J. DIFFERENTIAL GEOMETRY
30 (1989) 303–394

RICCI DEFORMATION OF THE METRIC ON COMPLETE NONCOMPACT RIEMANNIAN MANIFOLDS

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1. Main result

Suppose (M, g_{ij}) is an n-dimensional complete Riemannian manifold with metric

$$ds^2 = g_{ij}dx^i dx^j > 0.$$

It is well known that the curvature tensor $\mathrm{Rm} = \{R_{ijkl}\}$ can be decomposed into the orthogonal components which have the same symmetries as Rm:

$$\mathrm{Rm} = W + V + U \quad \text{or} \quad R_{ijkl} = W_{ijkl} + V_{ijkl} + U_{ijkl}, \tag{1}$$

where $W = \{W_{ijkl}\}$ is the Weyl conformal curvature tensor, and $V = \{V_{ijkl}\}$ and $U = \{U_{ijkl}\}$ denote the traceless Ricci part and the scalar curvature part respectively.

We know that the Ricci curvature is

$$R_{ij} = g^{kl}R_{ikjl},$$

and the scalar curvature is

$$R = g^{ij}R_{ij} = g^{ij}g^{kl}R_{ikjl}.$$

Under these notations we can write U, V, W as follows:

$$\begin{aligned}
U_{ijkl} &= \frac{1}{n(n-1)}R(g_{ik}g_{jl} - g_{il}g_{jk}),\\
V_{ijkl} &= \frac{1}{n-2}(\overset{\circ}{R}_{ik}g_{jl} - \overset{\circ}{R}_{il}g_{jk} - \overset{\circ}{R}_{jk}g_{il} + \overset{\circ}{R}_{jl}g_{ik}),\\
W_{ijkl} &= R_{ijkl} - V_{ijkl} - W_{ijkl};
\end{aligned} \tag{2}$$

here $\overset{\circ}{R}_{ij} = R_{ij} - \frac{1}{n}g_{ij}$.

If we let

$$\overset{\circ}{\mathrm{Rm}} = \{\overset{\circ}{R}_{ijkl}\} = (R_{ijkl} - U_{ijkl}) = (V_{ijkl} + W_{ijkl}), \tag{3}$$

Received August 7, 1987.

then

$$
\begin{aligned}
&|\overset{\circ}{\mathrm{Rm}}|^2 = |\overset{\circ}{R}_{ijkl}|^2 = |W_{ijkl}|^2 + |V_{ijkl}|^2, \\
(4) \qquad &|U_{ijkl}|^2 = \frac{2}{n(n-1)} R^2, \\
&|\mathrm{Rm}|^2 = |R_{ijkl}|^2 = |\overset{\circ}{\mathrm{Rm}}|^2 + |U_{ijkl}|^2.
\end{aligned}
$$

Now suppose M is a complete noncompact Riemannian manifold of dimension n. Fix a point $x_0 \in M$, and for any $x \in M$ let $\gamma(x_0, x)$ denote the distance between x_0 and x. Let

(5) $$B(x, \gamma) = \{y \in M \mid \gamma(x, y) < \gamma\}$$

be the geodesic ball. Then we can state the main result of this paper as follows.

Main Theorem. *Let M be an n-dimensional complete noncompact Riemannian manifold, $n \geq 3$. For any $c_1, c_2 > 0$ and $\delta > 0$, there exists a constant $\varepsilon = \varepsilon(n, c_1, c_2, \delta) > 0$ such that if the curvature of M satisfies:*

(A) $\mathrm{Vol}(B(x, \gamma)) \geq c_1 \gamma^n$, $\forall x \in M$, $\gamma \geq 0$, *and*

(B) $|\overset{\circ}{\mathrm{Rm}}|^2 \leq \varepsilon R^2$, $0 < R \leq c_2/\gamma(x_0, x)^{2+\delta}$ $\forall x \in M$,

then the evolution equation

$$
\begin{cases} \frac{\partial}{\partial t} g_{ij}(t) = -2R_{ij}(t), \\ g_{ij}(0) = g_{ij} \end{cases}
$$

has a solution for all time $0 \leq t < +\infty$ and the metric $g_{ij}(t)$ converges to a smooth metric $g_{ij}(\infty)$ as time $t \to +\infty$ such that $R_{ijkl}(\infty) \equiv 0$ on M.

2. Notation and conventions

The notation we are going to use in this paper is basically the same as the notation used by Hamilton in [6].

We denote vectors as V^i, covectors as V_j, and mixed tensors as T^{ij}_{klm} etc. The summation convention will always hold. For the Riemannian metric g_{ij}, we let

(1) $$(g^{ij}) = (g_{ij})^{-1}.$$

The Levi-Civita connection is given by the Christoffel symbols

(2) $$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left(\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l} \right),$$

and the Riemannian curvature tensor is

$$(3)\qquad R^l{}_{ijk} = \frac{\partial}{\partial x^i}\Gamma^l_{jk} - \frac{\partial}{\partial x^j}\Gamma^l_{ik} + \Gamma^l_{ip}\Gamma^p_{jk} - \Gamma^l_{jp}\Gamma^p_{ik},$$

$$(4)\qquad R_{ijkl} = g_{pk}R^k_{ijl}.$$

We denote the covariant derivatives of a vector V^j and a covector V_j respectively by

$$(5)\qquad \nabla_i V^j = \frac{\partial}{\partial x^i}V^j + \Gamma^j_{ik}V^k,$$

$$(6)\qquad \nabla_i V_j = \frac{\partial}{\partial x^i}V_j - \Gamma^k_{ij}V_k.$$

This definition extends uniquely to tensors so as to preserve the product rule and contractions. For the interchange of two covariant derivatives we have

$$(7)\qquad \nabla_i\nabla_j V_k - \nabla_j\nabla_i V_k = g^{pq}R_{ijkp}V_q.$$

For any tensors such as $\{S_{ijkl}\}$ and $\{T_{ijkl}\}$, we have the inner product

$$(8)\qquad \langle S_{ijkl}, T_{ijkl}\rangle = g^{i\alpha}g^{j\beta}g^{k\gamma}g^{l\delta}S_{ijkl}T_{\alpha\beta\gamma\delta},$$

and the norm of $\{T_{ijkl}\}$ is defined as

$$(9)\qquad |T_{ijkl}|^2 = \langle T_{ijkl}, T_{ijkl}\rangle.$$

We use $\mathrm{inj}(M)$ to denote the injectivity radius of M.

3. Evolution equation and the short time existence of the solution

For any n-dimensional Riemannian manifold M with metric

$$(1)\qquad ds^2 = g_{ij}dx^i dx^j > 0,$$

consider the heat flow equation

$$(2)\qquad \tfrac{\partial}{\partial t}g_{ij} = -2R_{ij}$$

on M. We want to find the evolution equations for the curvature tensor and its covariant derivatives; we need these evolution equations in this paper.

Lemma 3.1. *If the metric $g_{ij}(t)$ satisfies the evolution equation* (2), *then*

$$(3)\qquad \begin{aligned}\frac{\partial}{\partial t}R_{ijkl} &= \Delta R_{ijkl} + 2(B_{ijkl} - B_{ijlk} - B_{iljk} + B_{ikjl})\\ &\quad - g^{pq}(R_{pjkl}R_{qi} + R_{ipkl}R_{qj} + R_{ijpl}R_{qk} + R_{ijkp}R_{ql}),\\ \frac{\partial}{\partial t}R_{ij} &= \Delta R_{ij} + 2R_{piqj}R^{pq} - 2g^{pq}R_{pi}R_{qj},\\ \frac{\partial R}{\partial t} &= \Delta R + 2g^{ik}g^{jl}R_{ij}R_{kl} = \Delta R + 2S,\end{aligned}$$

where

$$B_{ijkl} = g^{p\gamma} g^{qs} R_{piqj} R_{\gamma ksl}, \tag{4}$$

$$R^{pq} = g^{pi} g^{qj} R_{ij}, \tag{5}$$

$$S = |R_{ij}|^2 = g^{ik} g^{jl} R_{ij} R_{kl}. \tag{6}$$

Proof. See Hamilton [6].

If A and B are two tensors, we write $A * B$ for the linear combination of terms formed by contraction on $A_{i\dots j} B_{k\dots l}$ using the g^{ik}, and write $\nabla^m A$ for the mth covariant derivatives of A with respect to the metric g_{ij}. Then we have the following lemma.

Lemma 3.2. *If the metric $g_{ij}(t)$ satisfies the evolution equation* (2), *then for any integer $m \geq 0$ we have*

$$\begin{aligned} \frac{\partial}{\partial t} \nabla^m \mathrm{Rm} &= \Delta(\nabla^m \mathrm{Rm}) + \sum_{i+j=m} \nabla^i \mathrm{Rm} * \nabla^j \mathrm{Rm}, \\ \frac{\partial}{\partial t} |\nabla^m \mathrm{Rm}|^2 &= \Delta |\nabla^m \mathrm{Rm}|^2 - 2 |\nabla^{m+1} \mathrm{Rm}|^2 \\ &\quad + \sum_{i+j=m} \nabla^i \mathrm{Rm} * \nabla^j \mathrm{Rm} * \nabla^m \mathrm{Rm}. \end{aligned} \tag{7}$$

Proof. This is Theorem 13.2 and Corollary 13.3 in Hamilton [6].

Lemma 3.3. *Suppose (M, g_{ij}) is a noncompact complete n-dimensional Riemannian manifold with sectional curvature $0 < R_{ijij} \leq k_0$. Then the injectivity radius of M satisfies*

$$\mathrm{inj}(M) \geq \pi / \sqrt{k_0}. \tag{8}$$

Proof. This is a well-known fact. Actually one can use the arguments of [2] to prove this lemma. For example, use Lemma 5.6 and Corollary 5.7 in [2].

The following short time existence theorem for the evolution equation (2) is a special case of the theorem proved in [12].

Theorem 3.4. *Suppose $(M, g_{ij}(x))$ is an n-dimensional complete noncompact Riemannian manifold with its sectional curvature satisfying $0 < R_{ijij} \leq k_0$. Then there exists a constant $T_0 = T_0(n, k_0) > 0$ depending only on n and k_0 such that the evolution equation*

$$\frac{\partial}{\partial t} g_{ij}(x,t) = -2R_{ij}(x,t), \qquad g_{ij}(x,0) = g_{ij}(x) \tag{9}$$

has a smooth solution $g_{ij}(x,t) > 0$ on $0 \leq t \leq T_0$ and satisfies the following estimates: For any integer $m \geq 0$, there exist constants

$c_{m+1} = c_{m+1}(n, k_0) > 0$ *depending only on* n *and* k_0 *such that*

$$\sup_M |\nabla^m R_{ijkl}(x,t)|^2 \le c_{m+1}/t^m, \qquad 0 \le t \le T_0. \tag{10}$$

Proof. The sectional curvature of M satisfies $0 < R_{ijij} \le k_0$. By using formula (1.10) of [2] we have

$$|R_{ijkl}|^2 \le 100 n^4 k_0^2 \quad \text{on } M. \tag{11}$$

Thus from Theorem 1.1 of [12] it follows that the theorem is true.

4. Maximal principle of the heat equation on noncompact manifolds

In the case where M is a compact Riemannian manifold, the maximal principle of the heat equation on M is easy to prove, just as Hamilton did in [6].

In the case where M is a noncompact complete Riemannian manifold, the maximal principle for the parabolic heat equation on M is much more complicated and is not always true except if we make some curvature assumption on M and some growth assumption of the solution near the infinite of M. The proof of such maximal principles is not so easy; for details we refer the reader to the papers of D. G. Aronson [1], H. Donnelly [5], L. Karp and P. Li [9], P. Li and S. T. Yau [10], and M. H. Protter and H. F. Weinberger [11].

Let $(M, g_{ij}(x))$ be an n-dimensional complete noncompact Riemannian manifold with its sectional curvature satisfying

$$0 < R_{ijij} \le k_0.$$

Then from Theorem 3.4 in §3 we can find a metric

$$ds^2 = g_{ij}(x,t)\, dx^i dx^j > 0 \quad \text{on } M \times [0, T_0]$$

such that

$$\begin{cases} \frac{\partial}{\partial t} g_{ij}(x,t) = -2R_{ij}(x,t), \\ g_{ij}(x,0) \equiv g_{ij}(x), \end{cases} \qquad x \in M,\ 0 \le t \le T_0,$$

and

$$\sup_M |\nabla^m R_{ijkl}(x,t)|^2 \le c_{m+1}/t^m, \qquad 0 \le t \le T_0,\ m \ge 0.$$

If $m = 1$, we get

$$|\nabla_p R_{ijkl}|^2 \le c_2/t, \qquad x \in M,\ 0 \le t \le T_0, \tag{1}$$

$$|\nabla_p R_{ijkl}(x,t)| \le c_2^{1/2}/\sqrt{t}, \qquad x \in M,\ 0 \le t \le T_0; \tag{2}$$

thus

$$\begin{aligned}(3)\qquad &\int_0^{T_0} |\nabla_p R_{ijkl}(x,t)|\,dt \le c_2^{1/2}\int_0^{T_0} dt/\sqrt{t} = 2\sqrt{T_0 c_2}, \qquad x\in M,\\ &\sup_{x\in M}\int_0^{T_0} |\nabla_p R_{ijkl}(x,t)|\,dt \le 2\sqrt{T_0c_2} < +\infty.\end{aligned}$$

In this section we make the following assumption.

Assumption A. M is an n-dimensional complete noncompact Riemannian manifold with respect to the metric

$$ds^2 = g_{ij}(x,t)\,dx^i dx^j > 0$$

on $C^\infty(M\times[0,T])$, where $0 < T < +\infty$ is some constant such that

$$(4)\qquad \frac{\partial}{\partial t} g_{ij}(x,t) = -2R_{ij}(x,t) \quad \text{on } M\times[0,T];$$

$$\begin{aligned}(5)\qquad &0 < R_{ijij}(x,0) \le k_0, \qquad x\in M,\\ &|R_{ijkl}(x,t)|^2 \le c_1, \qquad x\in M,\ 0\le t\le T,\\ &\int_0^T |\nabla_p R_{ijkl}(x,t)|\,dt \le c_2, \qquad x\in M,\end{aligned}$$

where $0 < c_1, c_2 < +\infty$ are two constants.

Under Assumption A, we let

$$(6)\qquad ds_t^2 = g_{ij}(x,t)\,dx^i dx^j > 0, \qquad 0\le t\le T,$$

and use ∇ or ∇^t to denote the connection of ds_t^2, Δ or Δ_t the Laplacian operator of ds_t^2, and $\gamma_t(x,y)$ the distance between x and y with respect to metric ds_t^2 for any two points $x, y\in M$.

Lemma 4.1. *Under Assumption* A, *we have*

$$\begin{aligned}(7)\qquad &e^{-2\sqrt{nc_1}t} ds_0^2 \le ds_t^2 \le e^{2\sqrt{nc_1}t} ds_0^2, \qquad 0\le t\le T,\\ &e^{-\sqrt{nc_1}t}\gamma_0(x,y) \le \gamma_t(x,y) \le e^{\sqrt{nc_1}t}\gamma_0(x,y), \qquad x\in M, y\in M.\end{aligned}$$

Thus for each t, $0\le t\le T$, ds_t^2 is equivalent to ds_0^2.

Proof. Since

$$|R_{ijkl}(x,t)|^2 \le c_1 \quad \text{on } M\times[0,T],$$

we get

$$(8)\qquad |R_{ij}(x,t)|^2 \le nc_1 \quad \text{on } M\times[0,T].$$

Thus from

$$\frac{\partial}{\partial t} g_{ij}(x,t) = -2R_{ij}(x,t),$$

it follows that

$$|\tfrac{\partial}{\partial t} g_{ij}(x,t)| \le 2|R_{ij}(x,t)| \le 2\sqrt{nc_1}; \tag{9}$$

that is,

$$-2\sqrt{nc_1}\, g_{ij}(x,t) \le \tfrac{\partial}{\partial t} g_{ij}(x,t) \le 2\sqrt{nc_1}\, g_{ij}(x,t). \tag{10}$$

We now have

$$e^{-2\sqrt{nc_1}t} g_{ij}(x,0) \le g_{ij}(x,t) \le e^{2\sqrt{nc_1}t} g_{ij}(x,0), \tag{11}$$

$$e^{-2\sqrt{nc_1}t} ds_0^2 \le ds_t^2 \le e^{2\sqrt{nc_1}t}\, ds_0^2. \tag{12}$$

Using (12) we get

$$e^{-\sqrt{nc_1}t} \gamma_0(x,y) \le \gamma_t(x,y) \le e^{\sqrt{nc_1}t} \gamma_0(x,y) \tag{13}$$

for any $x, y \in M$, and this completes the proof of Lemma 4.1.

In particular, we have

$$\begin{gathered} e^{-2\sqrt{nc_1}T} g_{ij}(x,0) \le g_{ij}(x,t) \le e^{2\sqrt{nc_1}T} g_{ij}(x,0), \\ e^{-T\sqrt{nc_1}} \gamma_0(x,y) \le \gamma_t(x,y) \le e^{T\sqrt{nc_1}} \gamma_0(x,y) \end{gathered} \tag{14}$$

for $0 \le t \le T$ and $x, y \in M$.

Lemma 4.2. *Under Assumption* A, *for a fixed point $x_0 \in M$, we can find a function $\psi(x) \in C^\infty(M)$ such that*

$$\begin{gathered} c_3\{1 + \gamma_0(x_0,x)\} \le \psi(x) \le c_4\{1 + \gamma_0(x_0,x)\}, \\ |\nabla_i^0 \psi(x)|^2 \le c_5, \\ \nabla_i^0 \nabla_j^0 \psi(x) \le c_5 g_{ij}(x,0) \end{gathered} \tag{15}$$

for all $x \in M$, where c_3, c_4, and c_5 are some positive constants.

Proof. Let

$$\varphi(x) = 1 + \gamma_0(x_0,x), \qquad x \in M. \tag{16}$$

Then at the smooth point of $\varphi(x)$ we have

$$|\nabla_i^0 \varphi(x)| \le 1. \tag{17}$$

If we compare $\varphi(x)$ with the distance function on $\mathbf{R}^n$ with respect to standard Euclidean metric, then, by using the Hessian comparison theorem in Riemannian geometry, we know that

$$\nabla_\xi^0 \nabla_\xi^0 \varphi(x) \le \frac{1}{\gamma_0(x_0,x)} \quad \text{for any } \xi \in T_x M, |\xi|^2 = 1, \tag{18}$$

because by Assumption A

$$0 < R_{ijij}(x,0) \le k_0, \qquad x \in M.$$

From (18) it follows that at the smooth point of $\varphi(x)$

$$\nabla_i^0\nabla_j^0\varphi(x) \le \frac{1}{\gamma_0(x_0,x)} g_{ij}(x,0), \qquad x \in M. \tag{19}$$

The problem is $\varphi(x)$ may not be smooth at some points of M.

We choose a cut-off function $\chi(x) \in C_0^\infty(\mathbf{R})$ such that

$$\begin{gathered} 0 \le \chi(x) \le 1 \quad \forall x \in \mathbf{R}, \\ \chi(x) \equiv 0 \quad \text{if } x \notin [-1,2], \\ \chi(x) \equiv 1 \qquad 0 \le x \le 1, \\ |\chi'(x)| \le 2 \quad \forall x \in \mathbf{R}, \\ |\chi''(x)| \le 8 \quad \forall x \in \mathbf{R}, \end{gathered} \tag{20}$$

and set

$$\alpha(x,y) = \chi\left(\frac{\gamma_0(x,y)}{\sqrt{k_0}}\right) \quad \forall x, y \in M. \tag{21}$$

By Assumption A we have

$$0 < R_{ijij}(x,0) \le k_0, \qquad x \in M.$$

Thus we know, from Lemma 3.3, that

$$\operatorname{inj}(M) \ge \pi/\sqrt{k_0} \tag{22}$$

with respect to metric ds_0^2, and, from (20) and (21), that

$$\alpha(x,y) \in C^\infty(M \times M). \tag{23}$$

Now we can use the so-called mollifier technique to modify $\varphi(x)$. Define

$$\psi(x) = \int_M \alpha(x,y)\varphi(y)\,dy, \qquad x \in M. \tag{24}$$

Then

$$\psi(x) = \int_M \chi\left(\frac{\gamma_0(x,y)}{\sqrt{k_0}}\right)\psi(y)\,dy, \qquad x \in M. \tag{25}$$

From (23) we know that $\psi(x) \in C^\infty(M)$, and from (17), (19), and (20) we know that

$$\begin{aligned} &c_3\{1+\gamma_0(x_0,x)\} \le \psi(x) \le c_4\{1+\gamma_0(x_0,x)\}, \\ &|\nabla_i^0\psi(x)|^2 \le c_5, \qquad x \in M, \\ &\nabla_i^0\nabla_j^0\psi(x) \le c_5 g_{ij}(x,0). \end{aligned}$$

Hence the proof of Lemma 4.2 is complete.

Lemma 4.3. *For $\psi(x) \in C^\infty(M)$, which was found in Lemma 4.2, there exists a constant $c_6 = c_6(T) > 0$ such that*

$$\Delta_t \psi \le c_6 \quad \text{on } M \times [0, T]. \tag{26}$$

Proof. For any $0 \le t \le T$ and $x \in M$, we want to compute $\Delta_t \psi(x)$ at x. Choose a coordinate system such that

$$\frac{\partial g_{ij}(x,0)}{\partial x^k} = 0 \quad \text{at } x, \tag{27}$$

and let

$$\Gamma_{ij}^k = \frac{1}{2} g^{k\gamma} \left(\frac{\partial g_{j\gamma}}{\partial x^i} + \frac{\partial g_{i\gamma}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^\gamma} \right). \tag{28}$$

Then, by the definition of covariant derivative, we have

$$\begin{aligned} \nabla_i^t \psi(x) &= \frac{\partial \psi(x)}{\partial x^i}, \\ \nabla_i^t \nabla_j^t \psi(x) &= \frac{\partial^2 \psi(x)}{\partial x^i \partial x^j} - \Gamma_{ij}^k(x,t) \frac{\partial \psi(x)}{\partial x^k}, \\ \Delta_t \psi(x) &= g^{ij}(x,t) \frac{\partial^2 \psi}{\partial x^i \partial x^j} - g^{ij}(x,t) \Gamma_{ij}^k(x,t) \frac{\partial \psi}{\partial x^k}. \end{aligned} \tag{29}$$

Since by (27),

$$\Gamma_{ij}^k(x,0) = 0, \tag{30}$$

we have $\nabla_i^0 \nabla_j^0 \psi(x) = \partial^2 \psi(x)/\partial x^i \partial x^j$.

From Lemma 4.2 we know that

$$\begin{aligned} & g^{ij}(x,0) \frac{\partial \psi(x)}{\partial x^i} \cdot \frac{\partial \psi(x)}{\partial x^j} \le c_5, \\ & \frac{\partial^2 \psi(x)}{\partial x^i \partial x^j} \le c_5 g_{ij}(x,0), \end{aligned} \tag{31}$$

which together with (14) implies

$$g^{ij}(x,t) \frac{\partial^2 \psi(x)}{\partial x^i \partial x^j} \le c_5 g^{ij}(x,t) g_{ij}(x,0) \le c_5 n e^{2T\sqrt{nc_1}}. \tag{32}$$

For each t, $[\Gamma_{ij}^k(x,t) - \Gamma_{ij}^k(x,0)]$ is a tensor on M. Define

$$\begin{aligned} u(x,t) = {} & g^{i\alpha}(x,t) g^{j\beta}(x,t) g_{k\gamma}(x,t) g_{k\gamma}(x,t) [\Gamma_{ij}^k(x,t) - \Gamma_{ij}^k(x,0)] \\ & \cdot [\Gamma_{\alpha\beta}^\gamma(x,t) - \Gamma_{\alpha\beta}^\gamma(x,0)]. \end{aligned} \tag{33}$$

Then $u(x,t) \in C^\infty(M \times [0,T])$.

Since $\Gamma_{ij}^k(x,0)=0$ at x, we have

$$u(x,t)=g^{i\alpha}g^{j\beta}g_{k\gamma}\Gamma_{ij}^k(x,t)\Gamma_{\alpha\beta}^\gamma(x,t), \tag{34}$$

$$\begin{aligned}\frac{\partial u(x,t)}{\partial t}&=2\frac{\partial g^{i\alpha}}{\partial t}g^{j\beta}g_{k\gamma}\Gamma_{ij}^k\Gamma_{\alpha\beta}^\gamma+g^{i\alpha}g^{i\beta}\frac{\partial g_{k\gamma}}{\partial t}\Gamma_{ij}^k\Gamma_{\alpha\beta}^\gamma\\&\quad+2g^{i\alpha}g^{j\beta}g_{k\gamma}\Gamma_{\alpha\beta}^\gamma\frac{\partial}{\partial t}\Gamma_{ij}^k.\end{aligned} \tag{35}$$

Since

$$\frac{\partial}{\partial t}g_{ij}=-2R_{ij},$$

we have

$$\frac{\partial}{\partial t}g^{ij}=2g^{i\alpha}g^{j\beta}R_{\alpha\beta}, \tag{36}$$

and from (28) it follows that

$$\begin{aligned}\frac{\partial}{\partial t}\Gamma_{ij}^k&=\frac{1}{2}g^{k\gamma}\left[\nabla_i\left(\frac{\partial g_{j\gamma}}{\partial t}\right)+\nabla_j\left(\frac{\partial g_{i\gamma}}{\partial t}\right)-\nabla_\gamma\left(\frac{\partial g_{ij}}{\partial t}\right)\right]\\&=g^{k\gamma}(\nabla_\gamma^tR_{ij}-\nabla_i^tR_{j\gamma}-\nabla_j^tR_{i\gamma}),\end{aligned}$$

$$\frac{\partial}{\partial t}\Gamma_{ij}^k=g^{k\gamma}(\nabla_\gamma R_{ij}-\nabla_iR_{j\gamma}-\nabla_jR_{i\gamma}). \tag{37}$$

Substituting (36) and (37) into (35) gives

$$\begin{aligned}\frac{\partial u(x,t)}{\partial t}&=4g^{i\sigma}g^{\alpha\tau}R_{\sigma\tau}g^{j\beta}g_{k\gamma}\Gamma_{ij}^k\Gamma_{\alpha\beta}^\gamma-2g^{i\alpha}g^{j\beta}R_{k\gamma}\Gamma_{ij}^k\Gamma_{\alpha\beta}^\gamma\\&\quad+2g^{i\alpha}g^{j\beta}g_{k\gamma}\Gamma_{\alpha\beta}^\gamma g^{kl}(\nabla_lR_{ij}-\nabla_iR_{jl}-\nabla_jR_{il}).\end{aligned} \tag{38}$$

By (8) we get

$$\frac{\partial}{\partial t}u(x,t)\le 6\sqrt{nc_1}|\Gamma_{ij}^k|^2+6|\Gamma_{\alpha\beta}^\gamma|\,|\nabla_iR_{jk}|, \tag{39}$$

where $|\Gamma_{ij}^k|^2=u(x,t)$. Thus

$$\frac{\partial}{\partial t}|\Gamma_{ij}^k|^2\le 6\sqrt{nc_1}|\Gamma_{ij}^k|^2+6|\Gamma_{\alpha\beta}^\gamma|\cdot|\nabla_iR_{jk}|,$$

$$\frac{\partial}{\partial t}|\Gamma_{ij}^k|\le 3\sqrt{nc_1}|\Gamma_{ij}^k|+3|\nabla_iR_{jk}|,$$

$$\frac{\partial}{\partial t}(e^{-3\sqrt{nc_1}t}|\Gamma_{ij}^k|)\le 3e^{-3\sqrt{nc_1}t}|\nabla_iR_{jk}|\le 3|\nabla_iR_{jk}|,$$

and therefore

$$
\begin{aligned}
e^{-3\sqrt{nc_1}t}|\Gamma_{ij}^k| - |\Gamma_{ij}^k(x,0)| &= \int_0^t \frac{\partial}{\partial t}[e^{-3\sqrt{nc_1}t}|\Gamma_{ij}^k|]\,dt \\
&\le 3\int_0^t |\nabla_i R_{jk}|\,dt \le 3\int_0^T |\nabla_i R_{jk}|\,dt \\
&\le 3n\int_0^T |\nabla_p R_{ijkl}|\,dt,
\end{aligned}
$$

which together with (5) gives

$$e^{-3\sqrt{nc_1}t}|\Gamma_{ij}^k(x,t)| - |\Gamma_{ij}^k(x,0)| \le 3nc_2.$$

Then by definition (33) we have

$$|\Gamma_{ij}^k(x,0)| \equiv 0 \quad \text{on } M,$$

and therefore

$$|\Gamma_{ij}^k(x,t)| \le 3nc_2e^{3\sqrt{nc_1}t} \le 3nc_2e^{3T\sqrt{nc_1}},$$

$$|\Gamma_{ij}^k(x,t)| \le 3nc_2e^{3T\sqrt{nc_1}}, \qquad x\in M,\ 0\le t\le T. \tag{40}$$

By (14) and (31) we get

$$g^{ij}(x,t)\frac{\partial\psi(x)}{\partial x^i}\frac{\partial\psi(x)}{\partial x^j} \le c_5e^{2T\sqrt{nc_1}}. \tag{41}$$

and therefore

$$|\nabla_i^t\psi(x)|^2 \le c_5e^{2T\sqrt{nc_1}}, \qquad 0\le t\le T. \tag{42}$$

From (40) and (42) it follows that

$$
\begin{aligned}
-g^{ij}(x,t)\Gamma_{ij}^k(x,t)\frac{\partial\psi}{\partial x^k} &\le n|\Gamma_{ij}^k(x,t)|\cdot|\nabla_k^t\psi(x)| \\
&\le 3n^2c_2c_5e^{5T\sqrt{nc_1}},
\end{aligned} \tag{43}
$$

which together with (32) gives

$$g^{ij}(x,t)\frac{\partial^2\psi}{\partial x^i\partial x^j} - g^{ij}(x,t)\Gamma_{ij}^k(x,t)\frac{\partial\psi}{\partial x^k} \le c_5ne^{2T\sqrt{nc_1}} + 3n^2c_2c_5e^{5T\sqrt{nc_1}}.$$

