta = \log u/(2\pi i)$, then $ds^2_{\text{hyp}} = (\alpha(\zeta)|d\zeta|)^2 = (|d\zeta|/\operatorname{Im}\zeta)^2$, $K_r = \alpha^{r-1}\frac{\partial}{\partial\zeta}\alpha^r$, $L_{-r} = \overline{K_r}$ and $\varphi = \sum_{n=1}^\infty a_n e^{2\pi i n\zeta}(d\zeta)^2$, where $a_0 = 0$ is the simple pole hypothesis. Choose $\chi(y)$ an approximate characteristic function of $\{\operatorname{Im}\zeta \leq 0\}$: $\zeta = x + iy$, $\chi = 1$ for $y \leq 0$ and $\chi = 0$ for $y \geq 1$. For a parameter γ write $\nu = \alpha^{-2}\chi(y-\gamma)\overline{\varphi}$. Now $K_{-2}\nu = \alpha^{-3}(\chi\overline{\varphi})_\zeta$ and given the exponential vertical decay of φ, any invariant derivative of $K_{-2}\nu$ is bounded by: (polynomial in γ) $\times\, e^{-2\pi\gamma}$. Given $\varepsilon > 0$ and an integer $k \geq 0$ we can choose γ sufficiently large such that $\|K_{-2}\nu\|_k < \varepsilon$, the desired estimate.

5.5. We now show that the hyperbolic metric on the vertical line bundle of $\Pi\colon \overline{\mathscr{C}_g} \to \overline{\mathscr{M}_g}$ (see §§1.2 and 2.4) is good (see §1.3). In particular, if Ω_{hyp} is the curvature 2-form, computed on $\mathscr{C}_g$, then $c_{1,\text{hyp}} = \frac{i}{2\pi}\Omega_{\text{hyp}}$ defines by integration a closed $(1,1)$ current on $\overline{\mathscr{C}_g}$ that represents the Chern class in rational cohomology. For the sake of reference Baily discusses the basics of divisors, vector bundles and characteristic classes on V-manifolds in [4], [5] and a review of currents on V-manifolds is given in [35].

Theorem 5.8. *The hyperbolic metric on the vertical line bundle over the universal curve is continuous and good.*

Proof. Recall the description of 2.4.M and 2.4.C, R is a Riemann surface with m nodes, $R_0 = R - \{\text{nodes}\}$ and $\{R_{s,t}\}$ is a degenerating family. Choose $U_0 \subset R_0$ to support smooth Beltrami differentials, $\nu_1, \cdots, \nu_n$, $\nu(s) = \sum_k s_k\nu_k$. As plumbing data we now specialize and take the R_0 rs coordinates at each puncture, data $= (U_j, V_j, u_j, v_j, t_j)$. As always $U_0 \cap (\bigcup_j U_j \cup V_j) = \varnothing$ and we fix a finite atlas $\{(\mathscr{O}_\alpha, z_\alpha)\}$ of $\overline{R_0}$ such that for $\mathscr{O}_\alpha \cap U_0 \neq \varnothing$ then $\mathscr{O}_\alpha$ does not contain a puncture. Also fix on each $\mathscr{O}_\alpha$, $\mathscr{O}_\alpha \cap U_0 \neq \varnothing$, a solution $f_\alpha(z,s)$, depending holomorphically on s of $f_{\bar{z}} = \nu(s) f_z$. Similarly for each plumbing collar of $R_{s,t}$, with $t_j \neq 0$, take the map $f(z,\varepsilon)$ described in 5.4.T. In brief, the small deformations of $R_{s,t}$, each $t_j \neq 0$, are given by Beltrami differentials and $(s,t) = (s_1, \cdots, s_n, t_1, \cdots, t_m)$ is a coordinate for the local manifold cover at the *point* R in $\overline{\mathscr{M}_g}$. Let D be the polydisc in the s-t-variables parametrizing the neighborhood and finally let $dg^2_{s,t}$ on $R_{s,t}$ be the primary grafting of the model metric and the R_s hyperbolic metric.

The first issue is to check that the hyperbolic metric is actually continuous. We start by observing that the grafted metric $dg^2_{s,t}$ is continuous. This is immediate away from the plumbing collars. On the collars the grafted metric coincides with the model metric, which was found to be continuous in §1.2. Now the hyperbolic metric is certainly continuous on $\mathscr{C}_g$, it only remains to check continuity at $\overline{\mathscr{C}_g} - \mathscr{C}_g$. Expansion 4.2 expresses the hyperbolic metric in terms of the continuous metric $dg^2_{s,t}$ and a quantity tending to zero, the curvature correction. The conclusion is immediate, the hyperbolic metric is continuous on the vertical line bundle.

As the preliminary step to obtaining the higher estimates for the plumbing collars, we use the change of variables of 5.4.T. For the jth plumbing collar (we now drop the subscript) $\{uv = t\} \subset \{R_{s,t}\}$, we must bound the connection and curvature forms on the vectors $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$. Recall the strip-covering of section 5.4.T, $z = \log u / \log|t_0|$, $w = \log v / \log|t_0|$, where t_0 is a fixed value of t. The estimates will be homogeneous in the vectors; we can replace $\frac{\partial}{\partial u}, \frac{\partial}{\partial v}$ by multiples or equivalently by $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial w}$. The vectors $\frac{\partial}{\partial z}, \frac{\partial}{\partial w}$ are relative to the local coordinate (z,w), $z + w = 1 + \varepsilon$, for the family of strips. By comparison the discussion of 5.4.T is in terms of the local coordinate (ζ, ε), with $\zeta = z$. The change of basis is, $\frac{\partial}{\partial z} = \frac{\partial}{\partial \zeta} + \frac{\partial}{\partial \varepsilon}$ and $\frac{\partial}{\partial w} = \frac{\partial}{\partial \varepsilon}$. We are almost in the context of Lemmas 5.6 and 5.7. The lemmas are formulated in terms of $\frac{\partial}{\partial \zeta}$, a tangent to the surface and $\frac{\partial}{\partial \sigma} = f_s \frac{\partial}{\partial \zeta} + \frac{\partial}{\partial s}$, an initial s-tangent to a family, where $f(\zeta, s)$ is a solution of the Beltrami equation. To relate the two formulations take $s = \varepsilon$, $\zeta = z$ and thus