Let $c_6 = c_5ne^{2T\sqrt{nc_1}} + 3n^2c_2c_5e^{5T\sqrt{nc_1}}$. Then from (29) we have

$$\Delta_t\psi(x) \le c_6, \qquad x\in M,\ 0\le t\le T. \tag{44}$$

Lemma 4.4. *Under Assumption* A, *for any $c_7 > 0$ we can find a constant $c_8 > 0$ and a function*

$$\theta(x,t)\in C^\infty(M\times[0,T])$$

such that the following are true:

$$0 < \theta(x,t) \leq 1 \quad \forall (x,t) \in M \times [0,T], \tag{45}$$

$$\frac{c_8^{-1}}{1+\gamma_0(x_0,x)} \leq \theta(x,t) \leq \frac{c_8}{1+\gamma_0(x_0,x)}, \qquad x \in M,\ 0 \leq t \leq T, \tag{46}$$

$$\frac{\partial \theta}{\partial t} \leq \Delta\theta - \frac{2|\nabla_p \theta|^2}{\theta} - c_7\theta \quad \text{on } M \times [0,T]. \tag{47}$$

Proof. From Lemma 4.3 we know that

$$\Delta_t \psi(x) \leq c_6, \qquad x \in M,\ 0 \leq t \leq T.$$

Let

$$\xi(x,t) = e^{c_7 t}\{\psi(x) + c_6 t\}, \qquad x \in M,\ 0 \leq t \leq T. \tag{48}$$

Then

$$\frac{\partial \xi}{\partial t} = c_7 e^{c_7 t}\{\psi(x) + c_6 t\} + c_6 e^{c_7 t},$$

$$\frac{\partial \xi}{\partial t} = c_7 \xi + c_6 e^{c_7 t}, \qquad 0 \leq t \leq T. \tag{49}$$

Since

$$\Delta\xi = e^{c_7 t}\Delta\{\psi(x) + c_6 t\} = e^{c_7 t}\Delta\psi \leq c_6 e^{c_7 t},$$

from (49) it follows that

$$\frac{\partial \xi}{\partial t} \geq \Delta\xi + c_7\xi \quad \text{on } M \times [0,T]. \tag{50}$$

By (48) we get

$$\psi(x) \leq \xi(x,t) \leq e^{c_7 T}\psi(x) + e^{c_7 T} \cdot c_6 T,$$

and therefore, in consequence of (15),

(51)

$$\begin{aligned} c_3[1+\gamma_0(x_0,x)] &\leq \psi(x) \leq \xi(x,t) \\ &\leq e^{c_7 T} \cdot c_4[1+\gamma_0(x_0,x)] + c_6 T e^{c_7 T} \quad \text{on } M \times [0,T], \\ c_3[1+\gamma_0(x_0,x)] &\leq \xi(x,t) \leq (c_4 e^{c_7 T} + c_6 T e^{c_7 T})[1+\gamma_0(x_0,x)] \\ &\qquad \text{on } M \times [0,T]. \end{aligned}$$

Let

$$\tilde{\theta}(x,t) = \frac{1}{\xi(x,t)} \quad \text{on } M \times [0,T]. \tag{52}$$

Then

$$\frac{\partial \tilde{\theta}}{\partial t} = -\frac{1}{\xi^2}\frac{\partial \xi}{\partial t} \le -\frac{1}{\xi^2}[\Delta\xi + c_7\xi]$$
$$= -\frac{\Delta\xi}{\xi^2} - \frac{c_7}{\xi} = \Delta\tilde{\theta} - \frac{2}{\tilde{\theta}}|\nabla_p\tilde{\theta}|^2 - c_7\tilde{\theta},$$

$$\frac{\partial \tilde{\theta}}{\partial t} \le \Delta\tilde{\theta} - \frac{2}{\tilde{\theta}}|\nabla_p\tilde{\theta}|^2 - c_7\tilde{\theta} \quad \text{on } M \times [0, T]. \tag{53}$$

From (51) we have

$$\frac{1}{c_3[1 + \gamma_0(x_0, x)]} \ge \tilde{\theta}(x, t) \ge \frac{1}{(c_4 e^{c_7 T} + c_6 T e^{c_7 T})[1 + \gamma_0(x_0, x)]}. \tag{54}$$

In particular,

$$\tilde{\theta}(x, t) \le \frac{1}{c_3} \quad \text{on } M \times [0, T].$$

Let

$$\theta(x, t) = c_3\tilde{\theta}(x, t) \quad \text{on } M \times [0, T]. \tag{55}$$

Then $0 < \theta(x, t) \le 1$ on $M \times [0, T]$.

From (53) we get

$$\frac{\partial \theta}{\partial t} \le \Delta\theta - \frac{2}{\theta}|\nabla_p\theta|^2 - c_7\theta \quad \text{on } M \times [0, T], \tag{56}$$

$$\frac{c_3}{(c_4 + c_6 T)e^{c_7 T}[1 + \gamma_0(x_0, x)]} \le \theta(x, t) \le \frac{1}{[1 + \gamma_0(x_0, x)]}. \tag{57}$$

Choose $c_8 > 0$ such that

$$c_8 \ge 1 + \frac{c_4 + c_6 T}{c_3} e^{c_7 T}.$$

Then (45), (46), and (47) are true.

Now we are going to prove the following maximal principle on noncompact manifold M.

Lemma 4.5. *Under Assumption* A, *suppose $\varphi(x, t)$ is a C^∞ function on $M \times [0, T]$ such that*

$$\begin{aligned} &\frac{\partial \varphi}{\partial t} = \Delta\varphi + Q(\varphi, x, t) \quad \text{on } M \times [0, T], \\ &|\varphi(x, t)| \le C_9 < +\infty \quad \text{on } M \times [0, T], \\ &\varphi(x, 0) \le 0 \quad \text{on } M, \\ &Q(\varphi, x, t) \le 0 \quad \text{for } \varphi \ge 0. \end{aligned} \tag{58}$$

Then we have

$$\varphi(x, t) \le 0 \quad \text{on } M \times [0, T]. \tag{59}$$

Proof. If this lemma is not true, then we can find some $(x_0, t_0) \in M \times [0, T]$ such that

$$\varphi(x_0, t_0) > 0. \tag{60}$$

Suppose $\theta(x, t) \in C^\infty(M \times [0, T])$ is the function obtained in Lemma 4.4, and define

$$\tilde{\varphi}(x, t) = \theta(x, t)\varphi(x, t) \quad \text{on } M \times [0, T]. \tag{61}$$

Since $0 < \theta(x, t) \le 1$ and $|\varphi(x, t)| \le c_9$, we have

$$|\tilde{\varphi}(x, t)| \le c_9 \quad \text{on } M \times [0, T], \tag{62}$$

$$\tilde{\varphi}(x_0, t_0) = \theta(x_0, t_0)\varphi(x_0, t_0) > 0. \tag{63}$$

Let

$$\alpha = \sup_{M \times [0, T]} \tilde{\varphi}(x, t). \tag{64}$$

Then from (62) and (63) it follows that

$$0 < \alpha \le c_9, \tag{65}$$

so that

$$|\tilde{\varphi}(x, t)| = \theta(x, t)|\varphi(x, t)| \le c_9\theta(x, t) \le \frac{c_8 c_9}{1 + \gamma_0(x_0, x)},$$

$$|\tilde{\varphi}(x, t)| \le \frac{c_8 c_9}{1 + \gamma_0(x_0, x)} \quad \text{on } M \times [0, T]. \tag{66}$$

Let

$$D = \{x \in M | \gamma_0(x_0, x) \le c_8 c_9 / \alpha\}. \tag{67}$$

Then $D \subseteq M$ is a compact subset.

If $(x, t) \notin D \times [0, T]$, then $\gamma_0(x_0, x) > \alpha^{-1} c_8 c_9$. From (66) we know that

$$|\tilde{\varphi}(x, t)| < \alpha \quad \text{for } (x, t) \notin D \times [0, T].$$

Since $D \times [0, T]$ is a compact set, we can find a point $(x_1, t_1) \in D \times [0, T]$ such that $\tilde{\varphi}(x_1, t_1) = \alpha$, so that

$$\tilde{\varphi}(x_1, t_1) = \sup_{M \times [0, T]} \tilde{\varphi}(x, t) > 0. \tag{68}$$

Thus we have

$$\frac{\partial \tilde{\varphi}}{\partial t}(x_1, t_1) \ge 0, \tag{69}$$

$$\Delta \tilde{\varphi}(x_1, t_1) \le 0, \tag{70}$$

$$\nabla \tilde{\varphi}(x_1, t_1) = 0, \tag{71}$$

where (69) comes from the fact that $\tilde{\varphi}(x,0) = \theta(x,0)\varphi(x,0) \leq 0$. Therefore we always have $t_1 > 0$.

From (58) it follows that

$$\begin{aligned}
\frac{\partial \tilde{\varphi}}{\partial t} &= \frac{\partial}{\partial t}(\theta\varphi) = \theta\frac{\partial \varphi}{\partial t} + \varphi\frac{\partial \theta}{\partial t} \\
&= \theta[\Delta\varphi + Q(\varphi, x, t)] + \varphi\frac{\partial \theta}{\partial t} \\
&= \theta\Delta\varphi + \varphi\frac{\partial \theta}{\partial t} + \theta Q(\varphi, x, t) \\
&= \Delta(\theta\varphi) - 2\nabla_k\theta \cdot \nabla_k\varphi - \varphi\Delta\theta + \varphi\frac{\partial \theta}{\partial t} + \theta Q(\varphi, x, t) \\
&= \Delta\tilde{\varphi} - \frac{2}{\theta}\nabla_k\theta \cdot \nabla_k(\theta\varphi) + \frac{2\varphi}{\theta}|\nabla_k\theta|^2 \\
&\quad - \varphi\Delta\theta + \varphi\frac{\partial \theta}{\partial t} + \theta Q(\varphi, x, t),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \tilde{\varphi}}{\partial t} &= \Delta\tilde{\varphi} - \frac{2}{\theta}\nabla_k\theta \cdot \nabla_k\tilde{\varphi} + \theta Q(\varphi, x, t) \\
&\quad + \left(\frac{\partial \theta}{\partial t} - \Delta\theta + \frac{2}{\theta}|\nabla_k\theta|^2\right)\varphi.
\end{aligned} \tag{72}$$

Let $(x,t) = (x_1, t_1)$. Then from (68) we get

$$\tilde{\varphi}(x_1, t_1) = \theta(x_1, t_1)\varphi(x_1, t_1) > 0.$$

Since

$$\theta(x_1, t_1) > 0, \tag{73}$$

we have

$$\varphi(x_1, t_1) > 0, \tag{74}$$

and, in consequence of (58),

$$Q(\varphi, x_1, t_1) \leq 0. \tag{75}$$

Let $c_7 = 1$ in Lemma 4.4. Then

$$\begin{aligned}
&\frac{\partial \theta}{\partial t} \leq \Delta\theta - \frac{2}{\theta}|\nabla_k\theta|^2 - \theta, \\
&\frac{\partial \theta}{\partial t} - \Delta\theta + \frac{2}{\theta}|\nabla_k\theta|^2 \leq -\theta.
\end{aligned} \tag{76}$$

From (73) and (75) it follows that

$$\theta(x_1, t_1) \cdot Q(\varphi, x_1, t_1) \leq 0, \tag{77}$$

and (74) together with (76) implies

$$\left(\frac{\partial\theta}{\partial t} - \Delta\theta + \frac{2}{\theta}|\nabla_k\theta|^2\right)\varphi \le -\theta\varphi = -\tilde{\varphi}(x_1, t_1). \tag{78}$$

Substituting (70), (71), (77), and (78) into (72), we get

$$\begin{gathered}\frac{\partial\tilde{\varphi}}{\partial t}(x_1, t_1) \le -\tilde{\varphi}(x_1, t_1) < 0,\\ \frac{\partial\tilde{\varphi}}{\partial t}(x_1, t_1) < 0.\end{gathered} \tag{79}$$

Since (79) contradicts (69), we have

$$\varphi(x,t) \le 0 \quad \text{on } M \times [0,T].$$

Theorem 4.6. *Under Assumption* A, *suppose $\varphi(x,t)$ is a C^∞ function on $M \times [0,T]$ such that*

$$\begin{gathered}\frac{\partial\varphi}{\partial t} = \Delta\varphi + c_{10}|\nabla_k\varphi|^2 + Q(\varphi, x, t) \quad \text{on } M \times [0,T],\\ \varphi(x,t) \le c_9 < +\infty \quad \text{on } M \times [0,T],\\ \varphi(x,0) \le 0 \quad \text{on } M,\\ Q(\varphi, x, t) \le c_{11}\varphi \quad \text{for } \varphi \ge 0,\end{gathered} \tag{80}$$

where $0 < c_9, c_{10}, c_{11} < +\infty$ are some constants. Then we have

$$\varphi(x,t) \le 0 \quad \text{on } M \times [0,T]. \tag{81}$$

Proof. Let

$$w(x,t) = e^{-\beta t}[e^{c_{10}\varphi(x,t)} - 1] \quad \text{on } M \times [0,T], \tag{82}$$

where $\beta > 0$ is a constant to be determined later. Then

$$\begin{aligned}\frac{\partial w}{\partial t} &= -\beta w + e^{-\beta t}\frac{\partial}{\partial t}e^{c_{10}\varphi(x,t)},\\ \frac{\partial w}{\partial t} &= -\beta w + c_{10}e^{-\beta t}e^{c_{10}\varphi(x,t)}\frac{\partial\varphi}{\partial t}\\ &= -\beta w + c_{10}e^{-\beta t}e^{c_{10}\varphi}[\Delta\varphi + c_{10}|\nabla_k\varphi|^2 + Q(\varphi, x, t)]\\ &= -\beta w + e^{-\beta t}\Delta e^{c_{10}\varphi} + c_{10}e^{-\beta t}e^{c_{10}\varphi}Q(\varphi, x, t),\end{aligned}$$

$$\frac{\partial w}{\partial t} = \Delta w - \beta w + c_{10}e^{-\beta t}e^{c_{10}\varphi}Q(\varphi, x, t). \tag{83}$$

If $w(x,t) \ge 0$, from (82) it follows that $\varphi(x,t) \ge 0$. Therefore

$$Q(\varphi, x, t) \le c_{11}\varphi(x,t). \tag{84}$$

Since $\varphi(x,t) \le c_9 < +\infty$, we can find a constant $\delta > 0$ such that

$$\varphi(x,t) \le \delta[e^{c_{10}\varphi(x,t)} - 1] \quad \text{for } 0 \le \varphi(x,t) \le c_9, \tag{85}$$

which together with (84) gives

$$Q(\varphi,x,t) \le c_{11}\delta(e^{c_{10}\varphi(x,t)} - 1),$$

$$\begin{aligned} c_{10}e^{-\beta t}e^{c_{10}\varphi}Q(\varphi,x,t) &\le c_{10}c_{11}\delta e^{c_{10}\varphi}e^{-\beta t}(e^{c_{10}\varphi} - 1) \\ &= c_{10}c_{11}\delta e^{c_{10}\varphi}w \le c_{10}c_{11}\delta e^{c_{10}c_9}w. \end{aligned} \tag{86}$$

Let

$$\tilde{Q}(w,x,t) = -\beta w + c_{10}e^{-\beta t}e^{c_{10}\varphi}Q(\varphi,x,t). \tag{87}$$

Then

$$\frac{\partial w}{\partial t} = \Delta w + \tilde{Q}(w,x,t) \quad \text{on } M \times [0,T]. \tag{88}$$

If $w(x,t) \ge 0$, from (86) we get

$$\tilde{Q}(w,x,t) \le [c_{10}c_{11}\delta e^{c_{10}c_9} - \beta]w.$$

Choose

$$\beta = c_{10}c_{11}\delta e^{c_{10}c_9}; \tag{89}$$

then

$$\tilde{Q}(w,x,t) \le 0 \quad \text{for } w \ge 0. \tag{90}$$

Since $\varphi(x,0) \le 0$, from (82) it follows that

$$\begin{gathered} w(x,0) \le 0 \quad \text{on } M, \\ -1 \le w(x,t) \le e^{c_{10}c_9} - 1 \quad \text{on } M \times [0,T], \end{gathered} \tag{91}$$

so that

$$|w(x,t)| \le e^{c_{10}c_9} \quad \text{on } M \times [0,T]. \tag{92}$$

Using (88), (90), (91), (92) and Lemma 4.5 we get

$$w(x,t) \le 0 \quad \text{on } M \times [0,T],$$

so that, in consequence of (82),

$$\varphi(x,t) \le 0 \quad \text{on } M \times [0,T].$$

Now we are going to prove another maximal principle that is different from Theorem 4.6.

Lemma 4.7. *Under Assumption* A, *for any fixed point $x_0 \in M$ and $\varepsilon > 0$, we can find constants $c(\varepsilon) > 0$ and $c_{12} > 0$ such that for all $k \geq c_{12}$, there exists a function $\theta(x) \in C^\infty(M)$ satisfying the following*:

$$
\begin{aligned}
&0 \leq \theta(x) \leq 1 \quad \text{on } M,\\
&\theta(x) \equiv 1 \quad \forall x \in B_0(x_0, k),\\
&\theta(x) \equiv 0 \quad \forall x \in M \backslash B_0(x_0, 2k),\\
&\left|\nabla_i^0 \left(\frac{1}{\theta(x)}\right)\right| \leq \frac{c(\varepsilon)}{k}\left(\frac{1}{\theta(x)}\right)^{1+\varepsilon} \quad \forall x \in \Omega,\\
&\nabla_i^0\nabla_j^0 \left(\frac{1}{\theta(x)}\right) \leq \frac{c(\varepsilon)}{k}\left(\frac{1}{\theta(x)}\right)^{1+\varepsilon} g_{ij}(x,0) \quad \forall x \in \Omega,
\end{aligned} \tag{93}
$$

where

$$
\begin{aligned}
&B_0(x_0, k) = \{x \in M \mid \gamma_0(x_0, x) < k\},\\
&\Omega = \{x \in M \mid \theta(x) > 0\}.
\end{aligned} \tag{94}
$$

Proof. Suppose $\rho(t) \in C^\infty(\mathbf{R})$ is a function such that

$$
\begin{aligned}
&0 \leq \rho(t) \leq 1, && -\infty < t < +\infty,\\
&0 \leq \rho'(t) \leq 90, && -\infty < t < +\infty,\\
&\rho(t) \equiv 0, && -\infty < t \leq \tfrac{34}{24},\\
&0 < \rho(t) < 1, && \tfrac{34}{24} < t < \tfrac{35}{24},\\
&\rho(t) \equiv 1, && \tfrac{35}{24} \leq t < +\infty.
\end{aligned} \tag{95}
$$

It is easy to show that such a $\rho(t)$ exists. Then we define a function $\tilde{\chi}(t) \in C^\infty[0, \frac{7}{4})$ as follows:

$$
\begin{aligned}
&\tilde{\chi}(t) \equiv 1, \qquad 0 \leq t \leq \frac{5}{4},\\
&\tilde{\chi}(t) = 1 + \exp\left(-\frac{1}{t - \frac{5}{4}}\right), \qquad \frac{5}{4} < t \leq \frac{11}{8},\\
&\tilde{\chi}(t) = [1 - \rho(t)]\left[1 + \exp\left(-\frac{1}{t - \frac{5}{4}}\right)\right]\\
&\qquad + \rho(t)\exp\left[\frac{1}{(\frac{7}{4} - t)^2}\right], \qquad \frac{11}{8} < t < \frac{3}{2},\\
&\tilde{\chi}(t) = \exp\left[\frac{1}{(\frac{7}{4} - t)^2}\right], \qquad \frac{3}{2} \leq t < \frac{7}{4}.
\end{aligned} \tag{96}
$$

It is easy to see that

$$
\tilde{\chi}(t) \in C^\infty[0, \tfrac{7}{4}), \tag{97}
$$

from (95) and (96), and that

$$\begin{aligned}
&\tilde{\chi}(t) \geq 1, \qquad 0 \leq t < \tfrac{7}{4},\\
&0 \leq \tilde{\chi}'(t) \leq \tilde{c}(\varepsilon)\tilde{\chi}(t)^{1+\varepsilon}, \qquad 0 \leq t < \tfrac{7}{4},\\
&|\tilde{\chi}''(t)| \leq \tilde{c}(\varepsilon)\tilde{\chi}(t)^{1+\varepsilon}, \qquad 0 \leq t < \tfrac{7}{4},
\end{aligned} \tag{98}$$

where $\tilde{c}(\varepsilon) > 0$ depending only on $\varepsilon > 0$.

Let

$$\begin{aligned}
&\eta(t) = \frac{1}{\tilde{\chi}(t)}, \qquad 0 \leq t < \frac{7}{4},\\
&\eta(t) \equiv 0, \qquad \frac{7}{4} \leq t < +\infty.
\end{aligned} \tag{99}$$

Then $\eta(t) \in C^\infty[0, +\infty)$.

For $k > 0$, let $\chi(t) \in C^\infty[0, \frac{7}{4}k)$ as follows:

$$\chi(t) = \tilde{\chi}(t/k), \qquad 0 \leq t < \tfrac{7}{4}k. \tag{100}$$

Then

$$\begin{aligned}
&\chi(t) \geq 1, \qquad 0 \leq t < \frac{7}{4}k,\\
&0 \leq \chi'(t) \leq \frac{\tilde{c}(\varepsilon)}{k}\chi(t)^{1+\varepsilon}, \qquad 0 \leq t < \frac{7}{4}k,\\
&|\chi''(t)| \leq \frac{\tilde{c}(\varepsilon)}{k^2}\chi(t)^{1+\varepsilon}, \qquad 0 \leq t < \frac{7}{4}k.
\end{aligned} \tag{101}$$

Choose a cut-off function $\zeta(x) \in C_0^\infty(\mathbf{R})$ such that

$$\begin{aligned}
&0 \leq \zeta(x) \leq 1 \quad \forall x \in \mathbf{R},\\
&\zeta(x) \equiv 0 \quad \text{if } x \notin [-1, 2],\\
&\zeta(x) \equiv 1, \qquad 0 \leq x \leq 1,\\
&|\zeta'(x)| \leq 2 \quad \forall x \in \mathbf{R},\\
&|\zeta''(x)| \leq 8 \quad \forall x \in \mathbf{R},
\end{aligned} \tag{102}$$

and define

$$\psi(x) = \frac{\int_M \zeta(\gamma_0(x, y)/64\sqrt{k_0})[1 + \gamma_0(x_0, y)]\,dy}{\int_M \zeta(\gamma_0(x_0, y)/64\sqrt{k_0})\,dy} \quad \forall x \in M. \tag{103}$$

Then similar to the proof of Lemma 4.2 we know that $\psi(x) \in C^\infty(M)$, $\psi(x) > 0$, and we can find a constant $c_5 > 0$ such that

$$\begin{aligned}
&\tfrac{8}{9}\gamma_0(x_0, x) - c_5 \leq \psi(x) \leq \tfrac{10}{9}\gamma_0(x_0, x) + c_5,\\
&|\nabla_i^0 \psi(x)|^2 \leq c_5 \quad \forall x \in M,\\
&\nabla_i^0 \nabla_j^0 \psi(x) \leq c_5 g_{ij}(x, 0).
\end{aligned} \tag{104}$$

Now we define $\theta(x)$ as

$$\theta(x) = \begin{cases} \dfrac{1}{\chi(\psi(x))} & \text{if } 0 \le \psi(x) < \dfrac{7}{4}k, \\ 0 & \text{if } \psi(x) \ge \dfrac{7}{4}k. \end{cases} \tag{105}$$

Then from $\eta(t) \in C^\infty[0, +\infty)$ and $\psi(x) \in C^\infty(M)$ it is easy to show that

$$\begin{gathered} \theta(x) \in C^\infty(M), \\ 0 \le \theta(x) \le 1 \quad \forall x \in M. \end{gathered} \tag{106}$$

Let $c_{12} = 40c_5$. Then if $k \ge c_{12}$, for any $x \in B_0(x_0, k)$ we get

$$\begin{gathered} \gamma_0(x_0, x) \le k, \\ \psi(x) \le \tfrac{10}{9}\gamma_0(x_0, x) + c_5 \le \tfrac{10}{9}k + \tfrac{1}{40}k < \tfrac{5}{4}k. \end{gathered}$$

From (105) and (96) we have, respectively,

$$\begin{gathered} \theta(x) = \frac{1}{\chi(\psi(x))}, \quad \forall x \in B_0(x_0, k), \\ \theta(x) \equiv 1 \quad \forall x \in B_0(x_0, k). \end{gathered} \tag{107}$$

If $x \in M \backslash B_0(x_0, 2k)$, then $\gamma_0(x_0, x) \ge 2k$ and

$$\psi(x) \ge \tfrac{8}{9}\gamma_0(x_0, x) - c_5 \ge \tfrac{16}{9}k - \tfrac{1}{40}k > \tfrac{7}{4}k.$$

Thus

$$\theta(x) \equiv 0 \quad \forall x \in M \backslash B_0(x_0, 2k). \tag{108}$$

For $\Omega = \{x \in M | \theta(x) > 0\}$, we have

$$\begin{gathered} \frac{1}{\theta(x)} = \chi(\psi(x)) \quad \forall x \in \Omega, \\ \nabla_i^0\left(\frac{1}{\theta(x)}\right) = \chi'(\psi(x))\nabla_i^0\psi(x). \end{gathered}$$

From (101) and (104) we get

$$\begin{gathered} \left|\nabla_i^0\left(\frac{1}{\theta(x)}\right)\right| \le c_5^{1/2} \cdot \frac{\tilde{c}(\varepsilon)}{k}\chi(\psi(x))^{1+\varepsilon}, \\ \left|\nabla_i^0\left(\frac{1}{\theta}\right)\right| \le \frac{\tilde{c}(\varepsilon)\sqrt{c_5}}{k}\left(\frac{1}{\theta}\right)^{1+\varepsilon} \quad \text{on } \Omega, \end{gathered} \tag{109}$$

$$\begin{aligned}\nabla_i^0\nabla_j^0\left(\frac{1}{\theta}\right) &= \chi''(\psi)\nabla_i^0\psi\cdot\nabla_j^0\psi + \chi'(\psi)\nabla_i^0\nabla_j^0\psi \\ &\le c_5\cdot\frac{\tilde{c}(\varepsilon)}{k^2}\chi(\psi)^{1+\varepsilon}g_{ij}(x,0) + c_5\cdot\frac{\tilde{c}(\varepsilon)}{k}\chi(\psi)^{1+\varepsilon}g_{ij}(x,0) \\ &\le \left[\frac{\tilde{c}(\varepsilon)c_5}{k} + \tilde{c}(\varepsilon)c_5\right]\frac{1}{k}\chi(\psi)^{1+\varepsilon}g_{ij}(x,0) \\ &\le \left[\frac{\tilde{c}(\varepsilon)}{40} + \tilde{c}(\varepsilon)c_5\right]\frac{1}{k}\left(\frac{1}{\theta}\right)^{1+\varepsilon} g_{ij}(x,0),\end{aligned}$$

$$(110)\qquad \nabla_i^0\nabla_j^0\left(\frac{1}{\theta(x)}\right) \le \left[\frac{\tilde{c}(\varepsilon)}{40} + \tilde{c}(\varepsilon)c_5\right]\frac{1}{k}\left(\frac{1}{\theta(x)}\right)^{1+\varepsilon} g_{ij}(x,0) \quad \text{on } \Omega.$$

Let

$$c(\varepsilon) = \max\{\tilde{c}(\varepsilon)\sqrt{c_5}, \tilde{c}(\varepsilon)/40 + \tilde{c}(\varepsilon)c_5\}.$$

Then the lemma follows from (106), (107), (108), (109), and (110).

Lemma 4.8. *For the function $\theta(x)$ obtained in Lemma 4.7, we can find another constant $c_{13} > 0$ depending only on ε and the constants in Assumption A, such that*

$$(111)\qquad \Delta_t\left(\frac{1}{\theta(x)}\right) \le \frac{c_{13}}{k}\left(\frac{1}{\theta(x)}\right)^{1+\varepsilon} \quad \forall x\in\Omega,\ 0\le t\le T.$$

Proof. Similar to the proof of Lemma 4.3.

Now we are going to prove the following maximal principle on noncompact manifold M.