$$\frac{\partial}{\partial \sigma} = f_\varepsilon \frac{\partial}{\partial \zeta} + \frac{\partial}{\partial \varepsilon}, \qquad \frac{\partial}{\partial z} = \frac{\partial}{\partial \zeta} + \frac{\partial}{\partial \varepsilon} = (1 - f_\varepsilon)\frac{\partial}{\partial \zeta} + \frac{\partial}{\partial \sigma}$$

and

$$\frac{\partial}{\partial w} = -f_\varepsilon \frac{\partial}{\partial \zeta} + \frac{\partial}{\partial \sigma}.$$

Now for the first simplification, f_ε is bounded independent of t_0 and thus it will suffice to estimate for the vectors $\frac{\partial}{\partial \zeta}$ and $\frac{\partial}{\partial \sigma}$. The second simplification involves the comparison form ω of §1.4: for $uv = t$, $dt = udv + vdu$, thus

$$\frac{|dt|}{|t|\log(1/|t|)} \leq \frac{|du|}{|u|\log(1/|t|)} + \frac{|dv|}{|v|\log(1/|t|)},$$

and since $|u|$, $|v| \leq 1$ we have that

$$\left(\frac{|dt|}{|t|\log(1/|t|)}\right)^2 \leq 2\omega.$$

Thus for the change of variables $1+\varepsilon = \log t/\log|t_0|$, we have the inequality $|d\varepsilon|^2 \leq 2\omega$. The third simplification starts with the observation that the section of the vertical line bundle given in 1.2 is realized as $\frac{\partial}{\partial z}$ on the strip. On the strip the connection 1-form is simply $\partial \log\langle\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\rangle$.

We are ready to estimate the connection and curvature forms; by Lemmas 5.6 and 5.7, with $\frac{\partial}{\partial \sigma}$ corresponding to the Beltrami differential ν,

$$\Theta\left(\frac{\partial}{\partial z}\right) = 2\left(\log\left\langle\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\right\rangle\right)_z, \qquad \Theta\left(\frac{\partial}{\partial \sigma}\right) = -f_{zs} + O(\|\nu\|_*),$$

$$\Omega\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\right) = \frac{1}{2}\left\langle\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\right\rangle, \qquad \Omega\left(\frac{\overline{\partial}}{\partial \sigma}, \frac{\partial}{\partial \sigma}\right) = O(\|\nu\|_*^2)$$

and considered as Hermitian forms

$$\left|\Omega\left(\frac{\overline{\partial}}{\partial z}, \frac{\partial}{\partial \sigma}\right)\right|^2 \leq \left|\Omega\left(\frac{\overline{\partial}}{\partial z}, \frac{\partial}{\partial z}\right)\Omega\left(\frac{\overline{\partial}}{\partial \sigma}, \frac{\partial}{\partial \sigma}\right)\right|,$$

since Ω is negative-definite for smooth surfaces. We start with the evaluations involving $\frac{\partial}{\partial z}$, a tangent to a fiber of $\overline{\mathscr{C}_g}$. By the last inequality there is no need to consider cross terms. For tangents to the fibers, by Expansion 4.2 we can replace the hyperbolic metric with the grafted metric. Now in the collars the grafted metric coincides with the model metric, a case treated in §1.4 and away from the collars the grafted metric converges C^∞ uniformly to the hyperbolic metric of the fiber R_0. The connection and curvature are bounded for the $\frac{\partial}{\partial z}$-directions.

All that remains are the evaluations involving $\frac{\partial}{\partial \sigma}$. We are to bound the forms on $\mathscr{C}_g$. Again we note that the estimates will be homogeneous in the vectors. We can replace the vectors $\partial/\partial t_j$ by multiples or equivalently change parameters, $1+\varepsilon_j = \log t_j/\log|t_{o_j}|$, t_{o_j} a fixed value of t_j. Coordinates for a local manifold cover of $\mathscr{C}_g$ are given: (i) on $(\mathscr{O}_\alpha \times D) \cap \mathscr{C}_g$, $\mathscr{O}_\alpha \cap U_0 \neq \varnothing$, by the chart (W, s, ε), $W = f(z, s)$, a solution of the Beltrami equation $f_{\overline{z}} = \nu(s) f_z$ on $\mathscr{O}_\alpha$, z a local coordinate on R_0 and (ii) on the jth plumbing collar $(\{\text{collar}\}\times D)\cap\mathscr{C}_g$, by the chart (W,s,ε), $W=f(z_j,\varepsilon_j)$, f the map of 5.4.T. By the above estimate for the comparison form, $|d\varepsilon_j|^2\le 2\omega_j$. It will suffice to bound the connection and curvature by the comparison form $\omega_p=|dW|^2+|ds|^2+|d\varepsilon|^2$. As the first step, observe that the Beltrami differentials for the unit s and ε tangents are bounded, independent of t, in C^∞ norm on $R_{s,t}$ (see 5.4.T and 5.4.S). As the second step, note that the derivatives $f_{zs}, f_{z\varepsilon}$ are also bounded in C^∞ norm independent of t. The conclusion follows from the expressions for Θ and Ω. The proof is complete.

5.6. As the final application we describe the null directions for the curvature form. It is interesting to compare the present discussion with that of Mumford [28, especially Theorem 4.1]. We shall assume that the curvature form can be evaluated on analytic cycles in $\overline{\mathscr{C}_g}$ by integration. This would follow for instance from the hyperbolic metric being good on the restriction of the line bundle to analytic cycles, a slight generalization of Theorem 5.8. Given the assumption, the curvature is to be considered as a measure on discs embedded in the local manifold covers of $\overline{\mathscr{C}_g}$.

We start by reviewing the argument for showing that the Chern form $c_{1,\text{hyp}}=\frac{i}{2\pi}\Omega$ is strictly negative on $\mathscr{C}_g$. Each tangent space $T\mathscr{C}_g$ is the direct sum of the tangents to the fiber of $\mathscr{C}_g\to\mathscr{M}_g$, and the harmonic Beltrami differentials for the fiber. In fact by Lemma 5.7 a tangent to the fiber is orthogonal, relative to the curvature form, to the harmonic differentials. It suffices to examine the curvature for the two subspaces. For the tangents to the fiber the curvature is negative by the classical equation for the hyperbolic metric. By Lemma 5.7 the curvature for a harmonic differential ν is simply

$$c_{1,\text{hyp}}=\frac{i}{2\pi}\Omega\left(\frac{\overline{\partial}}{\partial\sigma},\frac{\partial}{\partial\sigma}\right)=\frac{1}{\pi}(D-2)^{-1}|\nu|^2,$$

since $K_{-2}\nu=0$. Recall that the Greens function for $(D-2)^{-1}$ is pointwise negative for a smooth surface. In brief $c_{1,\text{hyp}}$ is negative on $\mathscr{C}_g$, and it follows that $c_{1,\text{hyp}}$ will be nonpositive on $\overline{\mathscr{C}_g}$.

There are two basic cases for the local geometry at a point p on a noded fiber R of $\overline{\mathscr{C}_g}$: p a smooth point of the fiber and p a node. We start by considering a smooth point and use the description of $T_p\overline{\mathscr{C}_g}$ from the previous section. The tangent space is the direct sum of three subspaces: the tangents to the fiber, the t_j-tangents (represented by Beltrami differentials for $t_j\neq 0$) and the s-tangents (represented by Beltrami differentials supported away from the nodes). The first step is to adjust the choice of differentials such that the three subspaces are approximately orthogonal relative to the curvature form $c_{1,\text{hyp}}$ at p. It will be enough to adjust the s-tangents.

An s-tangent is given as $\nu = (ds^2_{\text{hyp}})^{-1}\chi\overline{\phi}$ for ds^2_{hyp} the $R_0 = R - \{\text{nodes}\}$ hyperbolic metric, χ an approximate characteristic function of a component of R_0, and ϕ a holomorphic quadratic differential on R_0 (at most simple poles at the punctures). As notation we write $B(S)$ for the Beltrami differentials of the form $(ds^2_{\text{hyp}})^{-1}\chi\overline{\phi}$ supported on a component S of R_0, and we write $R(p)$ for the component of R_0 containing p. We wish to fix the cutoff function χ for the component $R(p)$. The space of quadratic differentials ϕ is finite dimensional, and the Greens operator $-(D-2)^{-1}$ is positive on $R(p)$; given $\varepsilon > 0$, by 5.4.S we can choose χ such that for $\nu \in B(R(p))$, with $(-(D-2)^{-1}|\nu|^2)(p) > 1$ then $\|K_{-2}\nu\|_2$ and $\|\nu\|_1\|K_{-2}\nu\|_2$ are bounded by ε.

The sign of the curvature at a smooth point p is determined as follows. A vector $v_p \in T_p\overline{\mathscr{C}_g}$ can be expressed as a sum of five components: tangent to $R\oplus$ element of $B(R(p)) \oplus t$-tangents for nodes on $R(p)\oplus$ element of $B(R_0 - R(p)) \oplus t$-tangents for remaining nodes. Then V_p is negative for $c_{1,\text{hyp}}$ if and only if one of the first three components is nonzero. We start the proof by showing that if one of the first three components is nonzero then the direction is negative. First consider the t-tangent for a node on $R(p)$. The node will open for a disc in $\overline{\mathscr{C}_g}$ with this initial tangent; the tangent can be treated as in 5.4.T. As a technical artifice to obtain smooth surfaces open the remaining nodes by a small amount, approximate, and pass to the limit by closing the nodes. Now to check the magnitudes of the components of the curvature. If $\hat{\nu}_0$ represents the unit t-tangent, then by 5.4.T and Lemma 5.7

$$c_{1,\text{hyp}} = \frac{1}{\pi}(D-2)^{-1}|\hat{\nu}_0|^2 + O\left(\frac{1}{|t|^2(\log|t|)^4}\right),$$

and $|\hat{\nu}_0|^2$ has magnitude $1/(|t|\log|t|)^2$ in the collar core. By (appendix A.4.4) the $\hat{\nu}_0$ curvature has magnitude $1/|t|^2(\log(1/|t|))^3$ at p. This will be the dominant term. Note that a priori $c_{1,\text{hyp}}$ is nonpositive; to show that v_p is negative it is enough to show that a component of the vector is negative, and that the component dominates the cross terms involving the remaining components. We consider $\hat{\nu}_0$ to be the negative component of v_p and go through the list of possible cross terms. Start with $\hat{\nu}_0$ and a tangent $\frac{\partial}{\partial z}$ to R, then by Lemma 5.7

$$\left|\Omega\left(\frac{\overline{\partial}}{\partial z}, \hat{\nu}_0\right)\right| \le C\left\langle\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\right\rangle^{1/2} \|K_{-2}\hat{\nu}_0\|_1$$