Lemma 4.9. *Under Assumption A, suppose there exist constants $0 < \varepsilon, c_{14}, c_{15} < +\infty$, and $\varphi(x,t) \in C^\infty(M\times[0,T])$ such that*

$$(112)\qquad \begin{aligned} &\frac{\partial\varphi}{\partial t} = \Delta\varphi + Q(\varphi,x,t) \quad \text{on } M\times[0,T], \\ &\varphi(x,0) \le c_{14} \quad \text{on } M, \\ &Q(\varphi,x,t) \le -c_{15}\varphi^{1+\varepsilon} \quad \text{if } \varphi \ge c_{14}.\end{aligned}$$

Then we have

$$(113)\qquad \varphi(x,t) \le c_{14} \quad \text{on } M\times[0,T].$$

Proof. Fix a point $x_0 \in M$ and suppose this lemma is not true. Then we can find some $(x_2,t_2) \in M\times[0,T]$ such that

$$(114)\qquad \varphi(x_2,t_2) > c_{14}.$$

Choose $k \ge c_{12}$ large enough such that $x_2 \in B_0(x_0,k)$, and let $\theta(x) \in C^\infty(M)$ be the function constructed in Lemma 4.7. Then we define

$$(115)\qquad \tilde{\varphi}(x,t) = \theta(x)\varphi(x,t) \quad \text{on } M\times[0,T].$$

Since $x_2 \in B_0(x_0, k)$, we have $\theta(x_2) = 1$, and therefore

$$\tilde{\varphi}(x_2, t_2) = \varphi(x_2, t_2) > c_{14}. \tag{116}$$

If $(x,t) \in [M \backslash B_0(x_0, 2k)] \times [0, T]$, then $\theta(x) \equiv 0$. Thus from (115) we know that

$$\tilde{\varphi}(x,t) \equiv 0 \quad \text{on } \{M \backslash B_0(x_0, 2k)\} \times [0, T]. \tag{117}$$

Since $\overline{B_0(x_0, 2k)} \times [0, T]$ is a compact set, where $\overline{B_0(x_0, 2k)}$ is the closure of $B_0(x_0, 2k)$, from (116) and (117) it follows that there exists $(x_1, t_1) \in \overline{B_0(x_0, 2k)} \times [0, T]$ such that

$$\tilde{\varphi}(x_1, t_1) = \sup_{M \times [0,T]} \tilde{\varphi}(x,t) > c_{14}. \tag{118}$$

Thus we have

$$\Delta \tilde{\varphi}(x_1, t_1) \leq 0, \qquad \nabla \tilde{\varphi}(x_1, t_1) = 0. \tag{119}$$

Since $0 \leq \theta(x) \leq 1$,

$$\tilde{\varphi}(x, 0) = \theta(x)\varphi(x, 0) \leq c_{14}\theta(x) \leq c_{14}.$$

From (118) it follows that $t_1 > 0$, so that

$$\frac{\partial \tilde{\varphi}}{\partial t}(x_1, t_1) \geq 0. \tag{120}$$

On the other hand, by (115) we get

$$\begin{aligned}
\frac{\partial \tilde{\varphi}}{\partial t} &= \frac{\partial}{\partial t}(\theta \varphi) = \theta \frac{\partial \varphi}{\partial t} = \theta[\Delta \varphi + Q(\varphi, x, t)] = \theta \Delta \varphi + \theta Q(\varphi, x, t) \\
&= \Delta(\theta \varphi) - 2\nabla_p \theta \cdot \nabla_p \varphi - \varphi \Delta \theta + \theta Q(\varphi, x, t), \\
\frac{\partial \tilde{\varphi}}{\partial t} &= \Delta \tilde{\varphi} - \frac{2}{\theta} \nabla_p \theta \cdot \nabla_p(\theta \varphi) + \frac{2\varphi}{\theta} |\nabla_p \theta|^2 - \varphi \Delta \theta + \theta Q(\varphi, x, t),
\end{aligned}$$

$$\frac{\partial \tilde{\varphi}}{\partial t} = \Delta \tilde{\varphi} - \frac{2}{\theta} \nabla_p \theta \cdot \nabla_p \tilde{\varphi} + \left[\frac{2}{\theta} |\nabla_p \theta|^2 - \Delta \theta\right] \varphi + \theta Q. \tag{121}$$

Let $(x,t) = (x_1, t_1)$. Then from (118) we have

$$\theta(x_1)\varphi(x_1, t_1) > c_{14}. \tag{122}$$

Since $0 \leq \theta(x_1) \leq 1$, $\varphi(x_1, t_1) > c_{14}$. By (112) we get

$$Q(\varphi, x_1, t_1) \leq -c_{15}\varphi(x_1, t_1)^{1+\varepsilon}, \tag{123}$$

$$\begin{aligned}
&\left(\frac{2}{\theta} |\nabla_p \theta|^2 - \Delta \theta\right) \varphi + \theta Q(\varphi, x_1, t_1) \\
&\quad \leq \left(\frac{2}{\theta} |\nabla_p \theta|^2 - \Delta \theta\right) \varphi(x_1, t_1) - c_{15}\theta(x_1)\varphi(x_1, t_1)^{1+\varepsilon} \\
&\quad = \left[\frac{2}{\theta^3} |\nabla_p \theta|^2 - \frac{\Delta \theta}{\theta^2} - \frac{c_{15}}{\theta} \varphi(x_1, t_1)^{\varepsilon}\right] \varphi(x_1, t_1)\theta^2.
\end{aligned}$$

From (122) it follows that

$$\varphi(x_1,t_1)^\varepsilon > c_{14}^\varepsilon \left(\frac{1}{\theta(x_1)}\right)^\varepsilon,$$

$$\begin{aligned}\left(\frac{2}{\theta}|\nabla_p\theta|^2 - \Delta\theta\right)\varphi &+ \theta Q(\varphi, x_1, t_1)\\ &< \left[\frac{2}{\theta^3}|\nabla_p\theta|^2 - \frac{\Delta\theta}{\theta} - c_{15}c_{14}^\varepsilon\left(\frac{1}{\theta}\right)^{1+\varepsilon}\right]\varphi(x_1,t_1)\theta^2\\ &= \left[\Delta\left(\frac{1}{\theta}\right) - c_{15}c_{14}^\varepsilon\left(\frac{1}{\theta}\right)^{1+\varepsilon}\right]\varphi(x_1,t_1)\theta^2.\end{aligned}$$

By means of Lemma 4.8 we have

$$\begin{aligned}\left(\frac{2}{\theta}|\nabla_p\theta|^2 - \Delta\theta\right)\varphi &+ \theta Q(\varphi, x_1, t_1)\\ &< \left(\frac{c_{13}}{k} - c_{15}c_{14}^\varepsilon\right)\left(\frac{1}{\theta}\right)^{1+\varepsilon}\theta^2\varphi(x_1,t_1) \quad \text{at } (x_1,t_1).\end{aligned} \tag{124}$$

If we choose k large enough such that $c_{13}/k - c_{15}c_{14}^\varepsilon \le 0$, then

$$\left(\frac{2}{\theta}|\nabla_p\theta|^2 - \Delta\theta\right)\varphi + \theta Q(\varphi, x_1, t_1) < 0 \quad \text{at } (x_1,t_1). \tag{125}$$

From (119), (121), and (125) we know that

$$\frac{\partial\tilde\varphi}{\partial t}(x_1,t_1) < 0,$$

which contradicts (120); therefore the lemma is true.

Lemma 4.10. *Under Assumption* A, *suppose there exist constants* $0 < \varepsilon < +\infty$ *and* $0 < c_{14}, c_{15}, c_{16} < +\infty$, *and* $\varphi(x,t) \in C^\infty(M \times [0,T])$ *such that*

$$\frac{\partial\varphi}{\partial t} = \Delta\varphi + Q(\varphi,x,t) \quad \text{on } M\times[0,T],$$

$$\varphi(x,0) \le c_{14} \quad \text{on } M,$$

$$Q(\varphi,x,t) \le \frac{c_{16}|\nabla_i\varphi|^2}{\varphi} - c_{15}\varphi^{1+\varepsilon} \quad \text{for } \varphi \ge c_{14}.$$

Then we have

$$\varphi(x,t) \le c_{14} \quad \text{on } M\times[0,T].$$

Proof. Let α be an odd integer and $\alpha \ge 1 + c_{16}$. Define

$$\psi(x,t) = \varphi(x,t)^\alpha \quad \text{on } M\times[0,T]. \tag{126}$$

Then

$$\frac{\partial \psi}{\partial t} = \alpha\varphi^{\alpha-1}\frac{\partial \varphi}{\partial t} = \alpha\varphi^{\alpha-1}[\Delta\varphi + Q(\varphi, x, t)]$$
$$= \Delta\varphi^{\alpha} - \alpha(\alpha-1)\varphi^{\alpha-2}|\nabla_i\varphi|^2 + \alpha\varphi^{\alpha-1}Q(\varphi, x, t).$$

Thus

$$\frac{\partial \psi}{\partial t} = \Delta\psi + \tilde{Q}(\psi, x, t), \tag{127}$$

where

$$\tilde{Q}(\psi, x, t) = -\alpha(\alpha-1)\varphi^{\alpha-2}|\nabla_i\varphi|^2 + \alpha\varphi^{\alpha-1}Q(\varphi, x, t). \tag{128}$$

From (126) we get

$$\psi(x, 0) \le c_{14}^{\alpha} \quad \text{on } M. \tag{129}$$

If $\psi(x,t) \ge c_{14}^{\alpha}$, then $\varphi(x,t) \ge c_{14}$,

$$Q(\varphi, x, t) \le \frac{c_{16}}{\varphi}|\nabla_i\varphi|^2 - c_{15}\varphi^{1+\varepsilon},$$

$$\tilde{Q}(\psi, x, t) \le -\alpha(\alpha-1)\varphi^{\alpha-2}|\nabla_i\varphi|^2 + c_{16}\alpha\varphi^{\alpha-2}|\nabla_i\varphi|^2 - \alpha c_{15}\varphi^{\alpha+\varepsilon}$$
$$= \alpha(1 + c_{16} - \alpha)\varphi^{\alpha-2}|\nabla_i\varphi|^2 - c_{15}\alpha\psi^{1+\varepsilon/\alpha}.$$

Since $\alpha \ge 1 + c_{16}$,

$$\tilde{Q}(\psi, x, t) \le -c_{15}\alpha\psi^{1+\varepsilon/\alpha} \quad \text{for } \psi \ge c_{14}^{\alpha}. \tag{130}$$

From (127), (129), (130) and Lemma 4.9 it follows that

$$\psi(x,t) \le c_{14}^{\alpha} \quad \text{on } M \times [0, T].$$

By (126) we get

$$\varphi(x,t) \le c_{14} \quad \text{on } M \times [0, T].$$

Lemma 4.11. *Under Assumption* A, *suppose there exist constants* $0 < \varepsilon < +\infty$ *and* $0 < c_{14}, c_{15}, c_{16}, c_{17} < +\infty$, *and* $\varphi(x,t) \in C^{\infty}(M \times [0,T])$ *such that*

$$\frac{\partial \varphi}{\partial t} = \Delta\varphi + Q(\varphi, x, t) \quad \text{on } M \times [0, T],$$
$$\varphi(x, 0) \le c_{14} \quad \text{on } M,$$
$$Q(\varphi, x, t) \le \frac{c_{16}|\nabla_i\varphi|^2}{\varphi} + \psi_i\nabla_i\varphi - c_{17}\varphi|\psi_i|^2 - c_{15}\varphi^{1+\varepsilon} \quad \text{for } \varphi \ge c_{14},$$

where $\{\psi_i\}$ *is a tensor. Then we have*

$$\varphi(x,t) \le c_{14} \quad \text{on } M \times [0, T].$$

Proof. The proof follows from the inequality

$$\psi_i \nabla_i \varphi - c_{17}\varphi |\psi_i|^2 \le \frac{|\nabla_i \psi|^2}{4c_{17}\varphi}$$

and Lemma 4.10.

Theorem 4.12. *Under Assumption* A, *suppose there exist constants* $0 < \varepsilon < +\infty$ *and* $0 < c_{10}, c_{11}, c_{14}, c_{15}, c_{16}, c_{17} < +\infty$, *and the function* $\varphi(x,t) \in C^\infty(M \times [0,T])$ *such that*

$$(131)\qquad \begin{aligned} &\frac{\partial \varphi}{\partial t} = \Delta\varphi + Q(\varphi, x, t) \quad \text{on } M \times [0,T], \\ &\varphi(x,0) \le 0 \quad \text{on } M, \\ &Q(\varphi,x,t) \le c_{10}|\nabla_i \varphi|^2 + c_{11}\varphi \quad \text{for } 0 \le \varphi \le c_{14}, \\ &Q(\varphi,x,t) \le \frac{c_{16}}{\varphi}|\nabla_i\varphi|^2 + \psi_i \cdot \nabla_i \varphi - c_{17}\varphi|\psi_i|^2 - c_{15}\varphi^{1+\varepsilon} \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } \varphi \ge c_{14}, \end{aligned}$$

where $\{\psi_i\}$ *is a tensor. Then we have*

$$(132)\qquad \varphi(x,t) \le 0 \quad \text{on } M \times [0,T].$$

Proof. From Lemma 4.11 we know that

$$\varphi(x,t) \le c_{14} \quad \text{on } M \times [0,T].$$

Using Theorem 4.6 we thus complete the proof.

Now we are going to use the maximal principle derived above to prove some properties of curvature on M under the Ricci flow. First we have

Lemma 4.13. *Under Assumption* A, *we have*

$$(133)\qquad 0 < R(x,t) \le n^2\sqrt{c_1} \quad \text{on } M \times [0,T].$$

Proof. Using (5) and Lemma 3.1 we get respectively

$$(134)\qquad \begin{aligned} &|R(x,t)| \le n^2\sqrt{c_1} \quad \text{on } M \times [0,T], \\ &\frac{\partial R}{\partial t} = \Delta R + 2S, \end{aligned}$$

where $S = g^{ik}g^{jl}R_{ij}R_{kl} \ge 0$. Thus

$$(135)\qquad \frac{\partial R}{\partial t} \ge \Delta R \quad \text{on } M \times [0,T].$$

From (5) we have

$$(136)\qquad R(x,0) > 0 \quad \text{on } M,$$

and therefore, in consequence of (134), (135), and Theorem 4.6,

$$(137)\qquad R(x,t) > 0 \quad \text{on } M \times [0,T],$$

which together with (134) implies

$$0 < R(x,t) \le n^2\sqrt{c_1} \quad \text{on } M \times [0,T].$$

Next we are going to show that the Ricci deformation preserves the positivity of the curvature operator on the complete noncompact Riemannian manifold M. Hamilton [7] proved this for the case when M is a compact manifold. In the case when M is a noncompact complete manifold the proof basically is the same as the compact case, but we need to use some cut-off function technique, just as we did in Lemma 4.5. For more details, see Hamilton [7].

We regard the Riemannian curvature tensor $\mathrm{Rm} = \{R_{ijkl}\}$ as a symmetric bilinear form on the two-forms $\Lambda^2(M)$ by letting

$$\mathrm{Rm}(\varphi,\psi) = R_{ijkl}\phi_{ij}\psi_{kl}. \tag{138}$$

We say that the manifold has a positive curvature operator if $\mathrm{Rm}(\phi,\phi) > 0$ for all two-forms $\phi \ne 0$; in this case we denote

$$R_{ijkl} > 0 \quad \text{or} \quad \mathrm{Rm} > 0. \tag{139}$$

We say that the manifold has a nonnegative curvature operator if $\mathrm{Rm}(\phi,\phi) \ge 0$ for all two-forms ϕ and denote it by

$$R_{ijkl} \ge 0 \quad \text{or} \quad \mathrm{Rm} \ge 0. \tag{140}$$

We want to prove

Theorem 4.14. *Under Assumption* A, *if $R_{ijkl}(x,0) \ge 0$ on M, then*

$$R_{ijkl}(x,t) \ge 0 \quad \text{on } M \times [0,T]. \tag{141}$$

Moreover, if $R_{ijkl}(x,0) > 0$ on M, then

$$R_{ijkl}(x,t) > 0 \quad \text{on } M \times [0,T]. \tag{142}$$

Proof. Since (142) is an immediate consequence of (141), by using the local technique, which is exactly the same as the one used in the compact case, we only need to prove (141).

From Lemma 3.1 we have

$$\begin{aligned}\frac{\partial}{\partial t}R_{ijkl} &= \Delta R_{ijkl} + 2(B_{ijkl} - B_{ijlk} + B_{ikjl} - B_{iljk}) \\ &\quad - (R_{pi}R_{qjkl} + R_{pj}R_{iqkl} + R_{pk}R_{ijql} + R_{pl}R_{ijkq})g^{pq}.\end{aligned}$$

To simplify these equations we pick an abstract vector bundle V isomorphic to the tangent bundle TM, but with a fixed metric h_{ab} on the fibers. Choose an isometry $u = \{u_a^i\}$ between V and TM at the time $t = 0$, and let the isometry u evolve by the equation

$$\frac{\partial}{\partial t}u_a^i = g^{ij}R_{jk}u_a^k. \tag{143}$$

Then the pull-back metric

$$h_{ab} = g_{ij} u_a^i u_b^j \tag{144}$$

remains constant in time, since it is easy to see that $\frac{\partial}{\partial t} h_{ab} \equiv 0$, and u remains an isometry between the varying metric g_{ij} on TM and the fixed metric h_{ab} on V. Now we use u to pull back the curvature tensor to a tensor on V:

$$R_{abcd} = R_{ijkl} u_a^i u_b^j u_c^k u_d^l. \tag{145}$$

We can also pull back the Levi-Civita connection $\Gamma = \{\Gamma_{ij}^k\}$ on TM to get a connection $\tilde{\Gamma} = \{\tilde{\Gamma}_{jc}^a\}$ on V, where the covariant derivative of a section $w = \{w^a\}$ of V is given locally by

$$\nabla_i w^a = \frac{\partial w^a}{\partial x^i} + \tilde{\Gamma}_{ib}^a w^b. \tag{146}$$

We may take the covariant derivative of any tensor of V and TM, in particular we have

$$\nabla_i u_a^j = 0, \qquad \nabla_i h_{ab} = 0, \tag{147}$$

and let the Laplacian

$$\Delta R_{abcd} = g^{ij} \nabla_i \nabla_j R_{abcd} \tag{148}$$

be the trace of the second covariant derivative. From Hamilton [7] we know that

$$\frac{\partial}{\partial t} R_{abcd} = \Delta R_{abcd} + 2(B_{abcd} - B_{abdc} + B_{acbd} - B_{adbc}), \tag{149}$$

where $B_{abcd} = R_{aebf} R_{cedf}$.

We regard the two-forms Λ^2 on V as the Lie algebra so(n) of the Lie group of rotations of V. Choose a local chart on V where h_{ab} is the identity, the metric on Λ^2 is given by $|\phi|^2 = \langle \phi, \phi \rangle$, where $\langle \phi, \psi \rangle = \phi_{ab} \psi_{ab}$, and the Lie bracket is given by

$$[\phi, \psi]_{ab} = \phi_{ac} \psi_{bc} - \psi_{ac} \phi_{bc}. \tag{150}$$

It is easy to check that the trilinear form $\langle [\phi, \psi], w \rangle$ is fully antisymmetric; choose an orthonormal basis $\phi^\alpha = \{\phi_{ab}^\alpha\}$ for the 2-forms on V, then the inner product on $\Lambda^2(V)$,

$$h_{\alpha\beta} = \langle \phi^\alpha, \phi^\beta \rangle,$$

is the identity matrix in the local chart. The Lie bracket is given by

$$[\phi^\alpha, \phi^\beta] = c_\gamma^{\alpha\beta} \phi^\gamma,$$

where the $c_\gamma^{\alpha\beta}$ are the Lie structure constants relative to this basis. Note that $c^{\alpha\beta\gamma} = c_\delta^{\alpha\beta} h^{\gamma\delta}$ is fully antisymmetric since

$$c^{\alpha\beta\gamma} = \langle[\phi^\alpha, \phi^\beta], \phi^\gamma\rangle. \tag{151}$$

The tensor R_{abcd} on V may be regarded as a symmetric bilinear form $M_{\alpha\beta}$ on $\Lambda^2(V)$, where

$$R_{abcd} = M_{\alpha\beta}\phi_{ab}^\alpha \phi_{cd}^\beta. \tag{152}$$

Then from Hamilton [7] we know that

$$\frac{\partial}{\partial t} R_{abcd} = \Delta R_{abcd} + R_{abcd}^2 + R_{abcd}^\sharp, \tag{153}$$

where

$$\begin{aligned} R_{abcd}^2 &= R_{abef}R_{cdef} = 2(B_{abcd} - B_{abdc}) = M_{\alpha\gamma}M_{\gamma\beta}\phi_{ab}^\alpha\phi_{cd}^\beta, \\ R_{abcd}^\sharp &= 2(B_{acbd} - B_{adbc}) = c_{\alpha\gamma\eta}c_{\beta\delta\theta}M_{\gamma\delta}M_{\eta\theta}\phi_{ab}^\alpha\phi_{cd}^\beta, \end{aligned} \tag{154}$$

or equivalently

$$\frac{\partial}{\partial t} M_{\alpha\beta} = \Delta M_{\alpha\beta} + M_{\alpha\gamma}M_{\gamma\beta} + c_{\alpha\gamma\eta}c_{\beta\delta\theta}M_{\gamma\delta}M_{\eta\theta}. \tag{155}$$

For any symmetric bilinear form $A = \{A_{\alpha\beta}\}$ on $\Lambda^2(V)$, if we define

$$Q(A)_{\alpha\beta} = A_{\alpha\gamma}A_{\gamma\beta} + c_{\alpha\gamma\eta}c_{\beta\delta\theta}A_{\gamma\delta}A_{\eta\theta}, \tag{156}$$

then we get

$$\frac{\partial}{\partial t} M_{\alpha\beta} = \Delta M_{\alpha\beta} + Q(M)_{\alpha\beta}. \tag{157}$$

For any $\{A_{\alpha\beta}\}$, if $A_{\alpha\beta}w^\alpha w^\beta \geq 0$ for all $w = \{w^\alpha\}$, we denote

$$A_{\alpha\beta} \geq 0. \tag{158}$$

For any $\{A_{\alpha\beta}\}$ and $\{\tilde{A}_{\alpha\beta}\}$, if $A_{\alpha\beta} - \tilde{A}_{\alpha\beta} \geq 0$, we denote

$$A_{\alpha\beta} \geq \tilde{A}_{\alpha\beta}. \tag{159}$$

For any fixed $(x, t) \in M \times [0, T]$, we define

$$\varphi(x, t) = \sup_{\theta \in \mathbf{R}}\{\theta | M_{\alpha\beta}(x, t) \geq \theta\delta_{\alpha\beta}\}, \tag{160}$$

where

$$\delta_{\alpha\beta} = \begin{cases} 1, & \alpha = \beta, \\ 0, & \alpha \neq \beta. \end{cases}$$

Lemma 4.15. *For any $(x, t) \in M \times [0, T]$, $R_{ijkl}(x, t) \geq 0$ if and only if $M_{\alpha\beta}(x, t) \geq 0$.*

Proof. By the definition of $M_{\alpha\beta}$.

From the assumption of Theorem 4.14, we have $R_{ijkl}(x,0) \geq 0$ on M, and therefore

$$M_{\alpha\beta}(x,0) \geq 0 \quad \text{on } M. \tag{161}$$

Thus

$$\varphi(x,0) \geq 0 \quad \text{on } M. \tag{162}$$

If we can prove $\varphi(x,t) \geq 0$ on $M \times [0,T]$, then Theorem 4.14 follows from Lemma 4.15.

From Assumption A it follows that

$$|R_{ijkl}(x,t)|^2 \leq c_1 \quad \text{on } M \times [0,T].$$

By (143), (144), and (145) we get

$$|R_{abcd}(x,t)|^2 \leq \tilde{c}_1 \quad \text{on } M \times [0,T], \tag{163}$$

where $0 < \tilde{c}_1 < +\infty$ is some constant. Thus by the definition of $M_{\alpha\beta}$ there exists a constant $0 < c_{18} < +\infty$ such that

$$-c_{18}\delta_{\alpha\beta} \leq M_{\alpha\beta}(x,t) \leq c_{18}\delta_{\alpha\beta} \quad \text{on } M \times [0,T]. \tag{164}$$

In particular we have

$$\varphi(x,t) \geq -c_{18} \quad \text{on } M \times [0,T]. \tag{165}$$

Lemma 4.16. *For any symmetric bilinear form $\{A_{\alpha\beta}\}$, if $A_{\alpha\beta} \geq 0$, then $Q(A)_{\alpha\beta} \geq 0$.*

Proof. Just by the definition of $Q(A)_{\alpha\beta}$.

From (165) it follows

$$M_{\alpha\beta}(x,t) \geq \varphi(x,t)\delta_{\alpha\beta} \quad \text{on } M \times [0,T], \tag{166}$$

so that

$$A_{\alpha\beta} = M_{\alpha\beta} - \varphi\delta_{\alpha\beta} \geq 0 \quad \text{on } M \times [0,T]. \tag{167}$$

By Lemma 4.16 we get $Q(A)_{\alpha\beta} \geq 0$. Since $Q(A)_{\alpha\beta}$ actually are the quadratic polynomials of $A_{\alpha\beta}$, from (156) and (164) we have

$$Q(M)_{\alpha\beta} \geq -c_{19}[c_{18}|\varphi| + \varphi^2]\delta_{\alpha\beta} \quad \text{on } M \times [0,T],$$

and, in consequence of (164) again,

$$|\varphi(x,t)| \leq c_{18} \quad \text{on } M \times [0,T].$$

Thus

$$Q(M)_{\alpha\beta} \geq -2c_{19}c_{18}|\varphi|\delta_{\alpha\beta} \quad \text{on } M \times [0,T], \tag{168}$$

and therefore

$$Q(M)_{\alpha\beta} \geq 2c_{19}c_{18}\varphi\delta_{\alpha\beta} \quad \text{for } \varphi \leq 0, \tag{169}$$

where $0 < c_{19} < +\infty$ is some constant.

Now if we can find some $(x_0, t_0) \in M \times [0, T]$ such that

$$\varphi(x_0, t_0) < 0. \tag{170}$$

Let $\theta(x,t) \in C^\infty(M \times [0,T])$ be the function constructed in Lemma 4.4, and consider $\tilde{M}_{\alpha\beta}$ as follows:

$$\tilde{M}_{\alpha\beta}(x,t) = \theta(x,t)M_{\alpha\beta}(x,t) \quad \text{on } M \times [0,T]. \tag{171}$$

Let

$$\tilde{\varphi}(x,t) = \sup_{\theta_0 \in \mathbf{R}} \{\theta_0 | \tilde{M}_{\alpha\beta}(x,t) \geq \theta_0\delta_{\alpha\beta}\}. \tag{172}$$

Then

$$\tilde{\varphi}(x,t) \equiv \theta(x,t)\varphi(x,t) \quad \text{on } M \times [0,T]. \tag{173}$$

Since $0 < \theta(x,t) \leq 1$, from (170) we have

$$\tilde{\varphi}(x_0,t_0) = \theta(x_0,t_0)\varphi(x_0,t_0) < 0. \tag{174}$$

By (46) and (165) we know that

$$\tilde{\varphi}(x,t) \geq -\frac{c_{18}c_8}{1+\gamma_0(x_0,x)}. \tag{175}$$

Thus if $\gamma_0(x_0,x) > -c_{18}c_8/\tilde{\varphi}(x_0,t_0)$, then

$$\tilde{\varphi}(x,t) > \tilde{\varphi}(x_0,t_0). \tag{176}$$

Since $\overline{B_0(x_0, -c_{18}c_8/\tilde{\varphi}(x_0,t_0))} \times [0,T]$ is a compact subset of $M \times [0,T]$ and $\tilde{\varphi}(x,t)$ is a continuous function, from (176) it follows that there exists a point $(x_1,t_1) \in M \times [0,T]$ with

$$\gamma_0(x_0,x_1) \leq -c_{18}c_8/\tilde{\varphi}(x_0,t_0)$$

such that

$$\tilde{\varphi}(x_1,t_1) = \inf_{M\times[0,T]} \tilde{\varphi}(x,t) < 0. \tag{177}$$

On the other hand, by (172) one can find an index α_1 such that

$$\tilde{M}_{\alpha_1\alpha_1}(x_1,t_1) = \tilde{\varphi}(x_1,t_1), \tag{178}$$

$$\tilde{M}_{\alpha\alpha}(x,t) \geq \tilde{\varphi}(x_1,t_1) \quad \forall (x,t) \in M \times [0,T], \ \forall \text{ index } \alpha. \tag{179}$$

Thus

$$\Delta\tilde{M}_{\alpha_1\alpha_1}(x_1,t_1) \geq 0, \qquad \nabla\tilde{M}_{\alpha_1\alpha_1}(x_1,t_1) = 0. \tag{180}$$

Since $\varphi(x,0) \geq 0$, we have $\tilde{\varphi}(x,0) \geq 0$, and from (177) we get $t_1 > 0$. From (179) we have

$$\frac{\partial}{\partial t}\tilde{M}_{\alpha_1\alpha_1}(x_1,t_1) \leq 0. \tag{181}$$

On the other hand,

$$\begin{aligned}
\frac{\partial}{\partial t}\tilde{M}_{\alpha\beta} &= \frac{\partial}{\partial t}(\theta M_{\alpha\beta}) = \frac{\partial\theta}{\partial t}M_{\alpha\beta} + \theta\frac{\partial}{\partial t}M_{\alpha\beta} \\
&= \theta[\Delta M_{\alpha\beta} + Q(M)_{\alpha\beta}] + \frac{\partial\theta}{\partial t}M_{\alpha\beta} \\
&= \Delta(\theta M_{\alpha\beta}) - 2\nabla_p\theta\cdot\nabla_p M_{\alpha\beta} + \theta Q(M)_{\alpha\beta} + \frac{\partial\theta}{\partial t}M_{\alpha\beta} - M_{\alpha\beta}\Delta\theta \\
&= \Delta\tilde{M}_{\alpha\beta} - \frac{2}{\theta}\nabla_p\theta\cdot\nabla_p(\theta M_{\alpha\beta}) + \frac{2|\nabla_p\theta|^2}{\theta}M_{\alpha\beta} + \theta Q(M)_{\alpha\beta} \\
&\quad + \frac{\partial\theta}{\partial t}M_{\alpha\beta} - (\Delta\theta)M_{\alpha\beta} \\
&= \Delta\tilde{M}_{\alpha\beta} - \frac{2}{\theta}\nabla_p\theta\cdot\nabla_p\tilde{M}_{\alpha\beta} + \left(\frac{\partial\theta}{\partial t} - \Delta\theta + \frac{2}{\theta}|\nabla_p\theta|^2\right)M_{\alpha\beta} \\
&\quad + \theta Q(M)_{\alpha\beta}.
\end{aligned} \tag{182}$$

Now let $(x,t) = (x_1,t_1)$ and $\alpha = \beta = \alpha_1$. Since

$$\tilde{\varphi}(x_1,t_1) = \theta(x_1,t_1)\varphi(x_1,t_1) < 0,$$

we have

$$\varphi(x_1,t_1) < 0.$$

From (169) it follows that

$$Q(M)_{\alpha_1\alpha_1} \geq 2c_{18}c_{19}\varphi(x_1,t_1). \tag{183}$$

But $M_{\alpha_1\alpha_1}(x_1,t_1) = \varphi(x_1,t_1)$, so by (182) and (183) we get

$$\begin{aligned}
\frac{\partial}{\partial t}\tilde{M}_{\alpha_1\alpha_1} \geq \Delta\tilde{M}_{\alpha_1\alpha_1} &- \frac{2}{\theta}\nabla_p\theta\cdot\nabla_p\tilde{M}_{\alpha_1\alpha_1} \\
&+ \left(\frac{\partial\theta}{\partial t} - \Delta\theta + \frac{2}{\theta}|\nabla_p\theta|^2 + 2c_{18}c_{19}\theta\right)\varphi(x_1,t_1).
\end{aligned} \tag{184}$$

By Lemma 4.4 if we choose $c_7 > 2c_{18}c_{19}$, then

$$\frac{\partial\theta}{\partial t} - \Delta\theta + \frac{2}{\theta}|\nabla_p\theta|^2 + 2c_{18}c_{19}\theta < 0. \tag{185}$$

Since $\varphi(x_1,t_1) < 0$, from (180), (184), and (185) it follows that

$$\frac{\partial}{\partial t}\tilde{M}_{\alpha_1\alpha_1}(x_1,t_1) > 0,$$

which contradicts (181). Thus

$$\varphi(x,t) \geq 0 \quad \text{on } M \times [0,T],$$

and the proof of Theorem 4.14 is complete.