and by 5.4.T

$$\|K_{-2}\hat{\nu}_0\|_1 \leq \frac{C}{|t|(\log(1/|t|))^3}.$$

The cross term is dominated by the $\hat{\nu}_0$ principal term. Next we consider $\lambda \in B(R_0)$ arbitrary. Estimate the cross term $\Omega(\overline{\lambda}, \hat{\nu}_0)$ by Lemma 5.7. The support of $|\lambda\hat{\nu}_0|$ is contained in bands near the boundary of the collar for $\hat{\nu}_0$. The differential $|\lambda|$ is bounded, while $|\hat{\nu}_0|$ restricted to the bands and $\|K_{-2}\hat{\nu}_0\|_2$ each have magnitude $1/|t|(\log(1/|t|))^3$. In brief the λ cross term has the same magnitude and is thus dominated by the $\hat{\nu}_0$ term. The last $\hat{\nu}_0$ cross term is for the tangent $\hat{\nu}_1$ for opening a second node. Trivially the Beltrami differentials have disjoint support, thus by 5.4.T and Lemma 5.7

$$|\Omega(\overline{\hat{\nu}}_0, \hat{\nu}_1)| \leq \frac{1}{|tt_1|(\log|t|\log|t_1|)^3},$$

where t_1 is the parameter for the second node. On a disc thru v_p the parameter t_1 is a function of t and thus the $\hat{\nu}_1$ cross term is also dominated by the $\hat{\nu}_0$ term.

To summarize: if v_p has a nonzero component for opening a node on $R(p)$ then $c_{1,\mathrm{hyp}}$ is negative.

Now consider that such components are zero. Assume that v_p has a nonzero component μ_0 in $B(R(p))$. This will be the new dominant term. Given $\varepsilon > 0$ by Lemma 5.7 and the above remarks we can adjust the cutoff function such that

$$\left|\Omega\left(\frac{\overline{\partial}}{\partial z}, \mu_0\right)\right| \leq \varepsilon \left|\Omega\left(\frac{\overline{\partial}}{\partial z}, \frac{\partial}{\partial z}\right)\Omega(\overline{\mu}_0, \mu_0)\right|$$

for $\frac{\partial}{\partial z}$ an R_0-tangent. Thus the μ_0 term dominates the R_0-tangent cross term. Now, for $\lambda \in B(R_0 - R(p))$ or for λ representing opening a node not on $R(p)$, λ and μ_0 have disjoint support. By Lemma 5.7 each nonzero term of $\Omega(\overline{\mu}_0, \lambda)$ involves a factor of the form $(D-2)^{-1}$ (derivative of λ). Recall that the nodes on $R(p)$ have been opened for the sake of approximation. The support of λ is separated from $R(p)$ by a collar and thus the *uniform damping* (A.4.2) provides that $\Omega(\overline{\mu}_0, \lambda)$ tends to zero as the nodes are closed. In particular the μ_0 term dominates any λ cross term.

To summarize: if v_p has a nonzero component in $B(R(p))$ then $c_{1,\mathrm{hyp}}$ is negative.

Now consider that v_p has zero components for opening the nodes on $R(p)$ and a zero component in $B(R(p))$. Assume that v_p has a nonzero component $\frac{\partial}{\partial z}$ tangent to R. By assumption every other component of v_p is represented by a Beltrami differential λ with support separated from p by a collar. As with the above case $\Omega(\frac{\partial}{\partial z}, \lambda)$ at p is given in the form

$(D-2)^{-1}$ (derivative of λ) and the *uniform damping* provides that the term tends to zero as the nodes are closed. *The conclusion follows*: if v_p has a nonzero component tangent to the fiber of R then $c_{1,\mathrm{hyp}}$ is negative.

We wish to show that for the remaining case v_p is a null curvature vector. By hypothesis v_p is represented by a Beltrami differential λ with support separated from p by a collar. By Lemma 5.7 $\Omega(\overline{\lambda},\lambda)$ is given in the form $(D-2)^{-1}$ (expression in λ). The *uniform damping* (A.4.2) provides that as we close the nodes on $R(p)$, to complete the approximation, then $\Omega(\overline{\lambda},\lambda)$ tends to zero.

To summarize: v_p is a null vector for $c_{1,\mathrm{hyp}}$ if its first three components are zero.

This completes the discussion for p a smooth point of a noded fiber.

Now we consider the curvature at a node p of a noded fiber of $\overline{\mathscr{C}_g}$. Again we approximate and open any other nodes by a small amount. As local coordinates on $\overline{\mathscr{C}_g}$ we take the variables z, w for the $zw=t$ plumbing at p and Beltrami differentials representing the remaining deformations. As notation write TV for the span of the z and w tangents, and BD for the span of the Beltrami differentials. A vector $v_p \in T_p\overline{\mathscr{C}_g}$ is negative for $c_{1,\mathrm{hyp}}$ if and only if its TV component is nonzero. We start by considering a disc D with parameter ε, mapping into $\overline{\mathscr{C}_g}$ with a nonzero z-w-component. Specifically it will suffice to consider the map $z=\varepsilon$, $w=c\varepsilon^n$, $n\in\mathbb{Z}^+$, and thus $t=zw=c\varepsilon^{n+1}$. We shall treat the general case $c\neq 0$; the special case $c=0$ is treated in a similar fashion. Since the z coordinate on D is in general nonzero we can change coordinates for ε nonzero: replace (z,w) by (z,t). In particular modulo BD-components,

$$\frac{\partial}{\partial\varepsilon}=\frac{\partial z}{\partial\varepsilon}\frac{\partial}{\partial z}+\frac{\partial t}{\partial\varepsilon}\frac{\partial}{\partial t}=\frac{\partial}{\partial z}+c(n+1)\varepsilon^n\frac{\partial}{\partial t}.$$

By the classical curvature equation $c_{1,\mathrm{hyp}}$ in the $\frac{\partial}{\partial z}$ direction is simply the hyperbolic length. By Expansion 4.2 the length is comparable to that of the model metric:

$$\left\langle\frac{\partial}{\partial z},\frac{\partial}{\partial z}\right\rangle_{\mathrm{hyp}}=\left(\frac{\pi}{|\varepsilon|\log|c\varepsilon^{n+1}|}\csc\left(\frac{\pi\log|\varepsilon|}{\log|c\varepsilon^{n+1}|}\right)\right)^2\approx\frac{c_n}{(|\varepsilon|\log|\varepsilon|)^2}.$$