5. Long time existence

Let M be an n-dimensional Riemannian manifold with metric

$$ds^2 = g_{ij}dx^i dx^j > 0.$$

If the curvature satisfies

$$|\overset{\circ}{\mathrm{Rm}}|^2 \leq \delta_n(1-\varepsilon)^2 \frac{2}{n(n-1)} R^2, \tag{1}$$

where $\varepsilon > 0$, $\delta_3 > 0$, $\delta_4 = \frac{1}{5}$, $\delta_5 = \frac{1}{10}$, and

$$\delta_n = \frac{2}{(n-2)(n+1)}, \qquad n \geq 6, \tag{2}$$

then the curvature operator is positive, more precisely, in this case we have

$$R_{ijkl}u_{ij}u_{kl} \geq 2\varepsilon |u_{ij}|^2 \frac{R}{n(n-1)} \tag{3}$$

for any two-form $\{u_{ij}\}$. For the proof of this statement, see G. Huisken [8].

Now we choose constants $\beta_n \leq \delta_n/[2n(n-1)]$ depending only on n, and suppose the curvature of the manifold considered satisfies

$$|\overset{\circ}{\mathrm{Rm}}|^2 \leq \beta_n R^2. \tag{4}$$

In this case from (1) and (3) we know that for any $\{u_{ij}\}$,

$$R_{ijkl}u_{ij}u_{kl} \geq |u_{ij}|^2 \frac{R}{n(n-1)}. \tag{5}$$

Lemma 5.1. *Suppose M is an n-dimensional complete noncompact Riemannian manifold with its curvature satisfying condition* (4). *Then*

$$|R_{ijkl}|^2 \leq \left[\beta_n + \frac{2}{n(n-1)}\right] R^2 \qquad \textit{on } M. \tag{6}$$

Proof. It is easy to show that

$$|R_{ijkl}|^2 = |\overset{\circ}{\mathrm{Rm}}|^2 + \frac{2}{n(n-1)} R^2. \tag{7}$$

From (4) and (7) we get (6) immediately.

Theorem 5.2. *Suppose M is an n-dimensional complete noncompact Riemannian manifold with metric $g_{ij}(x)$. If the curvature of M satisfies*

$$(8)\qquad |\overset{\circ}{\mathrm{R}}\mathrm{m}|^2 \le \beta_n R^2, \qquad 0 < R \le c_0,$$

where $0 < c_0 < +\infty$ is a constant, then the evolution equation

$$(9)\qquad \frac{\partial}{\partial t} g_{ij}(x,t) = -2R_{ij}(x,t), \quad g_{ij}(x,0) = g_{ij}(x), \qquad x \in M,$$

has a solution for all time $0 \le t < +\infty$.

This long time existence theorem is what we want to prove in this section, but before we start the proof, we must prove several lemmas.

Using (8) and Lemma 5.1 we find

$$(10)\qquad |R_{ijkl}(x,0)|^2 \le \left[\beta_n + \frac{2}{n(n-1)}\right] c_0^2 \quad \forall x \in M.$$

From (5) we get

$$(11)\qquad R_{ijkl}(x,0) > 0 \quad \forall x \in M,$$

which together with (10) implies

$$(12)\qquad 0 < R_{ijij}(x,0) \le \left[\beta_n + \frac{2}{n(n-1)}\right]^{1/2} n^2 c_0 \quad \text{on } M.$$

Thus by using Theorem 3.4 we know that the evolution equation (9) has a solution for a short time $0 \le t \le T_0$, where $T_0 > 0$ depends only on n and c_0, and by using Lemma 3.4 we still have the short time estimate:

$$(13)\qquad \sup_M |\nabla^m R_{ijkl}(x,t)|^2 \le c_{m+1}(n,c_0)/t^m, \qquad 0 \le t \le T_0,\ m \ge 0.$$

Lemma 5.3. *The solution obtained above satisfies Assumption* A *of* §4 *on $M \times [0, T_0]$.*

Proof. Similarly to the proof of (3) in §4 by using (13).

Now we define $0 < T_1 \le +\infty$ as follows:

$$(14)\qquad \begin{aligned} T_1 = \sup_{\tau \in \mathbf{R}}\{\tau \mid &\text{the evolution equation (9) has a solution } g_{ij}(x,t) \\ &\text{on } M \times [0,\tau), \text{ and for any } 0 < T < \tau, \text{ the solution} \\ &g_{ij}(x,t) \text{ satisfies Assumption A of §4 on } M \times [0,T] \text{ and} \\ &\text{(13) holds on } M \times [0, \tfrac{1}{2}T_0]\}. \end{aligned}$$

Then we have

$$(15)\qquad 0 < T_0 \le T_1 \le +\infty.$$

What we need to prove is that $T_1 = +\infty$.

For $0 < T_2 < T_1$, suppose $g_{ij}(x,t)$ is a solution of the evolution equation on $M \times [0, T_2)$, and for any $0 < T < T_2$, the solution $g_{ij}(x,t)$ satisfies Assumption A of §4 on $M \times [0, T]$.

Thus for any $T < T_2$, the maximal principle Theorem 4.6 and Theorem 4.12 are true on $M \times [0, T]$, but since $T < T_2$ is arbitrary, we know that Theorem 4.6 and Theorem 4.12 actually hold on $M \times [0, T_2)$.

Lemma 5.4. *We have the following:*

$$R_{ijkl}(x,t) > 0, \quad R(x,t) > 0 \quad \text{on } M \times [0, T_2). \tag{16}$$

Proof. From (11) we complete the proof immediately by using Theorem 4.14.

Lemma 5.5. *Suppose* $0 \le \sigma < \frac{1}{2}$ *and*

$$f_\sigma(x,t) = \frac{|\overset{\circ}{\mathrm{Rm}}|^2}{R^{2-\sigma}}(x,t) \quad \text{on } M \times [0, T_2). \tag{17}$$

Then

$$\begin{aligned} \frac{\partial}{\partial t} f_\sigma = \Delta f_\sigma + \frac{2(1-\sigma)}{R} \nabla_k R \cdot \nabla_k f_\sigma - \frac{\sigma(1-\sigma)}{R^{4-\sigma}} |\overset{\circ}{\mathrm{Rm}}|^2 |\nabla_i R|^2 \\ - \frac{2}{R^{4-\sigma}} |R \nabla_p R_{ijkl} - R_{ijkl} \nabla_p R|^2 + \frac{4}{R^{3-\sigma}} \left(P + \frac{\sigma}{2} |\overset{\circ}{\mathrm{Rm}}|^2 S \right), \end{aligned} \tag{18}$$

where

$$P = 2R R_{ijkl} R_{imkn} R_{mjnl} + \tfrac{1}{2} R R_{ijkl} R_{klmn} R_{mnij} - |\mathrm{Rm}|^2 S.$$

Proof. This is Lemma 3.2 [8].

Lemma 5.6. *If*

$$|\overset{\circ}{\mathrm{Rm}}|^2 \le \delta_n (1-\varepsilon)^2 \frac{2}{n(n-1)} R^2, \qquad \varepsilon > 0,$$

then

$$P \le -\frac{\varepsilon}{n} R^2 |\overset{\circ}{\mathrm{Rm}}|^2.$$

PROOF. This is Theorem 3.3 [8].

Lemma 5.7. $|\overset{\circ}{\mathrm{Rm}}|^2 / R^2 \le \beta_n$ *for* $0 \le t < T_2$.

Proof. Let $f_0(x,t) = (|\overset{\circ}{\mathrm{Rm}}|^2 / R^2)(x,t)$. Then from (18) we have

$$\frac{\partial}{\partial t} f_0 = \Delta f_0 + \frac{2}{R} \nabla_k R \cdot \nabla_k f_0 - \frac{2}{R^4} |R \nabla_p R_{ijkl} - R_{ijkl} \nabla_p R|^2 + \frac{4}{R^3} P. \tag{19}$$

Let

$$\varphi(x,t) = R(f_0 - \beta_n). \tag{20}$$

Then

$$\varphi(x,t)=\frac{|\overset{\circ}{\mathrm{Rm}}|^2}{R}(x,t)-\beta_n R(x,t). \tag{21}$$

By (3) and (7) of §3 we get

$$\begin{aligned}\frac{\partial\varphi}{\partial t}&=\Delta\varphi-\frac{2}{R^3}|R\nabla_p R_{ijkl}-R_{ijkl}\nabla_p R|^2+\frac{4}{R^2}P\\&\quad+2S(f_0-\beta_n),\qquad 0\le t<T_2.\end{aligned} \tag{22}$$

From (8) it follows that

$$f_0(x,0)\le\beta_n,\qquad x\in M, \tag{23}$$

and that

$$\varphi(x,0)\le 0,\qquad x\in M. \tag{24}$$

Using Lemma 5.4 and formula (1.10) in [2] we get, respectively,

$$\begin{aligned}&0<R_{ijij}(x,t)\le R(x,t)\quad\text{on } M\times[0,T_2),\\&|R_{ijkl}(x,t)|^2\le 200n^4R(x,t)^2\quad\text{on } M\times[0,T_2).\end{aligned} \tag{25}$$

Since $|\overset{\circ}{\mathrm{Rm}}|^2\le|R_{ijkl}|^2$, we have

$$\begin{aligned}&|\overset{\circ}{\mathrm{Rm}}|^2\le 200n^4R^2\quad\text{on } M\times[0,T_2),\\&f_0(x,t)\le 200n^4\quad\text{on } M\times[0,T_2),\end{aligned} \tag{26}$$

$$\varphi(x,t)\le R(x,t)f_0(x,t)\le 200n^4R(x,t)\quad\text{on } M\times[0,T_2). \tag{27}$$

For any $T<T_2$, since Assumption A of §4 is true on $M\times[0,T]$, we can find a constant $c_1(T)>0$ such that

$$|R_{ijkl}(x,t)|^2\le c_1(T)\quad\text{on } M\times[0,T], \tag{28}$$

which together with Lemma 5.4 implies that

$$0<R(x,t)\le n^2\sqrt{c_1(T)}\quad\text{on } M\times[0,T]. \tag{29}$$

From (27) it follows that

$$\varphi(x,t)\le 200n^6\sqrt{c_1(T)}\quad\text{on } M\times[0,T]. \tag{30}$$

If $\varphi\ge 0$, from (20) we have $|\overset{\circ}{\mathrm{Rm}}|^2/R^2\ge\beta_n$. Define

$$\theta^2=\frac{\beta_nR^2}{|\overset{\circ}{\mathrm{Rm}}|^2}\quad\text{or}\quad\theta=\frac{R}{|\overset{\circ}{\mathrm{Rm}}|}\beta_n^{1/2}.$$

Then $0 < \theta \leq 1$ and

$$\beta_n = \frac{|\overset{\circ}{\mathrm{R}}\mathrm{m}|^2}{R^2}\theta^2, \tag{31}$$

$$\begin{aligned}\varphi(x,t) &= R\left(\frac{|\overset{\circ}{\mathrm{R}}\mathrm{m}|^2}{R^2} - \beta_n\right) = R\left(\frac{|\overset{\circ}{\mathrm{R}}\mathrm{m}|^2}{R^2} - \frac{|\overset{\circ}{\mathrm{R}}\mathrm{m}|^2}{R^2}\theta^2\right) \\ &= \frac{|\overset{\circ}{\mathrm{R}}\mathrm{m}|^2}{R}(1-\theta)(1+\theta) \geq \frac{|\overset{\circ}{\mathrm{R}}\mathrm{m}|^2}{R}(1-\theta) \\ &= \frac{|\overset{\circ}{\mathrm{R}}\mathrm{m}|}{R}(1-\theta)|\overset{\circ}{\mathrm{R}}\mathrm{m}| \geq \beta_n^{1/2}(1-\theta)|\overset{\circ}{\mathrm{R}}\mathrm{m}|.\end{aligned}$$

Thus

$$(1-\theta)|\overset{\circ}{\mathrm{R}}\mathrm{m}| \leq \frac{\varphi(x,t)}{\sqrt{\beta_n}}. \tag{32}$$

Let

$$\begin{aligned}\overset{\circ}{\tilde{R}}_{ijkl} &= \theta \overset{\circ}{R}_{ijkl}, \\ U_{ijkl} &= \frac{1}{n(n-1)}R(g_{ik}g_{jl} - g_{il}g_{jk}), \\ \tilde{R}_{ijkl} &= U_{ijkl} + \theta\overset{\circ}{R}_{ijkl} = U_{ijkl} + \overset{\circ}{\tilde{R}}_{ijkl}.\end{aligned} \tag{33}$$

Then

$$R_{ijkl} = U_{ijkl} + \overset{\circ}{R}_{ijkl}. \tag{34}$$

From §1 we know that

$$\begin{aligned}|R_{ijkl}|^2 &= |U_{ijkl}|^2 + |\overset{\circ}{R}_{ijkl}|^2, \\ |\tilde{R}_{ijkl}|^2 &= |U_{ijkl}|^2 + \theta^2|\overset{\circ}{R}_{ijkl}|^2.\end{aligned} \tag{35}$$

Since $0 < \theta \leq 1$, from (28) and (35) we have

$$|\tilde{R}_{ijkl}|^2 \leq |R_{ijkl}|^2 \leq c_1(T). \tag{36}$$

Define

$$\tilde{P} = 2\tilde{R}\tilde{R}_{ijkl}\tilde{R}_{imkn}\tilde{R}_{mjnl} + \tfrac{1}{2}\tilde{R}\tilde{R}_{ijkl}\tilde{R}_{klmn}\tilde{R}_{mnij} - |\tilde{R}_{ijkl}|^2\tilde{S}, \tag{37}$$

where

$$\tilde{R} = g^{ik}g^{jl}\tilde{R}_{ijkl} = R, \quad \tilde{S} = g^{ik}g^{jl}\tilde{R}_{ij}\tilde{R}_{kl}, \quad \tilde{R}_{ij} = g^{kl}\tilde{R}_{ikjl}. \tag{38}$$

Since

$$|\overset{\circ}{\tilde{R}}_{ijkl}|^2 = \theta^2|\overset{\circ}{R}_{ijkl}|^2 = \beta_n R^2 = \beta_n \tilde{R}^2,$$

and $\beta_n \leq \delta_n/[2n(n-1)]$, from Lemma 5.6 it follows that

$$\tilde{P} \leq -\frac{1}{2n}\tilde{R}^2|\mathring{\tilde{R}}_{ijkl}|^2 \leq 0. \tag{39}$$

By the definition of P and $\tilde{P}$,

$$P - \tilde{P} \leq c_2(|R_{ijkl}|^3 + |\tilde{R}_{ijkl}|^3)|R_{ijkl} - \tilde{R}_{ijkl}|,$$

which becomes, in consequence of (36),

$$\begin{aligned} P - \tilde{P} &\leq 2c_2|R_{ijkl}|^3 \cdot |R_{ijkl} - \tilde{R}_{ijkl}| \\ &= 2c_2|R_{ijkl}|^3 \cdot |\mathring{R}_{ijkl} - \theta\mathring{R}_{ijkl}| \\ &= 2c_2|R_{ijkl}|^3(1-\theta)|\mathring{\mathrm{R}}\mathrm{m}|. \end{aligned}$$

From (25) and (32) it follows that

$$P - \tilde{P} \leq 6000c_2n^6R^3 \cdot \frac{\varphi}{\sqrt{\beta_n}} \leq c_3R^3\varphi, \tag{40}$$

which together with (39) implies $P \leq c_3R^3\varphi$. By using (29) we get

$$\frac{4}{R^2}P \leq c_3 \cdot 4R\varphi \leq c_4\varphi \quad \text{for } \varphi \geq 0. \tag{41}$$

On the other hand, from (20) we have

$$2S(f_0 - \beta_n) = \frac{2S}{R}\varphi \leq \frac{2n^2|\mathrm{Rm}|^2}{R}\varphi,$$

which together with (25) yields

$$2S(f_0 - \beta_n) \leq 400n^6R\varphi.$$

By using (29) we get

$$2S(f_0 - \beta_n) \leq c_5\varphi \quad \text{if } \varphi \geq 0, \tag{42}$$

which together with (41) implies

$$\frac{4}{R^2}P + 2S(f_0 - \beta_n) \leq c_6\varphi \quad \text{if } \varphi \geq 0. \tag{43}$$

From (22), (24), (30), (43) and Theorem 4.6 we have

$$\varphi(x,t) \leq 0 \quad \text{on } M \times [0,T]$$

for any $T < T_2$. Thus

$$\varphi(x,t) \leq 0 \quad \text{on } M \times [0,T_2),$$

and, in consequence of (20),

$$f_0(x,t) \leq \beta_n \quad \text{on } M \times [0,T_2). \tag{44}$$

Hence the proof of Lemma 5.7 is complete.

Lemma 5.8. *We have the inequality*

$$|\nabla_i R_{jk}|^2 \geq \frac{3n-2}{2(n-1)(n+2)}|\nabla_i R|^2. \tag{45}$$

Proof. This is Lemma 4.3 in [8].

From Lemma 5.8 we get

$$\begin{aligned} |\nabla_i R_{jk}|^2 - \frac{1}{n}|\nabla_i R|^2 &\geq \frac{(n-2)^2}{2n(n-1)(n+2)}|\nabla_i R|^2, \\ |\nabla_i R_{jk}|^2 - \frac{1}{n}|\nabla_i R|^2 &\geq \frac{(n-2)^2}{n(3n-2)}|\nabla_i R_{jk}|^2. \end{aligned} \tag{46}$$

Lemma 5.9. *Let* $\overset{\circ}{R}_{ij} = R_{ij} - \frac{1}{n}Rg_{ij}$. *Then*

$$S - \frac{1}{n}R^2 = |\overset{\circ}{R}_{ij}|^2 \geq 0, \tag{47}$$

$$\frac{\partial S}{\partial t} = \Delta S - 2|\nabla_i R_{jk}|^2 + 4R_{ij}R_{kl}R_{ikjl}, \tag{48}$$

$$\begin{aligned} \frac{\partial}{\partial t}\left(S - \frac{1}{n}R^2\right) = \Delta\left(S - \frac{1}{n}R^2\right) - 2|\nabla_i R_{jk}|^2 + \frac{2}{n}|\nabla_i R|^2 \\ + 4\overset{\circ}{R}_{ij}R_{kl}R_{ikjl}. \end{aligned} \tag{49}$$

Proof. This is Lemma 4.2 in [8].

Lemma 5.10. *We have the inequality*

$$\begin{aligned} \frac{\partial}{\partial t}\left(S - \frac{1}{n}R^2\right) \leq \Delta\left(S - \frac{1}{n}R^2\right) - \frac{(n-2)^2}{n(n-1)(n+2)} \cdot |\nabla_i R|^2 \\ + c(n)|R_{ijkl}|^3. \end{aligned} \tag{50}$$

Proof. This is a direct corollary of Lemmas 5.8 and 5.9.

Lemma 5.11. *For* $\gamma > 0$ *we have*

$$\frac{\partial}{\partial t}|\nabla_i R|^2 = \Delta|\nabla_i R|^2 - 2|\nabla_i\nabla_j R|^2 + 4\nabla_i R \cdot \nabla_i S, \tag{51}$$

$$\frac{\partial}{\partial t}\left(\frac{1}{R^\gamma}\right) = \Delta\left(\frac{1}{R^\gamma}\right) - \frac{\gamma(\gamma+1)}{R^{\gamma+2}}|\nabla_i R|^2 - \frac{2\gamma}{R^{\gamma+1}}S, \tag{52}$$

$$\begin{aligned} \frac{\partial}{\partial t}\left(\frac{|\nabla_i R|^2}{R^\gamma}\right) = \Delta\left(\frac{|\nabla_i R|^2}{R^\gamma}\right) + \frac{\gamma}{R}\nabla_k R \cdot \nabla_k\left(\frac{|\nabla_i R|^2}{R}\right) \\ - \gamma\left(1 - \frac{\gamma}{2}\right)\frac{|\nabla_i R|^4}{R^{\gamma+2}} - \frac{2}{R^\gamma}|\nabla_i\nabla_j R - \frac{\gamma}{2R}\nabla_i R \cdot \nabla_j R|^2 \\ - \frac{2\gamma S}{R^{\gamma+1}}|\nabla_i R|^2 + \frac{4}{R^\gamma}\nabla_i R \cdot \nabla_i S, \end{aligned} \tag{53}$$

where $S = g^{ik}g^{jl}R_{ij}R_{kl}$.

Proof. From Lemma 3.1 we have

$$\frac{\partial R}{\partial t} = \Delta R + 2S.$$

Thus

$$\begin{aligned}
\frac{\partial}{\partial t}\nabla_i R &= \nabla_i\left(\frac{\partial R}{\partial t}\right) = \nabla_i[\Delta R + 2S] \\
&= \nabla_i(\Delta R) + 2\nabla_i S = \Delta(\nabla_i R) - R_{ik}\nabla_k R + 2\nabla_i S,
\end{aligned}$$

$$(54)\qquad \begin{aligned}
\frac{\partial}{\partial t}|\nabla_i R|^2 &= \Delta|\nabla_i R|^2 - 2|\nabla_i\nabla_j R|^2 + 4\nabla_i R\cdot\nabla_i S, \\
\frac{\partial}{\partial t}\left(\frac{1}{R^\gamma}\right) &= -\frac{\gamma}{R^{\gamma+1}}\frac{\partial R}{\partial t} = -\frac{\gamma}{R^{\gamma+1}}(\Delta R + 2S) \\
&= \Delta\left(\frac{1}{R^\gamma}\right) - \frac{\gamma(\gamma+1)}{R^{\gamma+2}}|\nabla_i R|^2 - \frac{2\gamma}{R^{\gamma+1}}S.
\end{aligned}$$

The third and fourth equations of (54) are (51) and (52) respectively.

Now, using (51) and (52), we get

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{|\nabla_i R|^2}{R^\gamma}\right) &= \Delta\left(\frac{|\nabla_i R|^2}{R^\gamma}\right) - 2\nabla_p\left(\frac{1}{R^\gamma}\right)\cdot\nabla_p|\nabla_i R|^2 - \frac{\gamma(\gamma+1)}{R^{\gamma+2}}|\nabla_i R|^4 \\
&\quad - \frac{2\gamma S}{R^{\gamma+1}}|\nabla_i R|^2 - \frac{2}{R^\gamma}|\nabla_i\nabla_j R|^2 + \frac{4}{R^\gamma}\nabla_i R\cdot\nabla_i S.
\end{aligned}$$

Since

$$\begin{aligned}
-2\nabla_p\left(\frac{1}{R^\gamma}\right)\cdot\nabla_p|\nabla_i R|^2 &= \frac{4\gamma}{R^{\gamma+1}}\nabla_k R\cdot\nabla_i R\cdot\nabla_k\nabla_i R, \\
\nabla_k\left(\frac{|\nabla_i R|^2}{R^r}\right) &= \frac{2}{R^\gamma}\nabla_i R\cdot\nabla_k\nabla_i R - \frac{\gamma}{R^{\gamma+1}}|\nabla_i R|^2\nabla_k R, \\
\frac{\gamma}{R}\nabla_k R\cdot\nabla_k\left(\frac{|\nabla_i R|^2}{R^\gamma}\right) &= \frac{2\gamma}{R^{\gamma+1}}\nabla_i R\cdot\nabla_k R\cdot\nabla_k\nabla_i R - \frac{\gamma^2}{R^{\gamma+2}}|\nabla_i R|^4,
\end{aligned}$$

we have

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{|\nabla_i R|^2}{R^\gamma}\right) &= \Delta\left(\frac{|\nabla_i R|^2}{R^\gamma}\right) + \frac{\gamma}{R}\nabla_k R\cdot\nabla_k\left(\frac{|\nabla_i R|^2}{R^\gamma}\right) \\
&\quad + \frac{2\gamma}{R^{\gamma+1}}\nabla_i R\cdot\nabla_k R\cdot\nabla_k\nabla_i R - \frac{2}{R^\gamma}|\nabla_i\nabla_j R|^2 \\
&\quad - \frac{\gamma}{R^{\gamma+2}}|\nabla_i R|^4 - \frac{2\gamma S}{R^{\gamma+1}}|\nabla_i R|^2 + \frac{4}{R^\gamma}\nabla_i R\cdot\nabla_i S,
\end{aligned}$$

which actually is (53).

Lemma 5.12. *If we define*

$$w = \frac{|\nabla_i R|^2}{R} + 4\left(S - \frac{1}{n}R^2\right),$$

then

$$
(55)\qquad \begin{aligned} \frac{\partial w}{\partial t} \le \Delta w + \frac{1}{R}\nabla_k R\cdot\nabla_k w - \frac{1}{4R}w^2 + \frac{8}{n}Rw \\ + \frac{\tilde{c}(n)}{R}\left(S - \frac{1}{n}R^2\right)^2 + \tilde{c}(n)|R_{ijkl}|^3, \end{aligned}
$$

where $\tilde{c}(n) > 0$ *is a constant depending only on* n.