The next step is to check that the cross terms are smaller. For the $\frac{\partial}{\partial t}$ cross term we refer to 5.4.T and Lemma 5.7. The Beltrami differential ν for $\frac{\partial}{\partial t}$ has norm $\|K_{-2}\nu\|_1$ bounded by $1/|t|(\log(1/|t|))^3$. The cross term is $\Omega(\frac{\partial}{\partial z},\varepsilon^n\frac{\partial}{\partial t})$ bounded by

$$\left\langle\frac{\partial}{\partial z},\frac{\partial}{\partial z}\right\rangle^{1/2}\frac{|\varepsilon^n|}{|t|(\log(1/|t|))^3}.$$

We have already evaluated the norm and $t = \varepsilon^{n+1}$. The final bound is $1/[|\varepsilon|^2(\log|\varepsilon|)^4]$ and the cross term is dominated. We estimate the BD cross terms by Lemma 5.7. For λ the BD-component of the disc tangent field the cross term is bounded by $\langle\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\rangle^{1/2}\|K_{-2}\lambda\|_1$, and as above the hyperbolic length has magnitude $1/[|\varepsilon|\log(1/|\varepsilon|)]$. For a $B(R_0)$-component of λ the K_{-2}-norm is bounded and thus the $B(R_0)$ cross term is dominated. Finally consider a component of λ for opening a second node. If the parameter is t_1 then $t_1 = c\varepsilon^m$, $m \in \mathbb{Z}^+$. The t_1-component of $\frac{\partial}{\partial\varepsilon}$ is $mc\varepsilon^{m-1}\frac{\partial}{\partial t_1}$. If ν_1 represents $\frac{\partial}{\partial t_1}$ then $\|K_{-2}\nu_1\|_1$ has magnitude $1/[|t_1|(\log(1/|t_1|))^3]$; $\varepsilon^{m-1}\frac{\partial}{\partial t_1}$ is represented by $\varepsilon^{m-1}\nu_1$ with K_{-2}-norm of magnitude $1/[|\varepsilon|(\log(1/|\varepsilon|))^3]$. In brief the t_1 cross term is dominated.

To summarize: if the TV component of v_p is nonzero then $c_{1,\mathrm{hyp}}$ is negative. In fact by our analysis $c_{1,\mathrm{hyp}}$ is negative on a disc with z-w-component, $z = \varepsilon^n$, $w = c\varepsilon^m$.

The final case is to check that a disc at p with trivial (to all orders) z-w-component is null for the curvature. Open all nodes to approximate by smooth surfaces, and estimate as above. By the *uniform damping* the contribution near p tends to zero as the node at p is closed. The disc is null.

Remarks. The curvature nullity at a smooth point p of a noded fiber R. From the discussion the dimension of a complement for the null space is: 1+number of distinct nodes on $R(p)$+ dimension of $B(R(p))$, for $R(p)$ the component of R containing p. In particular the nullity is determined by the topology of $R(p) \subset R$. The conullity is at least three for a smooth point of a noded fiber, and has full rank for a smooth point of a fiber with exactly one nonseparating node.

Appendices

A.1. The interior Schauder estimate (ISE) is essential for bounding solutions of Laplaces equations. We only need the following result, a consequence of the L^p approach [9, Chapter 5]. Given $B \subset \mathbb{R}^2$, a bounded domain, and $B_0 \subset\subset B$, there exists a constant $c = c(B, B_0)$ such that for $\Delta P = Q$ on B, Δ the Euclidean Laplacian, then

$$|P|_{1,B_0} \leq c(|P|_{0,B} + |Q|_{0,B}),$$

where $|\ |_{k,A}$ is the C^k norm for the domain A.

A.2. The following is the elementary maximum principle argument [17].

Estimate A.2. *Let R be a compact surface with metric ds^2 and Laplacian D. For $DP + S = Q$ if p is a maximum and q a minimum of P, then $S(p) \geq Q(p)$ and $S(q) \leq Q(q)$.*

Proof. Evaluate at p and at q. Note that $DP(p) \leq 0$ and $DP(q) \geq 0$.

Remark. In practice Q is a monotone increasing function of P and thus $Q_{\max} = Q(P(p)) \leq S_{\max}$ and $Q_{\min} = Q(P(q)) \geq S_{\min}$.

A.3. For a metric ds^2 with Laplacian D and the equation $DP = Q$, we recall the standard estimate for the C^{k+1} norm of P in terms of the C^0 norm of P and C^k norm of Q. Write $\| \|_r$ for the ds^2 C^r norm, $\| \|_{r,B}$ for its restriction to B and $B(q, \varepsilon)$ for a metric ball.

Estimate A.3. *Let R be a surface with complete hyperbolic metric. Given an integer $k \geq 0$, $c_0 > 0$, $\varepsilon_0 > 0$ and $\varepsilon_1, 0 < \varepsilon_1 < \varepsilon_0$, there exists a constant $c = c(k, c_0, \varepsilon_0, \varepsilon_1)$ such that if ds^2 is c_0 C^k-close to the hyperbolic metric on $B(p, \varepsilon_0)$ and $DP = Q$ on $B(p, \varepsilon_0)$, then*

$$\|P\|_{k+1,B(p,\varepsilon_1)} \leq c(\|P\|_{0,B(p,\varepsilon_0} + \|Q\|_{k,B(p,\varepsilon_0)}).$$

Note. It is not necessary that $B(p, \varepsilon_0)$ be embedded.