Proof. We have

$$
\frac{1}{R}\nabla_k R\cdot\nabla_k\left(S - \frac{1}{n}R^2\right) = \frac{1}{R}\nabla_k R\cdot\nabla_k S - \frac{2}{n}|\nabla_k R|^2.
$$

Thus from (50) it follows that

$$
(56)\qquad \begin{aligned} \frac{\partial}{\partial t}\left(S - \frac{1}{n}R^2\right) \le \Delta\left(S - \frac{1}{n}R^2\right) + \frac{1}{R}\nabla_k R\cdot\nabla_k\left(S - \frac{1}{n}R^2\right) \\ - \frac{1}{R}\nabla_k R\cdot\nabla_k S + \frac{2}{n}|\nabla_k R|^2 \\ - \frac{(n-2)^2}{n(n-1)(n+2)}|\nabla_i R|^2 + c(n)|R_{ijkl}|^3. \end{aligned}
$$

Now, letting $\gamma = 1$ and using (53) we get

$$
\begin{aligned} \frac{\partial}{\partial t}\left(\frac{|\nabla_i R|^2}{R}\right) = \Delta\left(\frac{|\nabla_i R|^2}{R}\right) + \frac{1}{R}\nabla_k R\cdot\nabla_k\left(\frac{|\nabla_i R|^2}{R}\right) \\ - \frac{1}{2R^3}|\nabla_i R|^4 - \frac{2}{R}\left|\nabla_i\nabla_j R - \frac{1}{2R}\nabla_i R\cdot\nabla_j R\right|^2 \\ - \frac{2S}{R^2}|\nabla_i R|^2 + \frac{4}{R}\nabla_i R\cdot\nabla_i S, \end{aligned}
$$

$$
(57)\qquad \begin{aligned} \frac{\partial}{\partial t}\left(\frac{|\nabla_i R|^2}{R}\right) \le \Delta\left(\frac{|\nabla_i R|^2}{R}\right) + \frac{1}{R}\nabla_k R\cdot\nabla_k\left(\frac{|\nabla_i R|^2}{R}\right) \\ - \frac{1}{2R^3}|\nabla_i R|^4 + \frac{4}{R}\nabla_i R\cdot\nabla_i S. \end{aligned}
$$

By means of (56) and (57) we have

$$\frac{\partial w}{\partial t} \le \Delta w + \frac{1}{R}\nabla_k R \cdot \nabla_k w - \frac{1}{2R^3}|\nabla_i R|^4 + \frac{8}{n}|\nabla_k R|^2 + 4c(n)|R_{ijkl}|^3,$$

$$\frac{\partial w}{\partial t} \le \Delta w + \frac{1}{R}\nabla_k R \cdot \nabla_k w - \frac{R^2}{2R^3}\left[w - 4\left(S - \frac{1}{n}R^2\right)\right]^2 + \frac{8}{n}Rw - \frac{32}{n}R\left(S - \frac{1}{n}R^2\right) + 4c(n)|R_{ijkl}|^3,$$

$$\frac{\partial w}{\partial t} \le \Delta w + \frac{1}{R}\nabla_k R \cdot \nabla_k w - \frac{1}{4R}w^2 + \frac{8}{n}Rw + \frac{\tilde{c}(n)}{R}\left(S - \frac{1}{n}R^2\right)^2 + \tilde{c}(n)|R_{ijkl}|^3.$$

Lemma 5.13. *For any $T < T_2$, there exists a constant $c = c(T) > 0$ such that*

$$\frac{|\nabla_i R|^2}{R} \le \frac{c}{t}, \qquad 0 \le t \le T.$$

Proof. Let

$$w = \frac{|\nabla_i R|^2}{R} + 4\left(S - \frac{1}{n}R^2\right).$$

Then from Lemma 5.12 we have

$$\frac{\partial w}{\partial t} \le \Delta w + \frac{1}{R}\nabla_k R \cdot \nabla_k w - \frac{1}{4R}w^2 + \frac{8}{n}Rw + \frac{\tilde{c}(n)}{R}\left(S - \frac{1}{n}R^2\right)^2 + \tilde{c}(n)|R_{ijkl}|^3. \tag{58}$$

Since

$$|\mathrm{Rm}|^2 = |\mathring{\mathrm{R}}\mathrm{m}|^2 + \frac{2}{n(n-1)}R^2,$$

from Lemma 5.7 it follows that

$$\frac{1}{R^2}|\mathrm{Rm}|^2 = \frac{1}{R^2}|\mathring{\mathrm{R}}\mathrm{m}|^2 + \frac{2}{n(n-1)} \le \beta_n + \frac{2}{n(n-1)},$$

$$\frac{1}{R^2}|R_{ijkl}|^2 \le \beta_n + \frac{2}{n(n-1)} \quad \text{on } 0 \le t < T_2. \tag{59}$$

If $0 \le t \le T < T_2$, then by (59) we get

$$\frac{\tilde{c}(n)}{R}\left(S - \frac{1}{n}R^2\right)^2 \le \frac{\tilde{c}(n)}{R}|R_{ijkl}|^4 \cdot n^4 \le \frac{\hat{c}(n)}{R}R^4 = \hat{c}(n)R^3,$$

where $\hat{c}(n) > 0$ depends only on n and β_n. We still have

$$\tilde{c}(n)|R_{ijkl}|^3 \le \hat{c}(n)R^3.$$

Thus by the above two equations (58) is reduced to

$$\frac{\partial w}{\partial t} \le \Delta w + \frac{1}{R}\nabla_k R \cdot \nabla_k w - \frac{1}{4R}w^2 + \frac{8}{n}Rw + c_3R^3, \qquad 0 \le t \le T, \tag{60}$$

which together with (29) implies

$$\frac{\partial w}{\partial t} \le \Delta w + \frac{1}{R}\nabla_k R \cdot \nabla_k w - \frac{1}{4R}w^2 + c_2 w + c_4, \qquad 0 \le t \le T, \tag{61}$$

where $0 < c_2, c_4 < +\infty$ are constants depending on T. Let

$$F(x,t) = tw(x,t) = t\left[\frac{|\nabla_i R|^2}{R} + 4\left(S - \frac{1}{n}R^2\right)\right].$$

Then

$$\begin{aligned}\frac{\partial F}{\partial t} &\le \Delta F + \frac{1}{R}\nabla_k R \cdot \nabla_k F - \frac{1}{4tR}F^2 + c_2F + c_4t + \frac{F}{t}\\ &\le \Delta F + \frac{1}{R}\nabla_k R \cdot \nabla_k F - \frac{1}{4tR}F^2 + \frac{1+Tc_2}{t}F + c_4T.\end{aligned}$$

From (47) it follows that

$$\frac{|\nabla_k R|^2}{R^2} = \frac{1}{tR}\left[F - 4t\left(S - \frac{1}{n}R^2\right)\right] \le \frac{F}{tR}.$$

Finally we have

$$\begin{aligned}\frac{\partial F}{\partial t} \le \Delta F &+ \frac{1}{R}\nabla_k R \cdot \nabla_k F - \frac{|\nabla_k R|^2}{8R^2}F - \frac{1}{8tR}F^2\\ &+ \frac{1+Tc_2}{t}F + c_4T, \qquad 0 \le t \le T.\end{aligned}$$

Let $c_5 = 1 + Tc_2$ and $c_6 = c_4T$. Then

$$\begin{aligned}\frac{\partial F}{\partial t} \le \Delta F &+ \frac{1}{R}\nabla_k R \cdot \nabla_k F - \frac{|\nabla_k R|^2}{8R^2}F\\ &+ \frac{F}{tR}\left(c_5R + \frac{c_6tR}{F} - \frac{F}{8}\right), \qquad 0 \le t \le T.\end{aligned}$$

By using (29) again we get

$$c_5R \le c_7, \quad c_6tR \le c_8 \quad \text{on } 0 \le t \le T.$$

Thus

$$\begin{aligned}\frac{\partial F}{\partial t} \le \Delta F &+ \frac{1}{R}\nabla_k R \cdot \nabla_k F - \frac{|\nabla_k R|^2}{8R^2}F\\ &+ \frac{F}{tR}\left(c_7 + \frac{c_8}{F} - \frac{F}{8}\right), \qquad 0 \le t \le T.\end{aligned} \tag{62}$$

By definition we know that

$$F(x,0) \equiv 0 \quad \text{on } M. \tag{63}$$

Then from (62), (63) and Lemma 4.11 it follows that

$$F(x,t) \le c, \qquad 0 \le t \le T, \tag{64}$$

where $c > 0$ depends on T. Thus we have

$$\frac{|\nabla_i R|^2}{R} \le \frac{c}{t}, \qquad 0 \le t \le T.$$

Lemma 5.14. *We can find $\sigma > 0$ and $c(\sigma) > 0$ such that*

$$\frac{1}{R^{2-\sigma}} |\overset{\circ}{\mathrm{R}}\mathrm{m}|^2 \le c(\sigma), \qquad 0 \le t < T_2.$$

Proof. Let $f_\sigma(x,t) = |\overset{\circ}{\mathrm{R}}\mathrm{m}|^2/R^{2-\sigma}$. Then Lemma 5.5 implies that

$$\begin{aligned} \frac{\partial f\sigma}{\partial t} \le \Delta f_\sigma &+ \frac{2(1-\sigma)}{R} \nabla_k R \cdot \nabla_k f_\sigma - \frac{\sigma(1-\sigma)}{R^{4-\sigma}} |\overset{\circ}{\mathrm{R}}\mathrm{m}|^2 |\nabla_i R|^2 \\ &+ \frac{4}{R^{3-\sigma}} \left(P + \frac{1}{2}\sigma |\overset{\circ}{\mathrm{R}}\mathrm{m}|^2 S \right). \end{aligned} \tag{65}$$

From Lemmas 5.6 and 5.7 it follows that

$$P \le -\frac{1}{2n} R^2 |\overset{\circ}{\mathrm{R}}\mathrm{m}|^2, \qquad 0 \le t < T_2, \tag{66}$$

$$S \le \frac{1}{n} R^2 + |\overset{\circ}{R}_{ij}|^2 \le \left(\frac{1}{n} + c(n) \right) R^2, \qquad 0 \le t < T_2,$$

and therefore

$$\begin{aligned} &\frac{4}{R^{3-\sigma}} \left(P + \frac{\sigma}{2} |\overset{\circ}{\mathrm{R}}\mathrm{m}|^2 S \right) \\ &\quad \le \frac{4}{R^{3-\sigma}} \left[-\frac{1}{2n} R^2 |\overset{\circ}{\mathrm{R}}\mathrm{m}|^2 + \frac{\sigma}{2} \left(\frac{1}{n} + c(n) \right) R^2 |\overset{\circ}{\mathrm{R}}\mathrm{m}|^2 \right] \\ &\quad \le \frac{4}{R^{1-\sigma}} |\overset{\circ}{\mathrm{R}}\mathrm{m}|^2 \left[-\frac{1}{2n} + \widetilde{c(n)}\sigma \right], \end{aligned}$$

$$\frac{4}{R^{3-\sigma}} \left(P + \frac{\sigma}{2} |\overset{\circ}{\mathrm{R}}\mathrm{m}|^2 S \right) \le \frac{4}{R^{1-\sigma}} |\overset{\circ}{\mathrm{R}}\mathrm{m}|^2 \left[-\frac{1}{2n} + \widetilde{c(n)}\sigma \right], \qquad 0 \le t < T_2. \tag{67}$$

Now if we choose σ such that

$$0 < \sigma < \frac{1}{2n\widetilde{c(n)}}, \tag{68}$$

then

$$\frac{4}{R^{3-\sigma}}\left(P+\frac{\sigma}{2}|\overset{\circ}{\mathrm{Rm}}|^2 S\right)\leq 0, \qquad 0\leq t<T_2. \tag{69}$$

Substituting (69) into (65) gives

$$\frac{\partial f_\sigma}{\partial t}\leq \Delta f_\sigma+\frac{2(1-\sigma)}{R}\nabla_k R\cdot\nabla_k f_\sigma-\frac{\sigma(1-\sigma)}{R^2}|\nabla_i R|^2 f_\sigma,$$

or

$$\frac{\partial f_\sigma}{\partial t}\leq \Delta f_\sigma+|\nabla_k f_\sigma|^2+[1-\sigma(1-\sigma)f_\sigma]\frac{|\nabla_i R|^2}{R^2}, \qquad 0\leq t<T_2. \tag{70}$$

Since

$$f_\sigma(x,0)=\frac{|\overset{\circ}{\mathrm{Rm}}|^2}{R^{2-\sigma}}(x,0)=R^\sigma(x,0)f_0(x,0)\leq \beta_n R^\sigma(x,0),$$

from (8) it follows that

$$f_\sigma(x,0)\leq \beta_n c_0^\sigma \quad \text{on } M, \tag{71}$$

$$f_\sigma(x,t)=\frac{|\overset{\circ}{\mathrm{Rm}}|^2}{R^{2-\sigma}}(x,t)=R^\sigma(x,t)f_0(x,t).$$

By using (44) we find

$$f_\sigma(x,t)\leq \beta_n R^\sigma(x,t) \quad \text{on } M\times[0,T]. \tag{72}$$

For any $T<T_2$ we use (29) and (72) to get

$$f_\sigma(x,t)\leq \beta_n n^{2\sigma}c_1(T)^{\sigma/2} \quad \text{on } M\times[0,T]. \tag{73}$$

Since σ satisfies (68) and $\widetilde{c(n)}=\frac{1}{n}+c(n)\geq\frac{1}{n}$,

$$0<\sigma<\tfrac{1}{2}. \tag{74}$$

Therefore

$$[1-\sigma(1-\sigma)f_\sigma]\frac{|\nabla_i R|^2}{R^2}\leq 0 \quad \text{if } f_\sigma\geq\frac{1}{\sigma(1-\sigma)}. \tag{75}$$

From (70), (71), (73), (75) and Theorem 4.6 we know that

$$f_\sigma(x,t)\leq \max\left[\beta_n c_0^\sigma, \frac{1}{\sigma(1-\sigma)}\right] \quad \text{on } M\times[0,T].$$

Since $T<T_2$ is arbitrary, we get

$$f_\sigma(x,t)\leq \max\left[\beta_n c_0^\sigma, \frac{1}{\sigma(1-\sigma)}\right], \qquad 0\leq t<T_2. \tag{76}$$

From Lemma 5.14 it follows that if σ satisfies (68), then

$$\frac{1}{R^{2-\sigma}}\left(S-\frac{1}{n}R^2\right)\le n^2c(\sigma),\qquad 0\le t<T_2, \tag{77}$$

which holds since $S-\frac{1}{n}R^2=|\overset{\circ}{R}_{ij}|^2\le n^2|\overset{\circ}{R}_{ijkl}|^2$.

Lemma 5.15. *We have the inequalities*:

$$\overset{\circ}{R}_{ij}R_{kl}R_{ikjl}\le R\left(S-\frac{1}{n}R^2\right),\qquad 0\le t<T_2, \tag{78}$$

$$\begin{aligned}\frac{\partial}{\partial t}\left(S-\frac{1}{n}R^2\right)&\le\Delta\left(S-\frac{1}{n}R^2\right)-\frac{2(n-2)^2}{n(3n-2)}|\nabla_iR_{jk}|^2\\&\quad+4R\left(S-\frac{1}{n}R^2\right),\qquad 0\le t<T_2,\end{aligned} \tag{79}$$

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{|\nabla_iR|^2}{R}\right)&=\Delta\left(\frac{|\nabla_iR|^2}{R}\right)-\frac{2}{R^3}|R\nabla_i\nabla_jR-\nabla_iR\cdot\nabla_jR|^2\\&\quad+\frac{4}{R}\nabla_iR\cdot\nabla_iS-\frac{2S}{R^2}|\nabla_iR|^2.\end{aligned} \tag{80}$$

Proof. From Lemma 5.7 one can check directly that

$$\overset{\circ}{R}_{ij}R_{kl}R_{ikjl}\le R\left(S-\frac{1}{n}R^2\right),$$

or one can see [8, p. 60].

Now (79) follows directly from (46), (49), and (78); (80) follows from (53).

We want to prove the following important lemma:

Lemma 5.16. *For any $\eta>0$, we can find a constant $c(\eta)>0$ depending only on n,β_n,c_0, and η, such that*

$$|\nabla_iR|^2\le\eta R^3+c(\eta),\qquad \eta\le t<T_2. \tag{81}$$

Proof. Since

$$\frac{4}{R}\nabla_iR\cdot\nabla_iS=\frac{8}{R}\nabla_iR\cdot R_{jk}\nabla_iR_{jk},$$

we have

$$\frac{4}{R}\nabla_iR\cdot\nabla_iS\le\frac{2}{R^2}|R_{jk}|^2|\nabla_iR|^2+8|\nabla_iR_{jk}|^2=\frac{2S}{R^2}|\nabla_iR|^2+8|\nabla_iR_{jk}|^2,$$

which reduces (80) to

$$\frac{\partial}{\partial t}\left(\frac{|\nabla_iR|^2}{R}\right)\le\Delta\left(\frac{|\nabla_iR|^2}{R}\right)+8|\nabla_iR_{jk}|^2. \tag{82}$$

From Lemma 5.14 it follows that if $0 < \delta < 1/[2n\widetilde{c(n)}]$, then we can find a constant $c_1 = c_1(\beta_n, c_0, \delta) > 0$ such that

$$S - \frac{1}{n}R^2 \le c_1 R^{2-\delta}, \qquad 0 \le t < T_2 \tag{83}$$

(actually this comes from (77)). By (79) we get

$$\begin{aligned}\frac{\partial}{\partial t}\left(S - \frac{1}{n}R^2\right) \le \Delta\left(S - \frac{1}{n}R^2\right) - \frac{2(n-2)^2}{n(3n-2)}|\nabla_i R_{jk}|^2 \\ + 4c_1 R^{3-\delta}, \qquad 0 \le t < T_2.\end{aligned} \tag{84}$$

Since $\frac{\partial R}{\partial t} = \Delta R + 2S$, $S - \frac{1}{n}R^2 \ge 0$, we have

$$\begin{aligned}\frac{\partial R}{\partial t} &\ge \Delta R + \frac{2}{n}R^2, \\ \frac{\partial}{\partial t}R^2 = 2R\frac{\partial R}{\partial t} &= 2R\Delta R + 4RS, \\ &= \Delta R^2 - 2|\nabla_i R|^2 + 4RS,\end{aligned} \tag{85}$$

and therefore

$$\frac{\partial R^2}{\partial t} \ge \Delta R^2 - 2|\nabla_i R|^2 + \frac{4}{n}R^3. \tag{86}$$

From (82), (84), and (86) it follows that for any $\eta > 0$

$$\begin{aligned}&\frac{\partial}{\partial t}\left[\frac{|\nabla_i R|^2}{R} + c_3\left(S - \frac{1}{n}R^2\right) - \eta R^2\right] \\ &\le \Delta\left[\frac{|\nabla_i R|^2}{R} + c_3\left(S - \frac{1}{n}R^2\right) - \eta R^2\right] + 4c_1c_3R^{3-\delta} + 2\eta|\nabla_i R|^2 \\ &\quad + \left[8 - \frac{2(n-2)^2}{n(3n-2)}c_3\right]|\nabla_i R_{jk}|^2 - \frac{4}{n}\eta R^3, \qquad 0 \le t < T_2.\end{aligned} \tag{87}$$

If we choose c_3 such that

$$8 - \frac{2(n-2)^2}{n(3n-2)}c_3 \le -\frac{4\eta(n-1)(n+2)}{3n-2},$$

then from Lemma 5.8 we have

$$\begin{aligned}\left[8 - \frac{2(n-2)^2}{n(3n-2)}c_3\right]|\nabla_i R_{jk}|^2 &\le -\frac{4\eta(n-1)(n+2)}{3n-2}|\nabla_i R_{jk}|^2 \\ &\le -2\eta|\nabla_i R|^2, \\ \left[8 - \frac{2(n-2)^2}{n(3n-2)}c_3\right]|\nabla_i R_{jk}|^2 + 2\eta|\nabla_i R|^2 &\le 0,\end{aligned}$$

and therefore, in consequence of (87),

$$\frac{\partial}{\partial t}\left[\frac{|\nabla_i R|^2}{R} + c_3\left(S - \frac{1}{n}R^2\right) - \eta R^2\right]$$

(88)
$$\leq \Delta\left[\frac{|\nabla_i R|^2}{R} + c_3\left(S - \frac{1}{n}R^2\right) - \eta R^2\right] + 4c_1c_3R^{3-\delta} - \frac{4}{n}\eta R^3, \qquad 0 \leq t < T_2.$$

Let

$$F = \frac{|\nabla_i R|^2}{R} + c_3\left(S - \frac{1}{n}R^2\right) - \eta R^2 - CR.$$

Then from (85) and (88) we get

$$\frac{\partial F}{\partial t} \leq \Delta F + 4c_1c_3R^{3-\delta} - \frac{4}{n}\eta R^3 - \frac{2}{n}CR^2,$$

(89) $$\frac{\partial F}{\partial t} \leq \Delta F + \left[4c_1c_3R^{1-\delta} - \frac{4}{n}\eta R - \frac{2}{n}C\right]R^2, \qquad 0 \leq t < T_2.$$

If we choose C large enough, then

$$4c_1c_3R^{1-\delta} - \frac{4}{n}\eta R - \frac{2}{n}C \leq 0 \quad \text{for all } R \geq 0,$$

where C depends only on $\beta_n, c_0, \delta, \eta, c_1$, and c_3. We have

(90) $$\frac{\partial F}{\partial t} \leq \Delta F, \qquad 0 \leq t < T_2.$$

By the definition of F,

$$F \leq \frac{|\nabla_i R|^2}{R} + c_3\left(S - \frac{1}{n}R^2\right).$$

Suppose T_0 is the constant in Lemma 5.3. Then from Lemma 5.13 we know that

(91) $$\frac{|\nabla_i R|^2}{R} \leq \frac{c_5(c_0, n)}{t}, \qquad 0 \leq t \leq \frac{1}{2}T_0,$$

which implies

(92) $$\frac{|\nabla_i R|^2}{R} \leq c_6(\eta), \qquad \eta \leq t \leq \frac{1}{2}T_0,$$

and from Theorem 3.4 it follows that

$$0 \leq S - \frac{1}{n}R^2 \leq c_7(c_0, n), \qquad 0 \leq t \leq \frac{1}{2}T_0.$$

Thus

(93) $$F(x,t) \leq c_8(n, \beta_n, c_0, \eta), \qquad \eta \leq t \leq \frac{1}{2}T_0,\ x \in M.$$

For any $T < T_2$, by (28) and Lemma 5.13 we get

(94)
$$F(x,t) \leq \frac{|\nabla_i R|^2}{R} + c_3\left(S - \frac{1}{n}R^2\right) \leq c_9(n,\beta_n,c_0,\eta,T) \quad \text{on } \eta \leq t \leq T.$$

From (90), (93), (94) and Theorem 4.6 we have

$$F(x,t) \leq c_8(n,\beta_n,c_0,\eta), \qquad \eta \leq t \leq T.$$

Since $T < T_2$ is arbitrary, we have

(95)
$$F(x,t) \leq c_8, \qquad \eta \leq t < T_2,$$
$$\frac{|\nabla_i R|^2}{R} + c_3\left(S - \frac{1}{n}R^2\right) - \eta R^2 - CR \leq c_8(\eta), \qquad \eta \leq t < T_2,$$
$$|\nabla_i R|^2 \leq \eta R^3 + CR^2 + c_8(\eta)R, \qquad \eta \leq t < T_2.$$

If we replace η by $\frac{1}{2}\eta$, then

$$|\nabla_i R|^2 \leq \tfrac{1}{2}\eta R^3 + CR^2 + c_8(\eta)R, \qquad \eta \leq t < T_2,$$

and therefore

$$|\nabla_i R|^2 \leq \eta R^3 + C(\eta), \qquad \eta \leq t < T_2.$$

Note. $C(\eta) > 0$ in (81) depends only on n, β_n, c_0, η, and is independent of T_2.

Lemma 5.17. *There exists a constant $C > 0$ depending only on n, β_n, and c_0 such that*

(96)
$$0 < R(x,t) \leq C \quad \text{on } 0 \leq t < T_2.$$

Proof. From Lemma 5.4 and (13) we know respectively that $R(x,t) > 0$ on $0 \leq t < T_2$, and that

(97)
$$R(x,t) \leq c_1(n,c_0) \quad \text{on } 0 \leq t \leq \tfrac{1}{2}T_0.$$

For any $\eta > 0$, by using Lemma 5.16 we can find a constant $C(\eta) > 0$ such that

$$|\nabla_i R| \leq \tfrac{1}{2}\eta^2 R^{3/2} + C(\eta), \qquad \eta \leq t < T_2.$$

If $R_{\max} \to \infty$ as $t \to T_2$, we can find θ such that $\eta \leq \theta < T_2$ and

(98)
$$C(\eta) \leq \tfrac{1}{2}\eta^2 R_{\max}^{3/2}, \quad \text{while } t = \theta.$$

Thus

(99)
$$|\nabla_i R| \leq \eta^2 R_{\max}^{3/2} \quad \text{at } t = \theta.$$

Fix a point $x \in M$ such that

$$R(x,\theta) \geq (1-\eta)\mathrm{Max}_{y \in M} R(y,\theta).$$

Then on any geodesic out of x of length at most $S = \frac{1}{\eta} R_{\max}^{1/2}$ we have $R \geq (1 - 2\eta) R_{\max}$, and from Lemma 5.7 we know that there exists a fixed $\varepsilon_0 > 0$ such that $R_{ij} \geq \varepsilon_0 R g_{ij}$. Thus on any geodesic out of x of length at most $S = \frac{1}{\eta} R_{\max}^{1/2}$ we have

$$R_{ij} \geq \varepsilon_0 (1 - 2\eta) R_{\max} g_{ij}.$$

If $\eta > 0$ is small enough, it follows that every geodesic from x of length $S = \frac{1}{\eta} R_{\max}^{1/2}$ has a conjugate point by the well-known theorem of Myers, which can be found in [Theorem 1.26, Cheeger and Ebin [2]]. Thus

$$\gamma_\theta(x, y) \leq \frac{1}{\eta} R_{\max}^{1/2} \quad \forall y \in M. \tag{100}$$

Since $\theta < T_2$, Assumption A of §4 holds on $M \times [0, \theta]$. By using Lemma 4.1 we know that ds_θ^2 is equivalent to ds_0^2. Since M is a complete noncompact manifold with respect to ds_0^2, M is a complete noncompact manifold with respect to ds_θ^2; therefore (100) is impossible. This means that (98) cannot be true for any $\theta \in [\eta, T_2)$, so that

$$C(\eta) > \frac{1}{2} \eta^2 R_{\max}^{3/2}, \qquad \eta \leq t < T_2,$$

$$R_{\max} \leq \left(\frac{2C(\eta)}{\eta^2} \right)^{2/3}, \quad \eta \leq t < T_2.$$

Thus we can find $\tilde{C}(\eta) > 0$ such that

$$R(x, t) \leq \tilde{C}(\eta), \qquad \eta \leq t < T_2. \tag{101}$$

Fix $0 < \eta \leq \frac{1}{2} T_0$. Then (97) and (101) imply the lemma.

Proof of Theorem 5.2. Now we are going to prove the long time existence theorem. We need to prove that $T_1 = +\infty$.

Suppose $T_1 < +\infty$, from (15) we get

$$0 < T_0 \leq T_1 < +\infty. \tag{102}$$

By the definition of T_1 in (14), for any $\varepsilon > 0$ we can find a constant

$$T_1 - \varepsilon < T_2 \leq T_1 \tag{103}$$

and a solution $g_{ij}(x, t)$ of the evolution equation on $M \times [0, T_2)$ such that for any $T < T_2$, the solution $g_{ij}(x, t)$ satisfies Assumption A of §4 on $M \times [0, T]$, and (13) holds on $M \times [0, \frac{1}{2} T_0]$. Thus from Lemmas 5.4 and 5.17 we know that

$$\begin{cases} R_{ijkl}(x, t) > 0 \\ 0 < R(x, t) \leq C \end{cases} \quad \text{on } M \times [0, T_2). \tag{104}$$

By (59) and (104) we get

$$|R_{ijkl}|^2 \le \left[\beta_n + \frac{2}{n(n-1)}\right] C^2 \quad \text{on } M \times [0, T_2). \tag{105}$$

From (104) and (105) it follows that

$$0 < R_{ijij}(x,t) \le \left[\beta_n + \frac{2}{n(n-1)}\right]^{1/2} C \text{ on } M \times [0, T_2). \tag{106}$$

Now we consider the evolution equation

$$\frac{\partial}{\partial t}\tilde{g}_{ij}(x,t) = -2\tilde{R}_{ij}(x,t), \qquad \tilde{g}_{ij}(x,0) = g_{ij}(x, T_1 - \varepsilon). \tag{107}$$

Since $T_1 - \varepsilon < T_2$, from (106) we have

$$0 < R_{ijij}(x, T_1 - \varepsilon) \le \left[\beta_n + \frac{2}{n(n-1)}\right]^{1/2} C, \tag{108}$$

where C is independent of ε.

From Theorem 3.4 we know that (107) has a solution $\tilde{g}_{ij}(x,t)$ on $0 \le t < \delta$, $\delta = \delta(n, \beta_n, c_0, C)$ depending only on n, β_n, c_0, and C; in particular, δ is independent of ε. By Theorem 3.4 we still have

$$\sup_M |\nabla^m \tilde{R}_{ijkl}(x,t)|^2 \le \tilde{c}_{m+1}/t^m, \qquad 0 \le t \le \delta, x \in M,\ m \ge 0. \tag{109}$$

Define

$$\begin{aligned} g^*_{ij}(x,t) &= g_{ij}(x,t), & 0 \le t \le T_1 - \varepsilon, \\ g^*_{ij}(x,t) &= \tilde{g}_{ij}(x, t - T_1 + \varepsilon), & T_1 - \varepsilon < t \le T_1 - \varepsilon + \delta. \end{aligned} \tag{110}$$

Then $g^*_{ij}(x,t) > 0$ on $M \times [0, T_1 - \varepsilon + \delta]$, and

$$\begin{aligned} \frac{\partial}{\partial t} g^*_{ij}(x,t) &= -2R^*_{ij}(x,t), \qquad 0 \le t \le T_1 - \varepsilon + \delta, \\ g^*_{ij}(x,0) &= g_{ij}(x) \quad \text{on } M. \end{aligned} \tag{111}$$

By the regularity theorem of parabolic equation we know that

$$g^*_{ij}(x,t) \in C^\infty \quad \text{on } M \times [0, T_1 - \varepsilon + \delta].$$

Thus $g^*_{ij}(x,t)$ is a solution of evolution equation (9) on $M \times [0, T_1 - \varepsilon + \delta]$, and

$$g^*_{ij}(x,t) \equiv g_{ij}(x,t), \qquad 0 \le t \le T_1 - \varepsilon. \tag{112}$$

Since $\delta > 0$ depends only on n, β_n, c_0 and C, with C depending only on n, β_n and c_0, thus $\delta > 0$ depends only on n, β_n and c_0. If we choose $\varepsilon > 0$ small enough such that

$$0 < \varepsilon \le \min\left\{\frac{\delta}{2}, \frac{T_0}{2}\right\}, \tag{113}$$

then from (15), (112), and (113) we have

$$g^*_{ij}(x,t) \equiv g_{ij}(x,t), \qquad 0 \le t \le \tfrac{1}{2}T_0. \tag{114}$$

Since $g_{ij}(x,t)$ satisfies (13) on $M \times [0, \frac{1}{2}T_0]$, $g^*_{ij}(x,t)$ also satisfies (13) on $M \times [0, \frac{1}{2}T_0]$.

Because $T_1 - \varepsilon < T_2$, by the definition of $g_{ij}(x,t)$ and (112) we know that both $g_{ij}(x,t)$ and $g^*_{ij}(x,t)$ satisfy Assumption A of §4 on $M \times [0, T_1 - \varepsilon]$. Therefore we get the following:

$$\begin{aligned} &0 < R^*_{ijij}(x,0) \le k_0, \qquad x \in M, \\ &|R^*_{ijkl}(x,t)|^2 \le c^*_1, \qquad x \in M,\ 0 \le t \le T_1 - \varepsilon, \\ &\int_0^{T_1-\varepsilon} |\nabla_p R^*_{ijkl}(x,t)|\, dt \le c^*_2, \qquad x \in M. \end{aligned} \tag{115}$$

From (109) it follows that

$$\begin{aligned} &|R^*_{ijkl}(x,t)|^2 \le \tilde{c}_1 \quad \text{on } T_1 - \varepsilon \le t \le T_1 - \varepsilon + \delta, \\ &|\nabla_p R^*_{ijkl}(x,t)|^2 \le \tilde{c}_2/(t - T_1 + \varepsilon), \qquad T_1 - \varepsilon \le t \le T_1 - \varepsilon + \delta. \end{aligned} \tag{116}$$

Thus

$$\int_{T_1-\varepsilon}^{T_1-\varepsilon+\delta} |\nabla_p R^*_{ijkl}(x,t)|\, dt \le \int_{T_1-\varepsilon}^{T_1-\varepsilon+\delta} \frac{\tilde{c}_2^{1/2}}{\sqrt{t - T_1 + \varepsilon}}\, dt = c_3 < +\infty. \tag{117}$$

By (115), (116), and (117) we get

$$\begin{aligned} &0 < R^*_{ijij}(x,0) \le k_0, \qquad x \in M, \\ &|R^*_{ijkl}(x,t)|^2 \le \max\{c^*_1, \tilde{c}_1\}, \qquad x \in M, 0 \le t \le T_1 - \varepsilon + \delta, \\ &\int_0^{T_1-\varepsilon+\delta} |\nabla_p R^*_{ijkl}(x,t)|\, dt \le c^*_2 + c_3 \quad \forall x \in M. \end{aligned} \tag{118}$$

Therefore $g^*_{ij}(x,t)$ satisfies Assumption A of §4 on $M \times [0, T_1 - \varepsilon + \delta]$ and satisfies (13) on $M \times [0, \frac{1}{2}T_0]$. Thus from (14) and (113) we know respectively that $T_1 \ge T_1 - \varepsilon + \delta$, and that $T_1 \ge T_1 + \delta/2 > T_1$. Since this is impossible, $T_1 = +\infty$ and we can find a solution of evolution equation (9) on $M \times [0, +\infty)$. Hence the proof of Theorem 5.2 is complete.

Corollary 5.18. *Suppose $g_{ij}(x,t) > 0$ is the metric constructed in Theorem 5.2 on $M \times [0, +\infty)$. We still have*

$$\begin{aligned} &R_{ijkl}(x,t) > 0, \\ &0 < R(x,t) \le C, \qquad 0 \le t < +\infty, \\ &|\overset{\circ}{\mathrm{R}}\mathrm{m}|^2 \le \beta_n R^2, \end{aligned} \tag{119}$$

$$\sup_M |\nabla^m R_{ijkl}(x,t)|^2 \le c_{m+1}/t^m, \qquad 0 \le t \le \tfrac{1}{2}T_0,\ x \in M,\ m \ge 0, \tag{120}$$

where $C > 0$ *and* $c_{m+1} > 0$ *are constants depending only on* n, β_n *and* c_0. *Moreover, for any* $0 < T < +\infty$, $g_{ij}(x,t)$ *satisfies Assumption* A *of* §4 *on* $M \times [0, T]$.

Proof. We can prove this corollary by using Lemmas 5.4, 5.7, and 5.17, and (112) directly.

6. Controlling the scalar curvature

We have shown in the last section that the scalar curvature of M is positive and bounded from above for all time $0 \le t < +\infty$. In this section we want to show that the scalar curvature R actually tends to zero as time $t \to +\infty$.

Suppose M is an n-dimensional complete noncompact Riemannian manifold with metric $g_{ij}(x) > 0$. Then the curvature of M satisfies the following condition:

$$|\overset{\circ}{\mathrm{R}}\mathrm{m}|^2 \le \beta R^2, \qquad 0 < R \le c_0, \tag{1}$$

where β and c_0 are constants and $0 < \beta \le \delta_n/2n(n-1)$.

Now consider the evolution equation on M:

$$\begin{aligned} &\frac{\partial}{\partial t} g_{ij}(x,t) = -2R_{ij}(x,t), \\ &g_{ij}(x,0) = g_{ij}(x), \qquad x \in M. \end{aligned} \tag{2}$$

From Theorem 5.2 we can find a solution of this evolution equation for all time $0 \le t < +\infty$ and the solution satisfies the properties mentioned in Corollary 5.18. Thus we can find a constant $C > 0$ such that

$$|\overset{\circ}{\mathrm{R}}\mathrm{m}|^2 \le \beta R^2, \qquad 0 < R(x,t) \le C, \tag{3}$$

for all $0 \le t < +\infty$.

Let $0 \le \sigma < \frac{1}{2}$ and $f_\sigma(x,t) = (|\overset{\circ}{\mathrm{R}}\mathrm{m}|^2/R^{2-\sigma})(x,t)$. Then from Lemma 5.5 it follows that

$$\begin{aligned} \frac{\partial f_\sigma}{\partial t} &= \Delta f_\sigma + \frac{2(1-\sigma)}{R} \nabla_k R \cdot \nabla_k f_\sigma - \frac{\sigma(1-\sigma)}{R^{4-\sigma}} |\overset{\circ}{\mathrm{R}}\mathrm{m}|^2 |\nabla_i R|^2 \\ &\quad - \frac{2}{R^{4-\sigma}} |R \nabla_p R_{ijkl} - R_{ijkl} \nabla_p R|^2 + \frac{4}{R^{3-\sigma}} \left(P + \frac{\sigma}{2} |\overset{\circ}{\mathrm{R}}\mathrm{m}|^2 S \right). \end{aligned} \tag{4}$$

From Corollary 5.18 we have

$$R_{ij} > 0, \qquad 0 \le t < +\infty, \tag{5}$$

and therefore

$$\frac{1}{n} R^2 \le S \le R^2, \qquad 0 \le t < +\infty. \tag{6}$$

By (3) and Lemma 5.6 we get

$$P \le -\frac{1}{2n}|\mathring{\mathrm{R}}\mathrm{m}|^2 R^2, \qquad 0 \le t < +\infty, \tag{7}$$

which implies

$$\begin{aligned}\frac{4}{R^{3-\sigma}}\left(P+\frac{\sigma}{2}|\mathring{\mathrm{R}}\mathrm{m}|^2 S\right) &\le \frac{4}{R^{3-\sigma}}\left(-\frac{1}{2n}|\mathring{\mathrm{R}}\mathrm{m}|^2 R^2+\frac{\sigma}{2}|\mathring{\mathrm{R}}\mathrm{m}|^2 R^2\right)\\ &= \frac{2R^2}{R^{3-\sigma}}\left(\sigma-\frac{1}{n}\right)|\mathring{\mathrm{R}}\mathrm{m}|^2 = 2\left(\sigma-\frac{1}{n}\right)Rf_\sigma.\end{aligned}$$

Substituting the above equation into (4) gives

$$\frac{\partial f_\sigma}{\partial t} \le \Delta f_\sigma + \frac{2(1-\sigma)}{R}\nabla_k R\cdot\nabla_k f_\sigma - 2\left(\frac{1}{n}-\sigma\right)Rf_\sigma - \frac{|\nabla_i R|^2}{R^2}\sigma(1-\sigma)f_\sigma. \tag{8}$$

Lemma 6.1. *There exists a constant $c_1 > 0$ depending only on c_0, n, and σ such that for $0 < \sigma \le 1/(2n)$ we have*

$$\frac{|\mathring{\mathrm{R}}\mathrm{m}|^2}{R^{2-\sigma}} \le \frac{\beta c_1}{(t+1)^\sigma}, \qquad 0 \le t < +\infty. \tag{9}$$

Proof. Because $0 < \sigma \le 1/(2n)$, from (8) we have

$$\frac{\partial f_\sigma}{\partial t} \le \Delta f_\sigma + \frac{2(1-\sigma)}{R}\nabla_k R\cdot\nabla_k f_\sigma - \frac{1}{n}Rf_\sigma. \tag{10}$$

Let

$$\begin{aligned}\varphi(t) &= \left(\frac{1}{c_0}+\frac{t}{n\sigma}\right)^{-1}, \qquad 0 \le t < +\infty,\\ \psi(t) &= \beta\varphi(t)^\sigma, \qquad 0 \le t < +\infty.\end{aligned}$$

Then

$$\begin{aligned}\frac{\partial\psi}{\partial t} &= \beta\sigma\varphi(t)^{\sigma-1}\frac{\partial\varphi}{\partial t} = \beta\sigma\varphi(t)^{\sigma-1}\left[-\frac{1}{n\sigma}\left(\frac{1}{c_0}+\frac{t}{n\sigma}\right)^{-2}\right]\\ &= -\frac{1}{n}\beta\varphi(t)^{\sigma-1}\varphi(t)^2 = -\frac{1}{n}\beta\varphi(t)^{\sigma+1},\end{aligned}$$

$$\frac{\partial\psi}{\partial t} = -\frac{1}{n}\varphi\psi, \qquad 0 \le t < +\infty. \tag{11}$$

Thus

$$\frac{\partial\psi}{\partial t} = \Delta\psi + \frac{2(1-\sigma)}{R}\nabla_k R\cdot\nabla_k\psi - \frac{1}{n}\varphi\psi. \tag{12}$$

From (8), (12) and $0 < \sigma \le 1/(2n)$ it follows that

$$\frac{\partial}{\partial t}(f_\sigma - \psi) \le \Delta(f_\sigma - \psi) + \frac{2(1-\sigma)}{R}\nabla_k R \cdot \nabla_k(f_\sigma - \psi) + \frac{1}{n}\varphi\psi - \frac{1}{n}Rf_\sigma - \sigma(1-\sigma)\frac{|\nabla_k R|^2}{R^2}f_\sigma.$$

Let $F(x,t) = f_\sigma(x,t) - \psi(t)$ and

$$Q(F,x,t) = \frac{2(1-\sigma)}{R}\nabla_k R \cdot \nabla_k(f_\sigma - \psi) + \frac{1}{n}\varphi\psi - \frac{1}{n}Rf_\sigma - \sigma(1-\sigma)\frac{|\nabla_k R|^2}{R^2}f_\sigma.$$

Then

$$\frac{\partial F}{\partial t} \le \Delta F + Q(F,x,t), \qquad 0 \le t < +\infty. \tag{13}$$

Since $|\overset{\circ}{\mathrm{R}}\mathrm{m}|^2 \le \beta R^2$ on $0 \le t < +\infty$, we have

$$f_\sigma(x,t) = \frac{1}{R^{2-\sigma}}|\overset{\circ}{\mathrm{R}}\mathrm{m}|^2 \le \beta R^\sigma,$$

$$f_\sigma(x,t) \le \beta R^\sigma(x,t), \qquad 0 \le t < +\infty. \tag{14}$$

In particular,

$$f_\sigma(x,0) \le \beta R^\sigma(x,0) \le \beta c_0^\sigma$$

by (1). Since $\varphi(0) = c_0$,

$$F(x,0) = f_\sigma(x,0) - \psi(0) \le \beta c_0^\sigma - \beta\varphi(0)^\sigma \le 0, \qquad x \in M. \tag{15}$$

Therefore if $F(x,t) \ge 0$, then

$$0 \le F(x,t) = f_\sigma(x,t) - \psi(t) \le \beta R^\sigma(x,t) - \beta\varphi(t)^\sigma,$$

$$\varphi(t) \le R(x,t),$$

$$\begin{aligned} Q(F,x,t) &\le \frac{2(1-\sigma)}{R}\nabla_k R \cdot \nabla_k F + \frac{1}{n}R\psi - \frac{1}{n}Rf_\sigma \\ &\quad - \sigma(1-\sigma)\frac{|\nabla_k R|^2}{R^2}f_\sigma \\ &\le \frac{2(1-\sigma)}{R}\nabla_k R \cdot \nabla_k F - \frac{1}{n}RF - \sigma(1-\sigma)\frac{|\nabla_k R|^2}{R^2}F \\ &\quad - \sigma(1-\sigma)\frac{|\nabla_k R|^2}{R^2}\psi \\ &\le \frac{2(1-\sigma)}{R}\nabla_k R \cdot \nabla_k F - \sigma(1-\sigma)\frac{|\nabla_k R|^2}{R^2}F. \end{aligned}$$

Thus if $F > 0$, we get

$$Q(F,x,t) \leq \frac{|\nabla_k F|^2}{\sigma(1-\sigma)F}. \tag{16}$$

Suppose $m \geq 3$ is an odd integer, and define $H(x,t) = F(x,t)^m$. Then from (15) it follows that

$$H(x,0) \leq 0. \tag{17}$$

If $H(x,t) > 0$, then $F(x,t) > 0$, and we have

$$\begin{aligned}\frac{\partial H}{\partial t} &= F^{m-1} \cdot m\frac{\partial F}{\partial t} \leq mF^{m-1}[\Delta F + Q(F,x,t)] \\ &= \Delta H - m(m-1)F^{m-2}|\nabla_k F|^2 + mF^{m-1}Q(F,x,t) \\ &\leq \Delta H - m(m-1)F^{m-2}|\nabla_k F|^2 + \frac{m}{\sigma(1-\sigma)}F^{m-2}|\nabla_k F|^2.\end{aligned}$$

If $m \geq 1 + 1/[\sigma(1-\sigma)]$, then

$$\frac{\partial H}{\partial t} \leq \Delta H \quad \text{for } H \geq 0. \tag{18}$$

From (3) and (14) we know that

$$F(x,t) \leq f_\sigma(x,t) \leq \beta R^\sigma(x,t) \leq \beta c^\sigma, \qquad 0 \leq t < +\infty.$$

Thus

$$H(x,t) \leq \beta^m c^{m\sigma}, \qquad 0 \leq t < +\infty. \tag{19}$$

By (17), (18), (19) and Lemma 4.5 we get

$$H(x,t) \leq 0, \qquad 0 \leq t < +\infty;$$

thus

$$\begin{aligned} F(x,t) &\leq 0, \qquad 0 \leq t < +\infty, \\ f_\sigma(x,t) &\leq \psi(t), \qquad 0 \leq t < +\infty, \\ f_\sigma(x,t) &\leq \beta\left(\frac{1}{c_0} + \frac{t}{n\sigma}\right)^{-\sigma}, \qquad 0 \leq t < +\infty. \end{aligned} \tag{20}$$

Since

$$\left(\frac{1}{c_0} + \frac{t}{n\sigma}\right)^{-\sigma} \leq \frac{c_1(n,c_0,\sigma)}{(t+1)^\sigma}, \qquad 0 \leq t < +\infty;$$

we have

$$\frac{|\overset{\circ}{\mathrm{Rm}}|^2}{R^{2-\sigma}}(x,t) \leq \frac{\beta c_1}{(t+1)^\sigma}, \qquad 0 \leq t < +\infty, \tag{21}$$

which completes the proof of Lemma 6.1.

Now we want to estimate the gradient of the scalar curvature. From Lemma 5.11 we have

(22)
$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{|\nabla_i R|^2}{R^\gamma}\right) &= \Delta\left(\frac{|\nabla_i R|^2}{R^\gamma}\right) + \frac{\gamma}{R}\nabla_k R\cdot\nabla_k\left(\frac{|\nabla_i R|^2}{R^\gamma}\right)\\ &\quad -\gamma\left(1-\frac{\gamma}{2}\right)\frac{|\nabla_i R|^4}{R^{\gamma+2}} - \frac{2}{R^\gamma}|\nabla_i\nabla_j R - \frac{\gamma}{2R}\nabla_i R\cdot\nabla_j R|^2\\ &\quad -\frac{2\gamma S}{R^{\gamma+1}}|\nabla_i R|^2 + \frac{4}{R^\gamma}\nabla_i R\cdot\nabla_i S, \qquad 0\le t<+\infty.\end{aligned}$$

Let $1<\gamma<2$. Then

$$\begin{aligned}\frac{4}{R^\gamma}\nabla_i R\cdot\nabla_i S &= \frac{8}{R^\gamma}\nabla_i R\cdot R_{jk}\nabla_i R_{jk}\\ &\le \frac{2}{R^{\gamma+1}}|R_{jk}|^2|\nabla_i R|^2 + \frac{16}{R^{\gamma-1}}|\nabla_i R_{jk}|^2\\ &= \frac{2S}{R^{\gamma+1}}|\nabla_i R|^2 + \frac{16}{R^{\gamma-1}}|\nabla_i R_{jk}|^2\\ &\le \frac{2\gamma S}{R^{\gamma+1}}|\nabla_i R|^2 + \frac{16}{R^{\gamma-1}}|\nabla_i R_{jk}|^2,\end{aligned}$$

i.e.,

$$\frac{4}{R^\gamma}\nabla_i R\cdot\nabla_i S - \frac{2\gamma S}{R^{\gamma+1}}|\nabla_i R|^2 \le \frac{16}{R^{\gamma-1}}|\nabla_i R_{jk}|^2, \qquad 0\le t<+\infty. \tag{23}$$

Substituting (23) into (22) yields

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{|\nabla_i R|^2}{R^\gamma}\right) &\le \Delta\left(\frac{|\nabla_i R|^2}{R^\gamma}\right) + \frac{\gamma}{R}\nabla_k R\cdot\nabla_k\left(\frac{|\nabla_i R|^2}{R^\gamma}\right)\\ &\quad -\gamma\left(1-\frac{\gamma}{2}\right)\frac{|\nabla_i R|^4}{R^{\gamma+2}} + \frac{16}{R^{\gamma-1}}|\nabla_i R_{jk}|^2,\\ &\quad 0\le t<+\infty.\end{aligned} \tag{24}$$

From (79) we get

$$\begin{aligned}\frac{\partial}{\partial t}\left(S-\frac{1}{n}R^2\right) &\le \Delta\left(S-\frac{1}{n}R^2\right) - \frac{2(n-2)^2}{n(3n-2)}|\nabla_i R_{jk}|^2\\ &\quad + 4R\left(S-\frac{1}{n}R^2\right), \qquad 0\le t<+\infty.\end{aligned} \tag{25}$$

Since $\partial R/\partial t = \Delta R + 2S$,

$$\frac{\partial}{\partial t}R^{1-\gamma} = \Delta R^{1-\gamma} - \frac{\gamma(\gamma-1)}{R^{\gamma+1}}|\nabla_i R|^2 + \frac{2(1-\gamma)S}{R^\gamma}.$$

Therefore

$$\tag{26} \begin{aligned} &\frac{\partial}{\partial t}\left[R^{1-\gamma}\left(S-\frac{1}{n}R^2\right)\right] \\ &\quad\le \Delta\left[R^{1-\gamma}\left(S-\frac{1}{n}R^2\right)\right]-2\nabla_k R^{1-\gamma}\cdot\nabla_k\left(S-\frac{1}{n}R^2\right) \\ &\quad -\frac{2(n-2)^2}{n(3n-2)}R^{1-\gamma}|\nabla_i R_{jk}|^2+4R^{2-\gamma}\left(S-\frac{1}{n}R^2\right) \\ &\quad -\frac{\gamma(\gamma-1)}{R^{\gamma+1}}\left(S-\frac{1}{n}R^2\right)|\nabla_i R|^2+\frac{2(1-\gamma)}{R^\gamma}S\left(S-\frac{1}{n}R^2\right). \end{aligned}$$

Let $H=R^{1-\gamma}(S-\frac{1}{n}R^2)$. Then

$$\nabla_k H=R^{1-\gamma}\nabla_k\left(S-\frac{1}{n}R^2\right)+\left(S-\frac{1}{n}R^2\right)\frac{(1-\gamma)}{R^\gamma}\nabla_k R,$$

$$\frac{1}{R^\gamma}\nabla_k\left(S-\frac{1}{n}R^2\right)=\frac{1}{R}\nabla_k H+\frac{(\gamma-1)}{R^{\gamma+1}}\left(S-\frac{1}{n}R^2\right)\nabla_k R,$$

$$\begin{aligned} -2\nabla_k R^{1-\gamma}\cdot\nabla_k\left(S-\frac{1}{n}R^2\right)&=\frac{2(\gamma-1)}{R}\nabla_k R\cdot\nabla_k H \\ &\quad+\frac{2(\gamma-1)^2}{R^{\gamma+1}}\left(S-\frac{1}{n}R^2\right)|\nabla_i R|^2, \end{aligned}$$

$$\tag{27} \begin{aligned} &-2\nabla_k R^{1-\gamma}\cdot\nabla_k\left(S-\frac{1}{n}R^2\right) \\ &\quad=\frac{\gamma}{R}\nabla_k R\cdot\nabla_k H+\frac{\gamma-2}{R}\nabla_k R\cdot\nabla_k H \\ &\quad\quad+\frac{2(\gamma-1)^2}{R^{\gamma+1}}\left(S-\frac{1}{n}R^2\right)|\nabla_i R|^2 \\ &\quad=\frac{\gamma}{R}\nabla_k R\cdot\nabla_k H+\frac{(\gamma-2)}{R}\nabla_k R \\ &\quad\quad\cdot\left[R^{1-\gamma}\nabla_k\left(S-\frac{1}{n}R^2\right)+\left(S-\frac{1}{n}R^2\right)\frac{(1-\gamma)}{R^\gamma}\nabla_k R\right] \\ &\quad\quad+\frac{2(\gamma-1)^2}{R^{\gamma+1}}\left(S-\frac{1}{n}R^2\right)|\nabla_i R|^2 \\ &\quad=\frac{\gamma}{R}\nabla_k R\cdot\nabla_k H+\frac{\gamma-2}{R^\gamma}\nabla_k R\cdot\nabla_k S-\frac{2(\gamma-2)}{nR^{\gamma-1}}|\nabla_k R|^2 \\ &\quad\quad+\frac{(2-\gamma)(\gamma-1)}{R^{\gamma+1}}\left(S-\frac{1}{n}R^2\right)|\nabla_k R|^2 \\ &\quad\quad+\frac{2(\gamma-1)^2}{R^{\gamma+1}}\left(S-\frac{1}{n}R^2\right)|\nabla_k R|^2. \end{aligned}$$

Since

$$\frac{1}{R^\gamma}\nabla_k R\cdot\nabla_k S=\frac{2}{R^\gamma}\nabla_k R\cdot R_{ij}\nabla_k R_{ij}\le\frac{2}{R^\gamma}|R_{ij}|\cdot|\nabla_k R|\cdot|\nabla_k R_{ij}|,$$

by Lemma 5.8 we get

$$|\nabla_k R|^2\le\frac{2(n-1)(n+2)}{3n-2}|\nabla_k R_{ij}|^2.$$

Thus

$$|\nabla_k R|\cdot|\nabla_k R_{ij}|\le\left(\frac{2(n-1)(n+2)}{3n-2}\right)^{1/2}|\nabla_k R_{ij}|^2.$$

From (6) it follows that $|R_{ij}|^2\le R^2$; thus

$$\frac{1}{R^\gamma}\nabla_k R\cdot\nabla_k S\le\frac{2}{R^\gamma}|R_{ij}|\cdot|\nabla_k R|\cdot|\nabla_k R_{ij}|\le\frac{c_1(n)}{R^{\gamma-1}}|\nabla_i R_{jk}|^2,$$

where $c_1>0$ depends only on n.

Using (27) we get

$$\begin{aligned}-2\nabla_k R^{1-\gamma}\cdot\nabla_k\left(S-\frac{1}{n}R^2\right)\le\frac{\gamma}{R}\nabla_k R\cdot\nabla_k H&+\frac{(2-\gamma)c_1}{R^{\gamma-1}}|\nabla_i R_{jk}|^2\\&+\frac{c_2}{R^{\gamma+1}}\left(S-\frac{1}{n}R^2\right)|\nabla_k R|^2,\\&\qquad c_2=\gamma(\gamma-1).\end{aligned}$$

From (26) it follows that

$$\begin{aligned}\frac{\partial H}{\partial t}\le\Delta H&+\frac{\gamma}{R}\nabla_k R\cdot\nabla_k H+\frac{(2-\gamma)c_1}{R^{\gamma-1}}|\nabla_i R_{jk}|^2\\&+\frac{c_2}{R^{\gamma+1}}\left(S-\frac{1}{n}R^2\right)|\nabla_k R|^2-\frac{2(n-2)^2}{n(3n-2)}R^{1-\gamma}|\nabla_i R_{jk}|^2\\&+4R^{2-\gamma}\left(S-\frac{1}{n}R^2\right)-\frac{\gamma(\gamma-1)}{R^{\gamma+1}}\left(S-\frac{1}{n}R^2\right)|\nabla_i R|^2\\&+\frac{2(1-\gamma)}{R^\gamma}S\left(S-\frac{1}{n}R^2\right).\end{aligned}$$

By (6) we have

$$\frac{2(1-\gamma)}{R^\gamma}S\left(S-\frac{1}{n}R^2\right)\le 2|\gamma-1|\cdot R^{2-\gamma}\left(S-\frac{1}{n}R^2\right),$$

and therefore

$$\begin{aligned}\frac{\partial H}{\partial t}\le\Delta H+\frac{\gamma}{R}\nabla_k R\cdot\nabla_k H&+\left[(2-\gamma)c_1-\frac{2(n-2)^2}{n(3n-2)}\right]R^{1-\gamma}|\nabla_i R_{jk}|^2\\&+6R^{2-\gamma}\left(S-\frac{1}{n}R^2\right).\end{aligned}$$

If we choose $1 < \gamma < 2$ such that

$$0 < 2 - \gamma \le \frac{(n-2)^2}{n(3n-2)c_1}, \tag{28}$$

then

$$(2-\gamma)c_1 - \frac{2(n-2)^2}{n(3n-2)} \le -\frac{(n-2)^2}{n(3n-2)}.$$

Thus

$$\begin{aligned} \frac{\partial H}{\partial t} \le \Delta H + \frac{\gamma}{R}\nabla_k R \cdot \nabla_k H - \frac{(n-2)^2}{n(3n-2)} R^{1-\gamma}|\nabla_i R_{jk}|^2 \\ + 6R^{2-\gamma}\left(S - \frac{1}{n}R^2\right). \end{aligned} \tag{29}$$

We still have

$$\begin{aligned} \frac{\partial}{\partial t}R^{3-\gamma} &= (3-\gamma)R^{2-\gamma}\frac{\partial R}{\partial t} = (3-\gamma)R^{2-\gamma}(\Delta R + 2S) \\ &= \Delta R^{3-\gamma} - (3-\gamma)(2-\gamma)R^{1-\gamma}|\nabla_i R|^2 + 2(3-\gamma)SR^{2-\gamma}, \\ \frac{\gamma}{R}\nabla_k R \cdot \nabla_k R^{3-\gamma} &= (3-\gamma)\gamma R^{1-\gamma}|\nabla_i R|^2, \\ \frac{\partial}{\partial t}R^{3-\gamma} &= \Delta R^{3-\gamma} + \frac{\gamma}{R}\nabla_k R \cdot \nabla_k R^{3-\gamma} - 2(3-\gamma)R^{1-\gamma}|\nabla_i R|^2 \\ &\quad + 2(3-\gamma)SR^{2-\gamma}. \end{aligned}$$

From (6) it follows that

$$\begin{aligned} \frac{\partial}{\partial t}R^{3-\gamma} \ge \Delta R^{3-\gamma} + \frac{\gamma}{R}\nabla_k R \cdot \nabla_k R^{3-\gamma} - 2(3-\gamma)R^{1-\gamma}|\nabla_i R|^2 \\ + \frac{2}{n}(3-\gamma)R^{4-\gamma}. \end{aligned} \tag{30}$$

Now we define

$$F(x,t) = \frac{|\nabla_i R|^2}{R^\gamma} + \alpha R^{1-\gamma}\left(S - \frac{1}{n}R^2\right) - \eta R^{3-\gamma}, \qquad 0 \le t < +\infty, \tag{31}$$

where $\alpha > 0$ and $\eta > 0$ are two constants to be defined later. Then by (24), (29), and (30) we get

$$\begin{aligned} \frac{\partial F}{\partial t} &\le \Delta F + \frac{\gamma}{R}\nabla_k R \cdot \nabla_k F - \gamma\left(1 - \frac{\gamma}{2}\right)\frac{|\nabla_i R|^4}{R^{2+\gamma}} \\ &\quad + \left[16 - \frac{(n-2)^2}{n(3n-2)}\alpha\right] R^{1-\gamma}|\nabla_i R_{jk}|^2 + 6\alpha R^{2-\gamma}\left(S - \frac{1}{n}R^2\right) \\ &\quad + 2\eta(3-\gamma)R^{1-\gamma}|\nabla_k R|^2 - \frac{2}{n}(3-\gamma)\eta R^{4-\gamma}, \qquad 0 \le t < +\infty. \end{aligned} \tag{32}$$

If we choose α such that

$$\alpha \geq \frac{32n(3n-2)}{(n-2)^2}, \tag{33}$$

then

$$16 - \frac{(n-2)^2}{n(3n-2)}\alpha \leq -\frac{(n-2)^2}{2n(3n-2)}\alpha, \tag{34}$$

$$\begin{aligned}\frac{\partial F}{\partial t} \leq \Delta F &+ \frac{\gamma}{R}\nabla_k R \cdot \nabla_k F - \gamma\left(1 - \frac{\gamma}{2}\right)\frac{|\nabla_i R|^4}{R^{2+\gamma}} \\ &- \frac{(n-2)^2\alpha}{2n(3n-2)}R^{1-\gamma}|\nabla_i R_{jk}|^2 + 2\eta(3-\gamma)R^{1-\gamma}|\nabla_k R|^2 \\ &+ 6\alpha R^{2-\gamma}\left(S - \frac{1}{n}R^2\right) - \frac{2}{n}(3-\gamma)\eta R^{4-\gamma}, \qquad 0 \leq t < +\infty.\end{aligned} \tag{35}$$