Proof. Fix k, lift to the upper half plane H such that p becomes i and write $B(q, \varepsilon)$ for the ds^2 neighborhoods in H. The metric ds^2 is c_0 C^k-close to the hyperbolic metric and since $B(1, \varepsilon_0) \subset H$ is relatively compact the hyperbolic metric is C^k-close to the Euclidean metric. Thus we may interchange the ds^2 C^k norm and the Euclidean norm $| |_k$. The goal is to estimate in terms of $|P|_{0,B(i,\varepsilon_0)}$ and $|Q|_{0,B(i,\varepsilon_0)}$. On $B(i, \varepsilon_0)$, $ds^2 = (\lambda(z)|dz|)^2$ and the equation is $\Delta P = \lambda^2 Q$ for Δ the z-Euclidean Laplacian. By the ISE (and since ds^2 is C^k-close to the Euclidean metric) given ε_1, $0 < \varepsilon_1 < \varepsilon_0$, there exists c_1 such that

$$|P|_{1,B(i,\varepsilon_1)} \leq c_1(|P|_{0,B(i,\varepsilon_0)} + |Q|_{0,B(i,\varepsilon_0)}),$$

the first estimate. Proceed by finite induction. Assume given ε_n, $0 < \varepsilon_n < \varepsilon_0$; there exists a constant c_n such that

$$|P|_{n,B(i,\varepsilon_n)} \leq c_n(|P|_{0,B(i,\varepsilon_0)} + |Q|_{n-1,B(i,\varepsilon_0)}).$$

If $\Delta_n = \partial^n/\partial x^p \partial y^{n-p}$ is a derivative of order n, then $\Delta\Delta_n P = \Delta_n \Delta P = \Delta_n(\lambda^2 Q)$. By the induction hypothesis and since ds^2 is C^k-close to the Euclidean metric, both $\Delta_n P$ and $\Delta_n(\lambda^2 Q)$ are C^0 bounded on $B(i, \varepsilon_n)$. Now by the ISE given ε_{n+1}, $0 < \varepsilon_{n+1} < \varepsilon_n$, there exists a c_{n+1} such that

$$|\Delta_n P|_{1,B(i,\varepsilon_{n+1})} \leq c_{n+1}(|P|_{0,B(i,\varepsilon_0)} + |Q|_{0,B(i,\varepsilon_0)}).$$

Since Δ_n is arbitrary of order n, we have the desired $(n + 1)$st estimate, completing the argument.

A.4. We would like to collect certain standard estimates for the Greens operator $(D-a)^{-1}$, $a>0$ (we will also write $a=s(s-1)$, note for $a=2$ then $s=2$). We start with a very specific estimate, since this will then allow us to treat the compact and noncompact case together. The first argument is a variant of that in [14, Chapter 1] and is very close to the considerations of [19, Chapter 3]. We recall Fay's discussion and use similar notation.

Let H be the hyperbolic plane, $\delta(z,z_0)$ the distance, $\Delta(z;r)$ the disc of hyperbolic radius r about z, dA the area form and D the Laplacian. Given r,s, $r>0$, there exists a positive constant $m(r,s)$ such that an eigenfunction f of D on $\Delta(z;r)$ with eigenvalue $s(s-1)$ has the mean value property: $f(z)=m(r,s)\int_{\Delta(z,r)} f\,dA$. The rotationally invariant potential for $(D-s(s-1))$ is given by an associated Legendre function, $Q_s(z,z_0)=Q_s(\delta(z,z_0))$ [14, Example, p. 155]; Q_s is negative and its behavior for large δ is $Q_s(\delta)=-(\operatorname{sech}(\delta/2))^{2s}\times$ analytic function of $(s,e^{-\delta})$. The sum of the translates of Q_s by a discontinuous group Γ (isometries of H) $G_s(z,z_0)=\sum_{\gamma\in\Gamma} Q_s(z,\gamma z_0)$ is the Greens function for Γ-invariant functions. We wish to derive uniform bounds (independent of Γ) for $-G_s(z,z_0)$. The first is the *trivial estimate* $-G_s(z,z_0)\geq -Q_s(z,z_0)$, valid since the sum is termwise positive.

Lemma A.4.1. *Given $\delta_0>0$ and $s>1$ there exists a constant c_0 such that for Γ a Fuchsian group the Greens function satisfies*

$$0<-G_s(z,z_0)<c_0e^{(1-s)\delta(z,z_0)}$$

for $\delta(z,z_0)>1$ and provided the injectivity radius at z or z_0 is at least δ_0.

Proof. Assume z_0 has injectivity radius at least $\delta_0<1$. Then for $\Delta=\Delta(z_0,\delta_0)$

$$-G_s(z,z_0)=-m(\delta_0,s)\int_\Delta G_s(z,w)\,dA(w).$$

The sum for $-G_s$ has positive terms and the orbit $\Gamma(\Delta)$ consists of disjoint discs, thus

$$\begin{aligned}-\int_\Delta G_s(z,w)\,dA(w)&=-\sum_\gamma\int_\Delta Q_s(z,\gamma w)\,dA(w)\\&=-\int_{\bigcup_\gamma\gamma(\Delta)} Q_s(z,w)\,dA(w).\end{aligned}$$

Each point γz_0 is at distance at least $\delta(z,z_0)$ to z, thus each $\gamma(\Delta)$, as well as $\bigcup_\gamma\gamma(\Delta)$, is in the complement of $\Delta(z,\delta_1)$, $\delta_1=\delta(z,z_0)-\delta_0$, and in particular

$$-G_s(z,z_0)\leq -m(\delta_0,s)\int_{H-\Delta(z;\delta_1)} Q_s(z,w)\,dA(w).$$

We can use polar coordinates to integrate; noting that $-Q_s(\delta) \leq c(\operatorname{sech}\frac{\delta}{2})^{2s} \leq ce^{-\delta s}$ for $\delta = \delta(z, w)$, $dA \approx ce^{\delta} d\delta\, d\theta$ for large δ, we find

$$-G_s(z, z_0) \leq c \int_{\delta_1}^{\infty} e^{(1-s)\delta}\, d\delta = \frac{c}{s-1} e^{(1-s)\delta_1} \leq c' e^{(1-s)\delta(z, z_0)}.$$

The estimate is complete.

We now derive the same lower bound for a special case where Γ is infinite cyclic. Start with H the upper half plane and a hyperbolic transformation $z \mapsto \lambda z$ (representing a geodesic of length $l = \log \lambda$). Let $z = i$, $z_0 = e^{i\theta}$ and consider

$$-G_s(z, z_0) = -\sum_n Q_s(i, \lambda^n e^{i\theta}) \geq c \sum e^{-s\delta(i, \lambda^n e^{i\theta})}.$$

The hyperbolic distance is given by

$$\cosh \delta(a, b) = 1 + \frac{|a-b|^2}{2 \operatorname{Im} a \operatorname{Im} b},$$

and thus

$$e^{-\delta} \geq PP(a, b) = \left(2 + \frac{|a-b|^2}{\operatorname{Im} a \operatorname{Im} b}\right)^{-1}.$$

We substitute to find

$$-G_s(i, e^{i\theta}) \geq c \sum_n PP(i, \lambda^n e^{i\theta})^s.$$

Evaluating

$$PP(i, ae^{i\theta}) = \frac{a \sin\theta}{a^2 + 1} = \frac{1}{2} \frac{\sin\theta}{\cosh\alpha}$$

for $\alpha = \log a$ and in particular for $l = \log \lambda$ the sum is $\sum_n (\frac{1}{2} \sin\theta / \cosh nl)^s$. We can bound the sum below by an integral:

$$\left(\frac{\sin\theta}{\cosh nl}\right)^s = \frac{(\sin\theta)^s}{l}((\cosh nl)^{-s} l)$$

and

$$\sum_n (\cosh nl)^{-s} l \geq \int_0^{\infty} (\cosh x)^{-s}\, dx.$$

Gathering the estimates, we have a constant c, depending on s, such that for an infinite cyclic group Γ, stabilizing the imaginary axis

$$-G_s(i, e^{i\theta}) \geq c \frac{(\sin\theta)^s}{l}. \tag{A.4.1}$$

Now we are ready to start the general discussion of $(D - a)^{-1}$. Let R be a compact surface minus a finite number (possibly zero) of points, a punctured surface with complete hyperbolic metric. For $a > 0$, $-(D-a)^{-1}$ is a bounded, positive operator that is selfadjoint on a dense subspace of $L^2(R)$ [14].

Lemma A.4.2. *Given an integer $k \geq 0$ and $a > 0$ there exists a constant $c_k(a)$ such that if f has compact support in R (a vacuous condition if R is compact) then*

$$\|(D-a)^{-1}f\|_{k+1} \leq c_k(a)\|f\|_k.$$

Proof. The argument is in two parts. The first is essentially the maximum principle. Since $\mathrm{supp}(f)$ is compact, the injectivity radius is bounded below on the support and by Lemma A.4.1 $(D-a)^{-1}f$ vanishes at the punctures. Thus $(D-a)^{-1}f$ *has* an interior maximum and minimum. The argument of Estimate A.2 can be used, conclusion $\|(D-a)^{-1}f\|_0 \leq \|f\|_0$. Now the higher estimates follow from Estimate A.3. The proof is complete.

Now let us recall the thick-thin decomposition of R. Fix δ_0 and let $R_{\text{thick}} \subset R$ be the subset of points of injectivity radius at least δ_0, $R_{\text{thin}} = R - R_{\text{thick}}$. R_{thin} is a disjoint union of *collars* about short geodesic and *horoball neighborhoods* of cusps. The components of R_{thick} have diameter bounded by the genus of R. The core geodesic γ of a collar has length $l_\gamma \leq \delta_0$, the width of the collar is $\approx 2\log c'/l_\gamma$ and each boundary has length $\approx c''$. If γ corresponds to the deck transformation $z \to \lambda z$ on H then the collar boundaries lift to the rays $\theta = l/c''$, $\pi - lc''$, for $z = re^{i\theta}$.

We now describe the estimates for $-G_s(z, z_0)$ in terms of the thick-thin decomposition. Lemma A.4.1 gives an upper bound if at least one of the pair z, z_0 is in the thick component. A lower bound for one point lying on the geodesic in a collar is obtained as follows: $-G_s$ is estimated below by the Greens function of any subgroup of $\pi_1(R)$, for instance the infinite cyclic subgroup corresponding to a geodesic. Thus for z on the geodesic in the center of the collar and z_0 on the collar boundary (for this case $\theta(z_0) \approx l_\gamma/c''$) then $\delta(z, z_0) \approx \log c'/l_\gamma$ and $-G_s(z, z_0) \geq ce^{(1-s)\delta(z,z_0)}$ (same order as the upper bound). More generally the component of R_{thick} adjoining the γ-collar has diameter bounded by the genus and so we have the estimate $-G_s(z, z_0) \geq c_g l_\gamma^{s-1}$ for z_0 in this component. Now by Example 4.3 if t, $|t| < 1$, is the plumbing coordinate for a degenerating collar then $l_\gamma \approx 2\pi^2/\log 1/|t|$ and thus for $s > 1$, z on the geodesic, z_0 in the adjoining component,

$$c_g(\log(1/|t|))^{1-s} \leq -G_s(z, z_0) \leq c(\log(1/|t|))^{1-s} \tag{A.4.2}$$

and more generally for z_1, z_2 separated by the collar width

$$-G_s(z_1, z_2) \leq c(\log(1/|t|))^{2-2s}.$$

We see that there is a *uniform damping* for the propagation of a signal through a collar.