By definition we have

$$\begin{aligned}\frac{|\nabla_i R|^2}{R^\gamma} &= F + \eta R^{3-\gamma} - \alpha R^{1-\gamma}\left(S - \frac{1}{n}R^2\right), \\ \frac{|\nabla_i R|^4}{R^{2\gamma}} &\geq \frac{F^2}{2} - \left[\eta R^{3-\gamma} - \alpha R^{1-\gamma}\left(S - \frac{1}{n}R^2\right)\right]^2 \\ &\geq \frac{F^2}{2} - 2\eta^2 R^{6-2\gamma} - 2\alpha^2 R^{2-2\gamma}\left(S - \frac{1}{n}R^2\right)^2.\end{aligned}$$

Thus

$$\begin{aligned}-\frac{\gamma}{2}\left(1 - \frac{\gamma}{2}\right)\frac{|\nabla_i R|^4}{R^{\gamma+2}} &\leq -\frac{\gamma}{4}\left(1 - \frac{\gamma}{2}\right)\frac{F^2}{R^{2-\gamma}} + \gamma\left(1 - \frac{\gamma}{2}\right)\eta^2 R^{4-\gamma} \\ &\quad + \gamma\left(1 - \frac{\gamma}{2}\right)\alpha^2 R^{-\gamma}\left(S - \frac{1}{n}R^2\right)^2 \\ &\leq -\frac{\gamma}{4}\left(1 - \frac{\gamma}{2}\right)\frac{F^2}{R^{2-\gamma}} + \frac{\gamma}{2}(2-\gamma)\eta^2 R^{4-\gamma} \\ &\quad + \frac{\gamma}{2}(2-\gamma)\alpha^2 R^{2-\gamma}\left(S - \frac{1}{n}R^2\right).\end{aligned} \tag{36}$$

Using Lemma 5.8 we get

$$-\frac{(n-2)^2\alpha}{2n(3n-2)}R^{1-\gamma}|\nabla_i R_{jk}|^2 \leq -\frac{(n-2)^2\alpha}{4n(n-1)(n+2)}R^{1-\gamma}|\nabla_i R|^2. \tag{37}$$

Substituting (36) and (37) into (35) yields

$$
\begin{aligned}
\frac{\partial F}{\partial t} \le{}& \Delta F + \frac{\gamma}{R}\nabla_k R\cdot\nabla_k F - \frac{\gamma}{2}\left(1-\frac{\gamma}{2}\right)\frac{|\nabla_i R|^4}{R^{2+\gamma}} - \frac{\gamma}{4}\left(1-\frac{\gamma}{2}\right)\frac{F^2}{R^{2-\gamma}}\\
&+\left[2\eta(3-\gamma) - \frac{(n-2)^2\alpha}{4n(n-1)(n+2)}\right]R^{1-\gamma}|\nabla_i R|^2\\
&+\left[6\alpha + \frac{\gamma}{2}(2-\gamma)\alpha^2\right]R^{2-\gamma}\left(S-\frac{1}{n}R^2\right)\\
&+\left[\frac{\gamma}{2}(2-\gamma)\eta^2 - \frac{2}{n}(3-\gamma)\eta\right]R^{4-\gamma}, \qquad 0\le t<+\infty.
\end{aligned}
\tag{38}
$$

Choose $\eta > 0$ small enough such that

$$
0<\eta\le \min\left\{\frac{(n-2)^2\alpha}{16(3-\gamma)n(n-1)(n+2)}, \frac{(3-\gamma)}{n\gamma(2-\gamma)}\right\}. \tag{39}
$$

Then

$$
\begin{gathered}
2\eta(3-\gamma) - \frac{(n-2)^2\alpha}{4n(n-1)(n+2)} \le -\frac{(n-2)^2\alpha}{8n(n-1)(n+2)},\\
\gamma(2-\gamma)\eta^2 - \frac{2}{n}(3-\gamma)\eta \le -\frac{1}{n}(3-\gamma)\eta \le -\frac{1}{n}\eta.
\end{gathered}
$$

Thus from (38) it follows that

$$
\begin{aligned}
\frac{\partial F}{\partial t} \le{}& \Delta F + \frac{\gamma}{R}\nabla_k R\cdot\nabla_k F - \frac{\gamma}{2}\left(1-\frac{\gamma}{2}\right)\frac{|\nabla_i R|^4}{R^{2+\gamma}}\\
&- \frac{\gamma}{4}\left(1-\frac{\gamma}{2}\right)\frac{F^2}{R^{2-\gamma}} - \frac{(n-2)^2\alpha}{8n(n-1)(n+2)}R^{1-\gamma}|\nabla_k R|^2\\
&+[6\alpha+(2-\gamma)\alpha^2]R^{2-\gamma}\left(S-\frac{1}{n}R^2\right) - \frac{1}{n}\eta R^{4-\gamma},\\
&\qquad\qquad 0\le t<\infty.
\end{aligned}
\tag{40}
$$

By the definition of F, we have

$$
\begin{aligned}
\frac{|\nabla_i R|^2}{R^2}F &= \frac{|\nabla_i R|^4}{R^{2+\gamma}} + \frac{\alpha}{R^{\gamma+1}}\left(S-\frac{1}{n}R^2\right)|\nabla_k R|^2 - \eta R^{1-\gamma}|\nabla_k R|^2,\\
-\frac{|\nabla_i R|^4}{R^{2+\gamma}} &\le -\frac{|\nabla_i R|^2}{R^2}F + \frac{\alpha}{R^{\gamma+1}}\left(S-\frac{1}{n}R^2\right)|\nabla_k R|^2\\
&\le -\frac{|\nabla_i R|^2}{R^2}F + \alpha R^{1-\gamma}|\nabla_k R|^2,\\
-\frac{\gamma}{2}\left(1-\frac{\gamma}{2}\right)\frac{|\nabla_i R|^4}{R^{2+\gamma}} &\le -\frac{\gamma}{2}\left(1-\frac{\gamma}{2}\right)\frac{|\nabla_i R|^2}{R^2}F + \frac{\gamma}{4}(2-\gamma)\alpha R^{1-\gamma}|\nabla_k R|^2.
\end{aligned}
$$

Substituting the last equation into (40) gives
(41)
$$\begin{aligned}\frac{\partial F}{\partial t} \le \Delta F &+ \frac{\gamma}{R}\nabla_k R\cdot\nabla_k F - \frac{\gamma}{2}\left(1-\frac{\gamma}{2}\right)\frac{|\nabla_i R|^2}{R^2}F - \frac{\gamma}{4}\left(1-\frac{\gamma}{2}\right)\frac{F^2}{R^{2-\gamma}}\\ &+ \left[\frac{\gamma}{4}(2-\gamma) - \frac{(n-2)^2}{8n(n-1)(n+2)}\right]\alpha R^{1-\gamma}|\nabla_k R|^2\\ &+ [6\alpha + (2-\gamma)\alpha^2]R^{2-\gamma}\left(S - \frac{1}{n}R^2\right) - \frac{1}{n}\eta R^{4-\gamma}.\end{aligned}$$

Let

$$0 < 2-\gamma \le \frac{(n-2)^2}{8n(n-1)(n+2)}.$$

Then

$$\frac{\gamma}{4}(2-\gamma) - \frac{(n-2)^2}{8n(n-1)(n+2)} \le -\frac{(n-2)^2}{16n(n-1)(n+2)},$$

$$\begin{aligned}\frac{\partial F}{\partial t} \le \Delta F &+ \frac{\gamma}{R}\nabla_k R\cdot\nabla_k F - \frac{\gamma}{4}(2-\gamma)\frac{|\nabla_i R|^2}{R^2}F\\ &- \frac{\gamma}{8}(2-\gamma)\frac{F^2}{R^{2-\gamma}} - \frac{(n-2)^2\alpha}{16n(n-1)(n+2)}R^{1-\gamma}|\nabla_k R|^2\\ &+ [6\alpha + (2-\gamma)\alpha^2]R^{2-\gamma}\left(S - \frac{1}{n}R^2\right) - \frac{1}{n}\eta R^{4-\gamma}.\end{aligned} \tag{42}$$

Since

$$R^{1-\gamma}|\nabla_k R|^2 = RF - \alpha R^{2-\gamma}\left(S - \frac{1}{n}R^2\right) + \eta R^{4-\gamma},$$

from (42) we get

$$\begin{aligned}\frac{\partial F}{\partial t} \le \Delta F &+ \frac{\gamma}{R}\nabla_k R\cdot\nabla_k F - \frac{\gamma}{4}(2-\gamma)\frac{|\nabla_i R|^2}{R^2}F - \frac{\gamma}{8}(2-\gamma)\frac{F^2}{R^{2-\gamma}}\\ &- \frac{(n-2)^2\alpha}{16n(n-1)(n+2)}RF + 8\alpha^2 R^{2-\gamma}\left(S - \frac{1}{n}R^2\right)\\ &- \frac{(n-2)^2}{16n(n-1)(n+2)}\alpha\eta R^{4-\gamma}, \qquad 0 \le t < +\infty.\end{aligned} \tag{43}$$

Lemma 6.2. *Suppose $m > 0$, $C > 0$ and $\varphi(x) = x + C/x^m$, $0 < x < +\infty$. Then*

$$\varphi(x) \ge \left(1 + \frac{1}{m}\right) m^{1/(m+1)} C^{1/(m+1)}, \qquad 0 < x < +\infty.$$

Proof. Let $\varphi'(x) = 0$. Then $\varphi'(x) = 1 - mC/x^{m+1} = 0$, and the solution is $x_0 = [mC]^{1/(m+1)}$. We get

$$\varphi(x) \geq \varphi(x_0) = x_0\left(1 + \frac{C}{x_0^{m+1}}\right) = \left(1 + \frac{1}{m}\right)x_0,$$
$$\varphi(x) \geq \left(1 + \frac{1}{m}\right)m^{1/(m+1)}C^{1/(m+1)}, \qquad 0 < x < +\infty.$$

Thus

$$\begin{aligned}
&\frac{\gamma}{8}(2-\gamma)\frac{F^2}{R^{2-\gamma}} + \frac{(n-2)^2\alpha}{16n(n-1)(n+2)}RF \\
&\quad = \frac{(n-2)^2\alpha F}{16n(n-1)(n+2)}\left[R + \frac{2n(n-1)(n+2)}{(n-2)^2\alpha}\gamma(2-\gamma)F\cdot\frac{1}{R^{2-\gamma}}\right] \\
&\quad \geq \frac{(n-2)^2\alpha F}{16n(n-1)(n+2)}\left(1 + \frac{1}{2-\gamma}\right)(2-\gamma)^{1/(3-\gamma)}\cdot F^{1/(3-\gamma)} \\
&\qquad\qquad \cdot\left[\frac{2n(n-1)(n+2)}{(n-2)^2\alpha}\gamma(2-\gamma)\right]^{1/(3-r)} \\
&\quad = \frac{(n-2)^2(3-\gamma)}{16n(n-1)(n+2)}\left[\frac{2n(n-1)(n+2)}{(n-2)^2}\right]^{1/(3-\gamma)} \\
&\qquad \times \gamma^{1/(3-\gamma)}(2-\gamma)^{(\gamma-1)/(3-\gamma)}\alpha^{(2-\gamma)/(3-\gamma)}F^{(4-\gamma)/(3-\gamma)}.
\end{aligned}$$

Substituting this into (43) yields, for $F > 0$,

$$\begin{aligned}
\frac{\partial F}{\partial t} &\leq \Delta F + \frac{\gamma}{R}\nabla_k R\cdot\nabla_k F - \frac{\gamma}{4}(2-\gamma)\frac{|\nabla_i R|^2}{R^2}F \\
&\quad - \frac{(n-2)^2(3-\gamma)}{16n(n-1)(n+2)}\left[\frac{2n(n-1)(n+2)}{(n-2)^2}\gamma\right]^{1/(3-\gamma)} \\
&\quad \times (2-\gamma)^{(\gamma-1)/(3-\gamma)}\alpha^{(2-\gamma)/(3-\gamma)}F^{(4-\gamma)/(3-\gamma)} \\
&\quad + 8\alpha^2R^{2-\gamma}\left(S - \frac{1}{n}R^2\right) - \frac{(n-2)^2}{16n(n-1)(n+2)}\alpha\eta R^{4-\gamma}, \\
&\qquad\qquad 0 \leq t < +\infty.
\end{aligned} \tag{44}$$

From Lemma 6.1 we know that for $0 < \sigma \leq 1/(2n)$

$$|\overset{\circ}{\mathrm{R}}\mathrm{m}|^2 \leq \frac{\beta c_1(\sigma)}{(t+1)^\sigma}R^{2-\sigma}, \qquad 0 \leq t < +\infty.$$

Thus

$$0 \le S - \frac{1}{n}R^2 \le \frac{n^2\beta c_1(\sigma)}{(t+1)^\sigma}R^{2-\sigma}, \qquad 0 \le t < +\infty, \tag{45}$$

$$\begin{aligned}
&8\alpha^2 R^{2-\gamma}\left(S - \frac{1}{n}R^2\right) - \frac{(n-2)^2}{16n(n-1)(n+2)}\alpha\eta R^{4-\gamma} \\
&\le \left[8\alpha n^2 c_1(\sigma)\beta\left(\frac{1}{t+1}\right)^\sigma - \frac{(n-2)^2}{16n(n-1)(n+2)}\eta R^\sigma\right] R^{4-\gamma-\sigma}\cdot\alpha \\
&\le 8\alpha^2 n^2\beta c_1(\sigma)\left(\frac{1}{t+1}\right)^\alpha \\
&\qquad\cdot\left[\frac{128n^3(n-1)(n+2)\alpha\cdot\beta c_1(\sigma)}{(n-2)^2\eta}\right]^{(4-\gamma)/\sigma-1}\left(\frac{1}{t+1}\right)^{4-\gamma-\sigma} \\
&\le 8n^2 c_1(\sigma)\left[\frac{128n^3(n-1)(n+2)c_1(\sigma)}{(n-2)^2}\right]^{(4-\gamma)/\sigma-1} \\
&\qquad\qquad\cdot\alpha\cdot\eta\left(\frac{\alpha\beta}{\eta}\right)^{(4-\gamma)/\sigma}\left(\frac{1}{t+1}\right)^{4-\gamma}.
\end{aligned}$$

Let $\sigma = 1/2n$. Then

$$\begin{aligned}
&8\alpha^2 R^{2-\gamma}\left(S - \frac{1}{n}R^2\right) - \frac{(n-2)^2}{16n(n-1)(n+2)}\alpha\eta R^{4-\gamma} \\
&\qquad\le c_3(n, c_0)\alpha\eta\left(\frac{\alpha\beta}{\eta}\right)^{2n(4-\gamma)}\left(\frac{1}{t+1}\right)^{4-\gamma},
\end{aligned} \tag{46}$$

where c_0 is the constant in (1).

Substituting (46) into (44), we get

$$\begin{aligned}
\frac{\partial F}{\partial t} \le{}& \Delta F + \frac{\gamma}{R}\nabla_k R\cdot\nabla_k F - \frac{\gamma}{4}(2-\gamma)\frac{|\nabla_i R|^2}{R^2}F \\
&- c_4(n)(2-\gamma)^{(\gamma-1)/(3-\gamma)}\alpha^{(2-\gamma)/(3-\gamma)}F^{(4-\gamma)/(3-\gamma)} \\
&+ c_3(n, c_0)\alpha\eta\left(\frac{\alpha\beta}{\eta}\right)^{2n(4-\gamma)}\left(\frac{1}{t+1}\right)^{4-\gamma}, \qquad 0 \le t < +\infty.
\end{aligned} \tag{47}$$

Define

$$w(x,t) = F(x,t)t^{3-\gamma}, \qquad 0 \le t < +\infty. \tag{48}$$

Then

$$\frac{\partial w}{\partial t} \le \Delta w + \frac{\gamma}{R}\nabla_k R \cdot \nabla_k w - \frac{\gamma}{4}(2-\gamma)\frac{|\nabla_i R|^2}{R^2} w$$

$$- c_4(n)(2-\gamma)^{(\gamma-1)/(3-\gamma)}\alpha^{(2-\gamma)/(3-\gamma)} w^{(4-\gamma)/(3-\gamma)} \cdot \left(\frac{1}{t}\right)$$

$$+ c_3(n, c_0)\alpha\eta\left(\frac{\alpha\beta}{\eta}\right)^{2n(4-\gamma)} \frac{1}{t}\left(\frac{t}{t+1}\right)^{4-\gamma} + \frac{(3-\gamma)}{t} w,$$

$$\frac{\partial w}{\partial t} \le \Delta w + \frac{\gamma}{R}\nabla_k R \cdot \nabla_k w - \frac{\gamma}{4}(2-\gamma)\frac{|\nabla_i R|^2}{R^2} w$$

$$+ \frac{w}{t}\left[(3-\gamma) + \frac{c_3(n, c_0)}{w}\alpha\eta\left(\frac{\alpha\beta}{\eta}\right)^{2n(4-\gamma)} \right. \tag{49}$$

$$\left. - c_4(n)(2-\gamma)^{(\gamma-1)/(3-\gamma)}\alpha^{(2-\gamma)/(3-\gamma)} w^{1/(3-\gamma)}\right]$$

for $w > 0$.

From (48) it follows that

$$w(x, 0) \equiv 0, \tag{50}$$

and therefore, in consequence of Theorem 4.12, that

$$w(x, t) \le y_0 \quad \text{on } M \times [0, \infty), \tag{51}$$

where $y_0 > 0$ is the root of

$$(3-\gamma) + \frac{c_3(n, c_0)}{y_0}\alpha\eta\left(\frac{\alpha\beta}{\eta}\right)^{2n(4-\gamma)} \tag{52}$$

$$- c_4(n)(2-\gamma)^{(\gamma-1)/(3-\gamma)}\alpha^{(2-\gamma)/(3-\gamma)} y_0^{1/(3-\gamma)} = 0.$$

Now if we fix γ such that (28) holds and let

$$\alpha = (1/\beta)^{1/3}, \quad \eta = \beta^{1/3}, \quad y_1 = \beta^{(2-\gamma)/4},$$

then

$$(3-\gamma) + \frac{c_3(n, c_0)}{y_1}\alpha\eta\left(\frac{\alpha\beta}{\eta}\right)^{2n(4-\gamma)}$$

$$- c_4(n)(2-\gamma)^{(\gamma-1)/(3-\gamma)}\alpha^{(2-\gamma)/(3-\gamma)} y_1^{1/(3-\gamma)}$$

$$= (3-\gamma) + c_3(n, c_0)\beta^{2n(4-\gamma)/3 - 2/3 - (2-\gamma)/4}$$

$$- c_5(n, \gamma)\left(\frac{1}{\beta}\right)^{(2-\gamma)/12(3-\gamma)} < 0.$$

If $\beta > 0$ is small enough, we have

$$y_0 \le y_1 = \beta^{(2-\gamma)/4}.$$

Thus there exists a constant $c_6 = c_6(n, c_0, \gamma) > 0$ such that

$$w(x,t) \le \beta^{(2-\gamma)/4} \quad \text{if } 0 < \beta \le c_6. \tag{53}$$

By the definition of $w(x,t)$ we get

$$F(x,t) \le \beta^{(2-\gamma)/4}/t^{3-\gamma}, \qquad 0 \le t < +\infty. \tag{54}$$

Also by definition we have

$$F(x,t) \ge \frac{|\nabla_i R|^2}{R^\gamma} - \eta R^{3-\gamma}. \tag{55}$$

Combining (54) and (55) gives

$$\frac{|\nabla_i R|^2}{R^\gamma} \le \eta R^{3-\gamma} + \beta^{(2-\gamma)/4}\left(\frac{1}{t}\right)^{3-\gamma}, \qquad 0 \le t < \infty.$$

Since $\eta = \beta^{1/3}$, we have

$$|\nabla_i R|^2 \le \beta^{1/3} R^3 + \beta^{(2-\gamma)/4} R^\gamma \left(\frac{1}{t}\right)^{3-\gamma}, \qquad 0 \le t < +\infty. \tag{56}$$

Lemma 6.3. *Suppose M is a complete noncompact Riemannian manifold of dimension n, and suppose there exists $\delta > 0$ such that*

$$R_{ij} \ge \delta R g_{ij} > 0 \quad \text{on } M.$$

Then there exists a constant $\eta_0 = \eta_0(n, \delta) > 0$ such that

$$\eta_0 \left[\sup_{x \in M} R(x)\right]^3 \le \sup_M |\nabla_i R|^2. \tag{57}$$

Proof. The proof of this lemma is analogous to that of Lemma 5.17.

Now we can prove the following scalar curvature decay theorem.

Theorem 6.4. *There exist constants $\delta = \delta(n) > 0$ depending only on n, and $c_6 = c_6(n, c_0) > 0$ depending only on n and c_0, such that if $0 < \beta \le c_6$, then*

$$R(x,t) \le C(n)\beta^\delta / t, \qquad 0 \le t < +\infty, \tag{58}$$

where $C(n) > 0$ depends only on n.

Proof. Let

$$R_{\max}(t) = \sup_{x \in M} R(x,t), \qquad 0 \le t < +\infty. \tag{59}$$

Since from (3) we have

$$0 \le S - \frac{1}{n}R^2 = |\overset{\circ}{R}_{ij}|^2 \le n^2 |\overset{\circ}{\mathrm{R}}\mathrm{m}|^2 \le n^2 \beta R^2, \qquad 0 \le t < \infty,$$

we can find $\tilde{c}_6 > 0$ depending only on n such that if $0 < \beta \le \tilde{c}_6$ then

$$(60) \qquad R_{ij} \ge \frac{1}{2n} R g_{ij}, \qquad 0 \le t < +\infty.$$

By Lemma 6.3 we can find a constant $\eta_0 = \eta_0(n) > 0$ such that

$$\eta_0 R_{\max}^3 \le \sup_M |\nabla_i R|^2.$$

From (56), if we fix $\gamma = \gamma(n) > 0$ and let $c_6 = c_6(n, c_0) \le \tilde{c}_6$, then for $0 < \beta \le c_6$, we have

$$\sup_M |\nabla_i R|^2 \le \beta^{1/3} R_{\max}^3 + \beta^{(2-\gamma)/4} \left(\frac{1}{t}\right)^{3-\gamma} R_{\max}^\gamma, \qquad 0 \le t < +\infty,$$

and therefore

$$\eta_0 R_{\max}^3 \le \beta^{1/3} R_{\max}^3 + \beta^{(2-\gamma)/4} \left(\frac{1}{t}\right)^{3-\gamma} R_{\max}^\gamma, \qquad 0 \le t < +\infty.$$

If $0 < \beta \le \min\{c_6, (\eta_0/2)^3\}$, then

$$\eta_0 - \beta^{1/3} \ge \frac{\eta_0}{2},$$

$$\frac{\eta_0}{2} R_{\max}^3 \le \beta^{(2-\gamma)/4} \left(\frac{1}{t}\right)^{3-\gamma} R_{\max}^\gamma, \qquad 0 \le t < +\infty.$$

Thus if $0 < \beta \le \min\{c_6, (\eta_0/2)^3\}$, then

$$R_{\max}(t) \le \left(\frac{2}{\eta_0}\right)^{1/(3-\gamma)} \beta^{(2-\gamma)/4(3-\gamma)}/t, \qquad 0 \le t < \infty.$$

Let $C(n) = (2/\eta_0)^{1/(3-\gamma)}$ and $\delta = (2-\gamma)/4(3-\gamma) > 0$. Then

$$(61) \qquad R(x,t) \le C(n)\beta^\delta/t, \qquad 0 \le t < +\infty.$$

Corollary 6.5. *For $\delta > 0$ and $c_6 > 0$ in Theorem 6.4, there exists a constant $c_7 = c_7(n, c_0, \beta) > 0$ such that for $0 < \beta \le c_6$, we have*

$$(62) \qquad R(x,t) \le \frac{C(n)\beta^\delta}{t+1} + \frac{c_7(n, c_0, \beta)}{(t+1)^2}, \qquad 0 \le t < +\infty,$$

$$(63) \qquad |\nabla_i R|^2 \le \frac{\tilde{C}(n)\beta^{3\delta}}{(t+1)t^2} + \frac{c_7(n, c_0, \beta)}{(t+1)^4}, \qquad 0 \le t < +\infty,$$

where $C(n) > 0$ and $\tilde{C}(n) > 0$ depend only on n.

Proof. (62) follows from Theorem 6.4 and (3), and (63) follows from (56) and (62).

Thus we know that as time $t \to \infty$, the scalar curvature $R(x,t)$ goes to zero in t^{-1} order, but this is not enough; we need faster decay of the scalar curvature than t^{-1} to guarantee the convergence of the metric $g_{ij}(t)$ as time $t \to \infty$.

7. Decay of the controlling function

In this section we want to prove that the scalar curvature of M actually decays in the order of $(1/t)^{1+\theta}$, $\theta > 0$, as time $t \to +\infty$, provided that M satisfies all of the conditions stated in the Main Theorem.

We still use the notation of the last section. Suppose M is an n-dimensional complete noncompact Riemannian manifold with metric $g_{ij}(x)$, the curvature of which satisfies the condition

$$(1) \qquad |\overset{\circ}{\mathrm{Rm}}|^2 \le \beta R^2, \qquad 0 < R \le c_0,$$

where β and c_0 are constants, and $0 < \beta \le \delta_n/2n(n-1)$.

From Corollary 6.5 we know that if $0 < \beta \le c_6$, then

$$(2) \qquad R(x,t) \le \frac{C(n)\beta^\delta}{t+1} + \frac{c_7(n,c_0,\beta)}{(t+1)^2}, \qquad 0 \le t < +\infty.$$

Let

$$(3) \qquad \varepsilon = C(n)\beta^\delta, \qquad C(\varepsilon) = c_7(n,c_0,\beta).$$

Then

$$(4) \qquad R(x,t) \le \frac{\varepsilon}{t+1} + \frac{c(\varepsilon)}{(t+1)^2}, \qquad 0 \le t < +\infty.$$

Suppose $u(x) \in C^\infty(M)$ is a function satisfying

$$(5) \qquad 0 < R(x,0) \le u(x) \le 2R(x,0) \quad \text{on } M.$$

We consider the following equation on M:

$$(6) \qquad \begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + \left(2 + \dfrac{1}{\sqrt{\varepsilon}}\right)\left[\dfrac{\varepsilon}{t+1} + \dfrac{C(\varepsilon)}{(t+1)^2}\right] u - \dfrac{1}{\sqrt{\varepsilon}} u^2, & 0 \le t < +\infty, \\ u(x,0) \equiv u(x). \end{cases}$$

From (5) we get

$$(7) \qquad 0 < R(x,0) \le u(x,0) \le 2R(x,0) \le 2c_0.$$

Therefore by using some simple technique we can find a positive solution $u(x,t) \in C^\infty(M \times [0,+\infty))$ of (6) such that

$$(8) \qquad 0 < u(x,t) \le c_1, \qquad 0 \le t < +\infty,$$

where $0 < c_1 < +\infty$ is some constant.