In brief our technique for estimating a degenerating Greens function is as follows. First use a combination of the *trivial estimate*, Lemma A.4.1 and (A.4.1) to estimate $G_s(x, y)$ in terms of the hyperbolic distance $\delta(x, y)$ and then use Expansion 4.2 to analyze the distance.

References

[1] L. V. Ahlfors, *Some remarks on Teichmüller's space of Riemann surfaces*, Ann. of Math. (2) **74** (1961) 171–191.

[2] L. V. Ahlfors & L. Bers, *Riemann's mapping theorem for variable metrics*, Ann. of Math. (2) **72** (1960) 385–404.

[3] S. Arakelov, *Families of algebraic curves with fixed degeneracies*, Izv. Akad. Nauk **35** (1971).

[4] W. L. Baily, *The decomposition theorem for V-manifolds*, Amer. J. Math. **78** (1956) 862–888.

[5] ——, *On the imbedding of V-manifolds in projective space*, Amer. J. Math. **79** (1957) 403–430.

[6] M. Berger, *On Riemannian structures of prescribed Gaussian curvature for compact 2-manifolds*, J. Differential Geometry **5** (1971) 325–332.

[7] L. Bers, *Fibre spaces over Teichmüller spaces*, Acta Math. **130** (1973) 89–126.

[8] ——, *Spaces of degenerating Riemann surfaces*, Ann. of Math. Studies, No. 79, Princeton University Press, Princeton, NJ, 1974, 43–55.

[9] L. Bers, F. John & M. Schechter, *Partial differential equations*, Interscience, New York, 1964.

[10] P. Deligne & D. Mumford, *The irreducibility of the space of curves of given genus*, Inst. Hautes Études Sci. Publ. Math. **36** (1979) 75–110.

[11] C. J. Earle & J. Eells, *A fibre bundle description of Teichmüller theory*, J. Differential Geometry **3** (1969) 19–43.

[12] C. Earle & A. Marden, *Geometric complex coordinates for Teichmüller space*, manuscript.

[13] J. D. Fay, *Theta functions on Riemann surfaces*, Lecture Notes in Math., Vol. 352, Springer, Berlin, 1973.

[14] ——, *Fourier coefficients of the resolvent for a Fuchsian group*, J. Reine Angew. Math. **293** (1977) 143–203.

[15] A. E. Fischer & A. J. Tromba, *On a purely "Riemannian" proof of the structure and dimension of the unramified moduli space of a compact Riemann surface*, Math. Ann. **267** (1984) 311–345.

[16] L. Gerritzen & F. Herrlich, *The extended Schottky space*, preprint.

[17] D. Gilbarg & N. S. Trudinger, *Elliptic partial differential equations of second order*, Springer, New York, 1977.

[18] D. A. Hejhal, *A continuity method for spectral theory of Fuchsian groups*, Modular Forms (R. A. Rankin, ed.), Ellis-Horwood, Chichester, 1984, 107–140.

[19] ——, *Regular b-groups, degenerating Riemann surfaces, and spectral theory*, preprint.

[20] J. L. Kazdan, *Prescribing the curvature of a Riemannian manifold*, CBMS, No. 57, Amer. Math. Soc., Providence, RI, 1984.

[21] J. L. Kazdan & F. W. Warner, *Curvature functions for compact 2-manifolds*, Ann. of Math. (2) (1974) 14–47.

[22] F. Knudsen, *The projectivity of the moduli space of stable curves*, Math. Scand. **39** (1976) 19–55, **52** (1983) 161–212.

[23] K. Kodaira, *On compact analytic surfaces.* II, Ann. of Math. (2) **77** (1963) 563–626.
[24] H. Masur, *The extension of the Weil-Petersson metric to the boundary of Teichmuller space*, Duke Math. J. **43** (1976) 623–635.
[25] J. Morrow & K. Kodaira, *Complex manifolds*, Holt, Rinehart and Winston, New York 1971.
[26] D. Mumford, *Stability of projective varieties*, Enseign. Math. (2) **23** (1977), 39–110.
[27] ——, *Hirzebruch's proportionality theorem in the noncompact case*, Invent. Math. **42** (1977) 239–272.
[28] ——, *Towards an enumerative geometry of the moduli space of curves*, in Arithmetic and Geometry (M. Artin and J. Tate, eds.), Birkhauser, Boston, Basel, 1983, 271–328.
[29] R. Richberg, *Stetige streng pseudokonvexe funktionen*, Math. Ann. **175** (1968) 257–286.
[30] I. Satake, *On a generalization of the notion of manifold*, Proc. Nat. Acad. Sci. U.S.A. **42** (1956) 359–363.
[31] A. J. Tromba, *On a natural algebraic affine connection on the space of almost complex structures and the curvature of Teichmüller space with respect to its Weil-Petersson metric*, preprint.
[32] M. Wolf, *The Teichmüller theory of harmonic maps*, J. Differential Geometry **29** (1989) 449–479.
[33] S. A. Wolpert, *On the symplectic geometry of deformations of a hyperbolic surface*, Ann. of Math. (2) **117** (1983) 207–234.
[34] ——, *On the homology of the moduli space of stable curves*, Ann. of Math. (2) **118** (1983) 491–523.
[35] ——, *On the Weil-Petersson geometry of the moduli space of curves*, Amer. J. Math. **107** (1985) 969–997.
[36] ——, *On obtaining a positive line bundle from the Weil-Petersson class*, Amer. J. Math. **107** (1985) 1485–1507.
[37] ——, *Chern forms and the Riemann tensor for the moduli space of curves*, Invent. Math. **85** (1986) 119–145.
[38] ——, *Asymptotics of the spectrum and the Selberg zeta function on the space of Riemann surfaces*, Comm. Math. Phys. **112** (1987) 283–315.
[39] ——, *The Bers embedding and the Weil-Petersson metric*, preprint.

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