Since $\partial R/\partial t = \Delta R + 2S$, from (6) of §6 we have

$$\frac{\partial R}{\partial t} \le \Delta R + 2R^2, \qquad 0 \le t < +\infty, \tag{9}$$

which together with (6) implies

$$\begin{aligned}\frac{\partial}{\partial t}(u - R) \ge \Delta(u - R) &+ \left[2R - \frac{1}{\sqrt{\varepsilon}}u\right](u - R) \\ &+ \left(2 + \frac{1}{\sqrt{\varepsilon}}\right)\cdot\left[\frac{\varepsilon}{t+1} + \frac{C(\varepsilon)}{(t+1)^2} - R\right]u.\end{aligned} \tag{10}$$

From (4) we get

$$\frac{\varepsilon}{t+1} + \frac{C(\varepsilon)}{(t+1)^2} - R \ge 0.$$

Since $u(x,t) > 0$, we have

$$\left(2 + \frac{1}{\sqrt{\varepsilon}}\right)\left[\frac{\varepsilon}{t+1} + \frac{C(\varepsilon)}{(t+1)^2} - R\right]u \ge 0,$$

which thus reduce (10) to

$$\frac{\partial}{\partial t}(u - R) \ge \Delta(u - R) + \left(2R - \frac{u}{\sqrt{\varepsilon}}\right)(u - R). \tag{11}$$

Let $\psi(x,t) = u - R$. Then

$$\frac{\partial \psi}{\partial t} \ge \Delta\psi + Q(\psi, x, t), \tag{12}$$

where

$$Q(\psi, x, t) = \left(2R - \frac{u}{\sqrt{\varepsilon}}\right)(u - R).$$

Furthermore, from (7) we have

$$\psi(x, 0) \ge 0. \tag{13}$$

Since $u(x,t) > 0$, we get

$$\psi(x,t) \ge -R \ge -\tilde{C}, \qquad 0 \le t < +\infty. \tag{14}$$

By using (4)and (8) we get

$$|Q(\psi, x, t)| \le \left(2R + \frac{u}{\sqrt{\varepsilon}}\right)\cdot|u - R| \le \left(2\tilde{C} + \frac{c_1}{\sqrt{\varepsilon}}\right)|\psi|,$$

$$|Q(\psi, x, t)| \le \left(2\tilde{C} + \frac{c_1}{\sqrt{\varepsilon}}\right)\cdot|\psi|, \qquad 0 \le t < +\infty,$$

$$Q(\psi, x, t) \ge -\left(2\tilde{C} + \frac{c_1}{\sqrt{\varepsilon}}\right)\cdot|\psi|, \qquad 0 \le t < +\infty. \tag{15}$$

From (12), (13), (14), (15) and Theorem 4.6 it follows that

$$\psi(x,t) \geq 0, \qquad 0 \leq t < +\infty,$$

so that

$$R(x,t) \leq u(x,t), \qquad 0 \leq t < +\infty. \tag{16}$$

Let

$$P(t) = \left(2 + \frac{1}{\sqrt{\varepsilon}}\right)\left[\frac{\varepsilon}{t+1} + \frac{C(\varepsilon)}{(t+1)^2}\right], \qquad 0 \leq t < +\infty. \tag{17}$$

Then

$$\frac{\partial u}{\partial t} = \Delta u + P(t)u - \frac{1}{\sqrt{\varepsilon}}u^2, \qquad 0 \leq t < +\infty. \tag{18}$$

Let

$$\varphi(t) = \frac{\delta}{t+1} + \frac{c_1}{(t+1)^2}, \qquad 0 \leq t < +\infty, \tag{19}$$

where $\delta > 0$ and $c_1 > 0$ are two constants to be determined later. If we let

$$w(x,t) = u(x,t) - \varphi(t),$$

then we have

$$\frac{\partial w}{\partial t} = \Delta w + P(t)[w + \varphi(t)] - \frac{1}{\sqrt{\varepsilon}}[w + \varphi(t)]^2 - \varphi'(t),$$

$$\begin{aligned} \frac{\partial w}{\partial t} = \Delta w &+ \left[P(t) - \frac{1}{\sqrt{\varepsilon}}w - \frac{2}{\sqrt{\varepsilon}}\varphi(t)\right] w + P(t)\varphi(t) \\ &- \frac{\varphi(t)^2}{\sqrt{\varepsilon}} - \varphi'(t), \qquad 0 \leq t < +\infty. \end{aligned} \tag{20}$$

From (19) we have

$$-\varphi'(t) = \frac{\delta}{(t+1)^2} + \frac{2c_1}{(t+1)^3},$$

$$\begin{aligned} P(t)\varphi(t) - \varphi'(t) &= \frac{\delta}{(t+1)^2} + \frac{2c_1}{(t+1)^3} \\ &\quad + \left(2 + \frac{1}{\sqrt{\varepsilon}}\right)\left[\frac{\varepsilon}{t+1} + \frac{C(\varepsilon)}{(t+1)^2}\right]\left[\frac{\delta}{t+1} + \frac{c_1}{(t+1)^2}\right] \\ &= \left[\left(2 + \frac{1}{\sqrt{\varepsilon}}\right)\varepsilon\delta + \delta\right]\frac{1}{(t+1)^2} + \left(2 + \frac{1}{\sqrt{\varepsilon}}\right)\frac{C(\varepsilon)c_1}{(t+1)^4} \\ &\quad + \left[\left(2 + \frac{1}{\sqrt{\varepsilon}}\right)(\varepsilon c_1 + \delta C(\varepsilon)) + 2c_1\right]\frac{1}{(t+1)^3}, \\ &\qquad 0 \leq t < +\infty, \end{aligned}$$

$$\frac{1}{\sqrt{\varepsilon}}\varphi(t)^2 = \frac{1}{\sqrt{\varepsilon}}\frac{\delta^2}{(t+1)^2} + \frac{2\delta c_1}{\sqrt{\varepsilon}}\frac{1}{(t+1)^3} + \frac{c_1^2}{\sqrt{\varepsilon}}\frac{1}{(t+1)^4}.$$

Let $\delta = 4\sqrt{\varepsilon}$. If c_1 is large enough, then

$$\left(2+\frac{1}{\sqrt{\varepsilon}}\right)\varepsilon\delta+\delta \le \frac{\delta^2}{\sqrt{\varepsilon}},$$
$$\left(2+\frac{1}{\sqrt{\varepsilon}}\right)[\varepsilon c_1+\delta C(\varepsilon)]+2c_1 \le \frac{2\delta c_1}{\sqrt{\varepsilon}},$$
$$\left(2+\frac{1}{\sqrt{\varepsilon}}\right)C(\varepsilon)c_1 \le \frac{c_1^2}{\sqrt{\varepsilon}}.$$

Thus

$$P(t)\varphi(t)-\varphi'(t) \le \frac{1}{\sqrt{\varepsilon}}\varphi(t)^2, \qquad 0 \le t < \infty,$$

$$P(t)\varphi(t)-\frac{1}{\sqrt{\varepsilon}}\varphi(t)^2-\varphi'(t) \le 0, \qquad 0 \le t < +\infty. \tag{21}$$

Substituting (21) into (20) yields

$$\frac{\partial w}{\partial t} \le \Delta w + \left[P(t)-\frac{1}{\sqrt{\varepsilon}}w-\frac{2}{\sqrt{\varepsilon}}\varphi(t)\right]w, \qquad 0 \le t < +\infty, \tag{22}$$

$$\varphi(t)=\frac{4\sqrt{\varepsilon}}{t+1}+\frac{c_1}{(t+1)^2}, \qquad 0 \le t < +\infty. \tag{23}$$

By definition of $w(x,t)$ we have

$$w(x,0)=u(x,0)-\varphi(0),$$
$$u(x,0) \le 2R(x,0) \le 2c_0,$$
$$\varphi(0)=4\sqrt{\varepsilon}+c_1,$$

and therefore

$$w(x,0) \le 2c_0-4\sqrt{\varepsilon}-c_1.$$

If we choose $c_1 \ge 2c_0$, then

$$w(x,0) \le 0, \qquad x \in M. \tag{24}$$

From (8) and (23) it follows that there exists a constant $c_2 > 0$ such that

$$\begin{aligned} &0 < u(x,t) \le c_2, \\ &0 < \varphi(t) \le c_2, \qquad 0 \le t < +\infty, \end{aligned} \tag{25}$$

so that

$$-c_2 \le w(x,t) \le c_2, \qquad 0 \le t < +\infty, \tag{26}$$

$$\left[P(t) - \frac{1}{\sqrt{\varepsilon}}w - \frac{2}{\sqrt{\varepsilon}}\varphi(t)\right] w \le c_3|w|, \qquad 0 \le t < +\infty. \tag{27}$$

By means of (22), (24), (26), (27) and Theorem 4.6 we get

$$\begin{aligned} w(x,t) &\le 0, \qquad 0 \le t < +\infty, \\ u(x,t) &\le \varphi(t), \qquad 0 \le t < +\infty, \end{aligned} \tag{28}$$

and finally the following:

$$u(x,t) \le \frac{4\sqrt{\varepsilon}}{t+1} + \frac{c_1}{(t+1)^2}, \qquad 0 \le t < +\infty.$$

Thus we have proved the following lemma.

Lemma 7.1. *Suppose $u(x,t) \in C^\infty(M \times [0,+\infty))$ is the solution of equation* (6). *Then*

$$0 < R(x,t) \le u(x,t) \le \frac{4\sqrt{\varepsilon}}{t+1} + \frac{c_1}{(t+1)^2}, \qquad 0 \le t < +\infty, \tag{29}$$

where $c_1 = c_1(\varepsilon, c_0) > 0$ depends only on ε and c_0.

Since

$$\frac{\partial u}{\partial t} = \Delta u + P(t)u - \frac{1}{\sqrt{\varepsilon}}u^2, \qquad 0 \le t < +\infty,$$

we have

$$\begin{aligned} \frac{\partial u_i}{\partial t} &= \left(\frac{\partial u}{\partial t}\right)_i = \left[\Delta u + P(t)u - \frac{1}{\sqrt{\varepsilon}}u^2\right]_i \\ &= u_{kki} + P(t)u_i - \frac{2}{\sqrt{\varepsilon}}uu_i \\ &= u_{ikk} - R_{ik}u_k + P(t)u_i - \frac{2}{\sqrt{\varepsilon}}uu_i, \end{aligned}$$

where $u_i = \nabla_i u$ is the covariant derivative,

$$\frac{\partial u_i}{\partial t} = \Delta u_i - R_{ik}u_k + P(t)u_i - \frac{2}{\sqrt{\varepsilon}}uu_i, \qquad 0 \le t < +\infty, \tag{30}$$

$$\frac{\partial}{\partial t}|\nabla_i u|^2 = \Delta|\nabla_i u|^2 - 2|u_{ij}|^2 + 2R_{ij}u_iu_j - 2R_{ik}u_iu_k + 2P(t)|\nabla_i u|^2 - \frac{4}{\sqrt{\varepsilon}}u|\nabla_i u|^2,$$

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla_i u|^2 = \Delta|\nabla_i u|^2 - 2|u_{ij}|^2 + 2P(t)|\nabla_i u|^2 \\ - \frac{4}{\sqrt{\varepsilon}}u|\nabla_i u|^2, \qquad 0 \le t < +\infty,\end{aligned} \tag{31}$$

$$\frac{\partial}{\partial t}\left(\frac{1}{u^\gamma}\right) = -\frac{\gamma}{u^{\gamma+1}}\frac{\partial u}{\partial t} = -\frac{\gamma}{u^{\gamma+1}}\left[\Delta u + P(t)u - \frac{1}{\sqrt{\varepsilon}}u^2\right],$$

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{1}{u^\gamma}\right) = \Delta\left(\frac{1}{u^\gamma}\right) - \frac{\gamma(\gamma+1)}{u^{\gamma+2}}|\nabla_i u|^2 - \frac{\gamma P(t)}{u^\gamma} \\ + \frac{\gamma}{\sqrt{\varepsilon}}u^{1-\gamma}, \qquad 0 \le t < +\infty.\end{aligned} \tag{32}$$

From (31) and (32) we get

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{|\nabla_i u|^2}{u^\gamma}\right) &= \Delta\left(\frac{|\nabla_i u|^2}{u^\gamma}\right) - 2\nabla_k\left(\frac{1}{u^\gamma}\right)\cdot\nabla_k|\nabla_i u|^2 \\ &\quad + \frac{\gamma}{\sqrt{\varepsilon}}u^{1-\gamma}|\nabla_i u|^2 + \left[2P(t) - \frac{4}{\sqrt{\varepsilon}}u\right]\frac{|\nabla_i u|^2}{u^\gamma} \\ &\quad - \frac{2}{u^\gamma}|u_{ij}|^2 - \frac{\gamma(\gamma+1)}{u^{\gamma+2}}|\nabla_i u|^4 - \frac{\gamma P(t)}{u^\gamma}|\nabla_i u|^2.\end{aligned}$$

Let $w(x,t) = |\nabla_i u|^2/u^\gamma$. Then

$$\begin{aligned}\frac{\partial w}{\partial t} &= \Delta w - 2\nabla_k\left(\frac{1}{u^\gamma}\right)\cdot\nabla_k|\nabla_i u|^2 - \frac{2}{u^\gamma}|u_{ij}|^2 - \frac{\gamma(\gamma+1)}{u^{\gamma+2}}|\nabla_i u|^4 \\ &\quad + \left[2P(t) - \frac{4}{\sqrt{\varepsilon}}u - \gamma P(t) + \frac{\gamma}{\sqrt{\varepsilon}}u\right]w.\end{aligned} \tag{33}$$

Since

$$-2\nabla_k\left(\frac{1}{u^\gamma}\right)\nabla_k|\nabla_i u|^2 = \frac{4\gamma}{u^{\gamma+1}}u_iu_ku_{ik},$$

$$w_k = \nabla_k w = \frac{2u_iu_{ik}}{u^\gamma} - \frac{\gamma}{u^{\gamma+1}}|\nabla_i u|^2u_k,$$

$$\frac{2\gamma}{u^{\gamma+1}}u_iu_ku_{ik} = \frac{\gamma}{u}u_kw_k + \frac{\gamma^2}{u^{\gamma+2}}|\nabla_i u|^4,$$

from (33) we have, for $0 \le t < +\infty$,

$$\begin{aligned}\frac{\partial w}{\partial t} &= \Delta w + \frac{\gamma}{u}\nabla_k u \cdot \nabla_k w - \gamma\left(1-\frac{\gamma}{2}\right)\frac{w^2}{u^{2-\gamma}} \\ &\quad - \frac{2}{u^\gamma}\left|u_{ij} - \frac{\gamma}{2u}u_i u_j\right|^2 + \left[(2-\gamma)P(t) + \frac{\gamma-4}{\sqrt{\varepsilon}}u\right]w.\end{aligned} \tag{34}$$

Now let $0 < \gamma < 2$. Then

$$\gamma\left(1-\frac{\gamma}{2}\right) > 0, \qquad \gamma - 4 < 0. \tag{35}$$

From Lemma 7.1 we know that

$$-\frac{\gamma}{2}\left(1-\frac{\gamma}{2}\right)\frac{w^2}{u^{2-\gamma}} \le -\frac{\gamma}{4}(2-\gamma)\left[\frac{4\sqrt{\varepsilon}}{t+1} + \frac{c_1}{(t+1)^2}\right]^{\gamma-2} w^2,$$

$$\left[(2-\gamma)P(t) + \frac{\gamma-4}{\sqrt{\varepsilon}}u\right]w \le (2-\gamma)P(t)w.$$

Substituting (17) and the above equations into (34) yields

$$\begin{aligned}\frac{\partial w}{\partial t} &\le \Delta w + \frac{\gamma}{u}\nabla_k u \cdot \nabla_k w - \frac{\gamma}{2}\left(1-\frac{\gamma}{2}\right)\frac{w^2}{u^{2-\gamma}} \\ &\quad - \frac{\gamma}{4}(2-\gamma)\left[4\sqrt{\varepsilon} + \frac{c_1}{t+1}\right]^{\gamma-2}(t+1)^{2-\gamma}w^2 \\ &\quad + (2-\gamma)\left(2+\frac{1}{\sqrt{\varepsilon}}\right)\left[\frac{\varepsilon}{t+1} + \frac{C(\varepsilon)}{(t+1)^2}\right]w, \qquad 0 \le t < +\infty,\end{aligned}$$

$$\begin{aligned}\frac{\partial w}{\partial t} &\le \Delta w + \frac{\gamma}{u}\nabla_k u \cdot \nabla_k w - \frac{\gamma}{4}(2-\gamma)\frac{|\nabla_k u|^2}{u^2}w \\ &\quad - c_2(\gamma,\varepsilon)(t+1)^{2-\gamma}w^2 + \frac{c_3(\gamma,\varepsilon)}{(t+1)}w, \qquad 0 \le t < +\infty.\end{aligned} \tag{36}$$

Let

$$F(x,t) = w(x,t)\cdot t = \frac{|\nabla_i u|^2}{u^\gamma}t, \qquad 0 \le t < +\infty.$$

Then from (36) we have

$$\begin{aligned}\frac{\partial F}{\partial t} &\le \Delta F + \frac{\gamma}{u}\nabla_k u \cdot \nabla_k F - \frac{\gamma}{4}(2-\gamma)\frac{|\nabla_k u|^2}{u^2}F \\ &\quad - c_2(\gamma,\varepsilon)(t+1)^{2-\gamma}\frac{F^2}{t} + \frac{c_3(\gamma,\varepsilon)}{(t+1)}F + \frac{F}{t},\end{aligned}$$

$$\begin{aligned}\frac{\partial F}{\partial t} &\le \Delta F + \frac{\gamma}{u}\nabla_k u \cdot \nabla_k F - \frac{\gamma}{4}(2-\gamma)\frac{|\nabla_k u|^2}{u^2}F \\ &\quad + \frac{F}{t}\left[1 + c_3(\gamma,\varepsilon)\left(\frac{t}{t+1}\right) - c_2(\gamma,\varepsilon)(t+1)^{2-\gamma}F\right], \\ &\qquad 0 \le t < +\infty.\end{aligned} \tag{37}$$

By the definition of F we know that

$$F(x,0)\equiv 0, \tag{38}$$

which together with (37) and Theorem 4.12 implies

$$F(x,t)\le c_4(\gamma,\varepsilon),\qquad 0\le t<+\infty.$$

Thus if $0<\gamma<2$, then we have

$$\frac{|\nabla_i u|^2}{u^\gamma}\le\frac{c_4(\gamma,\varepsilon)}{t},\qquad 0\le t<+\infty. \tag{39}$$

Let

$$H(x,t)=w(x,t)t^{3-\gamma}=\frac{|\nabla_i u|^2}{u^\gamma}t^{3-\gamma},\qquad 0\le t<+\infty.$$

Then from (36) it follows that, for $0\le t<+\infty$,

$$\begin{aligned}\frac{\partial H}{\partial t}\le{}&\Delta H+\frac{\gamma}{u}\nabla_k u\cdot\nabla_k H-\frac{\gamma}{4}(2-\gamma)\frac{|\nabla_k u|^2}{u^2}H\\&-c_2(\gamma,\varepsilon)(t+1)^{2-\gamma}H^2\left(\frac{1}{t}\right)^{3-\gamma}+\frac{c_3(\gamma,\varepsilon)}{(t+1)}H+\frac{3-\gamma}{t}H,\end{aligned}$$

$$\begin{aligned}\frac{\partial H}{\partial t}&\Delta H+\frac{\gamma}{u}\nabla_k u\cdot\nabla_k H-\frac{\gamma}{4}(2-\gamma)\frac{|\nabla_k u|^2}{u^2}H\\&+\frac{H}{t}\left[(3-\gamma)+c_3(\gamma,\varepsilon)\left(\frac{t}{t+1}\right)-c_2(\gamma,\varepsilon)\left(\frac{t+1}{t}\right)^{2-\gamma}H\right],\end{aligned} \tag{40}$$

which together with Theorem 4.12 yields

$$H(x,t)\le c_5(\gamma,\varepsilon),\qquad 0\le t<+\infty.$$

Thus if $0<\gamma<2$, then

$$\frac{|\nabla_i u|^2}{u^\gamma}\le c_5(\gamma,\varepsilon)\left(\frac{1}{t}\right)^{3-\gamma},\qquad 0\le t<+\infty. \tag{41}$$

Lemma 7.2. *For any $0<\gamma<2$, there exists a constant $c_6>0$ depending only on γ and ε such that*

$$\frac{|\nabla_i u|^2}{u^\gamma}\le\frac{c_6}{t}\left(\frac{1}{t+1}\right)^{2-\gamma},\qquad 0\le t<+\infty. \tag{42}$$

Proof. (42) follows from (39) and (41).

Now we prove the Harnack Inequality for the controlling function $u(x,t)$. We know that

$$\frac{\partial u}{\partial t}=\Delta u+P(t)u-\frac{1}{\sqrt{\varepsilon}}u^2,\qquad 0\le t<+\infty.$$

Let $f(x,t) = \log u(x,t)$. Then

$$\frac{\partial f}{\partial t} = \Delta f + |\nabla_i f|^2 + P(t) - \frac{1}{\sqrt{\varepsilon}} u, \qquad 0 \le t < +\infty, \tag{43}$$

$$\begin{aligned} \frac{\partial}{\partial t}\Delta f &= \frac{\partial}{\partial t}(g^{ij}\nabla_i\nabla_j f) = \left(\frac{\partial g^{ij}}{\partial t}\right)\nabla_i\nabla_j f + g^{ij}\frac{\partial}{\partial t}\nabla_i\nabla_j f \\ &= 2R_{ij}f_{ij} + g^{ij}\frac{\partial}{\partial t}\nabla_i\nabla_j f. \end{aligned} \tag{44}$$

Since

$$\nabla_i\nabla_j f = \frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial f}{\partial x^k},$$

we have

$$\frac{\partial}{\partial t}\nabla_i\nabla_j f = \nabla_i\nabla_j\left(\frac{\partial f}{\partial t}\right) - \frac{\partial \Gamma_{ij}^k}{\partial t}\nabla_k f,$$

$$g^{ij}\frac{\partial}{\partial t}\nabla_i\nabla_j f = \Delta\left(\frac{\partial f}{\partial t}\right) - g^{ij} f_k \frac{\partial \Gamma_{ij}^k}{\partial t}.$$

We have proved the following formula in §4:

$$\frac{\partial \Gamma_{ij}^k}{\partial t} = g^{km}(\nabla_m R_{ij} - \nabla_i R_{jm} - \nabla_j R_{im}), \tag{45}$$

which together with the Bianchi identity implies

$$g^{ij}\frac{\partial \Gamma_{ij}^k}{\partial t} = 0 \quad \text{for any } k. \tag{46}$$

Therefore

$$g^{ij}\frac{\partial}{\partial t}\nabla_i\nabla_j f = \Delta\left(\frac{\partial f}{\partial t}\right).$$

Substituting this into (44), we have

$$\begin{aligned} \frac{\partial}{\partial t}\Delta f &= 2R_{ij}f_{ij} + \Delta\left(\frac{\partial f}{\partial t}\right) \\ &= 2R_{ij}f_{ij} + \Delta\left[\Delta f + |\nabla_i f|^2 + P(t) - \frac{1}{\sqrt{\varepsilon}}u\right] \\ &= \Delta(\Delta f) + 2\nabla_i f \cdot \nabla_i(\Delta f) + 2f_{ij}^2 + 2R_{ij}f_i f_j \\ &\quad - \frac{1}{\sqrt{\varepsilon}}\Delta u + 2R_{ij}f_{ij}; \end{aligned}$$

the last step comes from

$$\begin{aligned} \Delta|\nabla_i f|^2 &= \left(\sum f_i^2\right)_{kk} = (2f_i f_{ik})_k = 2f_{ik}^2 + 2f_i f_{ikk} \\ &= 2f_{ij}^2 + 2f_{kki}f_i + 2R_{ik}f_i f_k, \end{aligned}$$

where we have used the formula

$$f_{ikk} = f_{kki} + R_{ik} f_k. \tag{47}$$

Thus we get

$$\begin{aligned}\frac{\partial}{\partial t}(\Delta f) &= \Delta(\Delta f) + 2\nabla_i f \cdot \nabla_i(\Delta f) + 2f_{ij}^2 + 2R_{ij} f_{ij} \\ &\quad + 2R_{ij} f_i f_j - \frac{1}{\sqrt{\varepsilon}}\Delta u, \qquad 0 \le t < +\infty.\end{aligned} \tag{48}$$

Since

$$\Delta f = \Delta \log u = \frac{\Delta u}{u} - \frac{|\nabla_i u|^2}{u^2}, \tag{49}$$

we have

$$\Delta u = u\Delta f + u|\nabla_i f|^2,$$

and therefore (48) becomes

$$\begin{aligned}\frac{\partial}{\partial t}(\Delta f) &= \Delta(\Delta f) + 2\nabla_i f \cdot \nabla_i(\Delta f) + 2f_{ij}^2 + 2R_{ij} f_{ij} \\ &\quad + 2R_{ij} f_i f_j - \frac{1}{\sqrt{\varepsilon}}u\Delta f - \frac{1}{\sqrt{\varepsilon}}u|\nabla_i f|^2.\end{aligned} \tag{50}$$

From (60) of §6 it follows that $R_{ij} > 0$, so that

$$\begin{aligned}&2R_{ij} f_i f_j \ge 0, \qquad 0 \le t < +\infty, \\ &|2R_{ij} f_{ij}| \le |R_{ij}|^2 + f_{ij}^2 = S + f_{ij}^2.\end{aligned} \tag{51}$$

Since $\frac{1}{n}R^2 \le S \le R^2$, from Lemma 7.1 we know that $R \le u$, $S \le R^2 \le u^2$, and

$$|2R_{ij} f_{ij}| \le u^2 + f_{ij}^2, \qquad 0 \le t < +\infty. \tag{52}$$

Combining (50), (51), and (52) gives

$$\frac{\partial}{\partial t}(\Delta f) \ge \Delta(\Delta f) + 2\nabla_i f \cdot \nabla_i(\Delta f) + f_{ij}^2 - u^2 - \frac{1}{\sqrt{\varepsilon}}u\Delta f - \frac{1}{\sqrt{\varepsilon}}u|\nabla_i f|^2. \tag{53}$$

On the other hand, we have

$$f_{ij}^2 \ge \frac{1}{n}\left(\sum f_{ii}\right)^2 = \frac{1}{n}(\Delta f)^2, \tag{54}$$

$$\left|-\frac{1}{\sqrt{\varepsilon}}u\Delta f\right| \le \frac{1}{4n}(\Delta f)^2 + \frac{n}{\varepsilon}u^2. \tag{55}$$

Substituting (54) and (55) into (53) yields

$$\begin{aligned}\frac{\partial}{\partial t}(\Delta f) &\ge \Delta(\Delta f) + 2\nabla_i f \cdot \nabla_i(\Delta f) + \frac{1}{2}f_{ij}^2 + \frac{1}{4n}(\Delta f)^2 \\ &\quad - \left(1 + \frac{n}{\varepsilon}\right)u^2 - \frac{1}{\sqrt{\varepsilon}}u|\nabla_i f|^2, \qquad 0 \le t < +\infty.\end{aligned} \tag{56}$$

From Lemma 7.2 we know that

$$0 \le u|\nabla_i f|^2 = \frac{|\nabla_i u|^2}{u} \le \frac{c_6(\varepsilon)}{t}\left(\frac{1}{t+1}\right), \qquad 0 \le t < +\infty. \tag{57}$$

Substituting (57) in (56), and using Lemma 7.1 we get

$$\begin{aligned}\frac{\partial}{\partial t}(\Delta f) \ge \Delta(\Delta f) + 2\nabla_i f \cdot \nabla_i(\Delta f) + \frac{1}{2}f_{ij}^2 + \frac{1}{4n}(\Delta f)^2 \\ - \frac{c_7(\varepsilon)}{t}\left(\frac{1}{t+1}\right), \qquad 0 \le t < +\infty.\end{aligned} \tag{58}$$

On the other hand, from (34) it follows that

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{|\nabla_i u|^2}{u^2}\right) = \Delta\left(\frac{|\nabla_i u|^2}{u^2}\right) + \frac{2}{u}\nabla_k u \cdot \nabla_k\left(\frac{|\nabla_i u|^2}{u^2}\right) \\ - \frac{2}{u^2}|u_{ij} - \frac{1}{u}u_i u_j|^2 - \frac{2}{\sqrt{\varepsilon}}\left(\frac{|\nabla_i u|^2}{u}\right).\end{aligned} \tag{59}$$

Recalling that $f = \log u$, we have

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla_i f|^2 = \Delta|\nabla_i f|^2 + 2\nabla_k f \cdot \nabla_k |\nabla_i f|^2 \\ - 2f_{ij}^2 - \frac{2}{\sqrt{\varepsilon}}\left(\frac{|\nabla_i u|^2}{u}\right), \qquad 0 \le t < +\infty.\end{aligned} \tag{60}$$

Suppose $0 < \alpha < \frac{1}{4}$ is a constant. By (58) and (60) we get

$$\begin{aligned}\frac{\partial}{\partial t}(\Delta f + \alpha|\nabla_i f|^2) \ge \Delta(\Delta f + \alpha|\nabla_i f|^2) + 2\nabla_k f \cdot \nabla_k(\Delta f + \alpha|\nabla_i f|^2) \\ + \left(\frac{1}{2} - 2\alpha\right) f_{ij}^2 + \frac{1}{4n}(\Delta f)^2 - \frac{2\alpha}{\sqrt{\varepsilon}}\left(\frac{|\nabla_i u|^2}{u}\right) \\ - \frac{c_7(\varepsilon)}{t}\left(\frac{1}{t+1}\right), \qquad 0 \le t < +\infty.\end{aligned} \tag{61}$$

Since $0 < \alpha < \frac{1}{4}$, we have $\frac{1}{2} - 2\alpha \ge 0$, and, in consequence of (61) and Lemma 7.2,

$$\begin{aligned}\frac{\partial}{\partial t}[\Delta f + \alpha|\nabla_i f|^2] \ge \Delta[\Delta f + \alpha|\nabla_i f|^2] + 2\nabla_k f \cdot \nabla_k[\Delta f + \alpha|\nabla_i f|^2] \\ + \frac{1}{4n}[\Delta f + \alpha|\nabla_i f|^2]^2 - \frac{\alpha}{2n}|\nabla_i f|^2[\Delta f + \alpha|\nabla_i f|^2] \\ - \frac{c_8(\varepsilon)}{t}\left(\frac{1}{t+1}\right), \qquad 0 \le t < +\infty.\end{aligned} \tag{62}$$

Let

$$F(x,t) = [\Delta f + \alpha|\nabla_i f|^2]t, \qquad 0 \le t < +\infty. \tag{63}$$

Then

$$\frac{\partial F}{\partial t} \geq \Delta F + 2\nabla_k f \cdot \nabla_k F - \frac{\alpha}{2n}|\nabla_k f|^2 F + \frac{1}{4nt}F^2 - c_8(\varepsilon)\left(\frac{1}{t+1}\right) + \frac{F}{t}, \qquad 0 \leq t < +\infty.$$

Finally we have

$$\frac{\partial F}{\partial t} \geq \Delta F + 2\nabla_k f \cdot \nabla_k F - \frac{\alpha}{2n}|\nabla_k f|^2 F + \frac{F}{t}\left[1 - c_8\left(\frac{t}{t+1}\right)\frac{1}{F} + \frac{F}{4n}\right]. \tag{64}$$

By the definition of F we know that

$$F(x,0) \equiv 0. \tag{65}$$

From (64), (65) and Theorem 4.12 it follows that

$$F(x,t) \geq -c_9(\varepsilon), \qquad 0 \leq t < +\infty, \tag{66}$$

where $c_9(\varepsilon)$ is a constant independent of α. Thus

$$\Delta f + \alpha|\nabla_i f|^2 \geq -c_9(\varepsilon)/t, \qquad 0 \leq t < +\infty, \tag{67}$$

for $0 < \alpha < \frac{1}{4}$, and letting $\alpha \to 0$ we get

$$\Delta f \geq -c_9(\varepsilon)/t, \qquad 0 \leq t < +\infty, \tag{68}$$

$$\frac{\Delta u}{u} - \frac{|\nabla_i u|^2}{u^2} \geq -c_9(\varepsilon)/t, \qquad 0 \leq t < +\infty. \tag{69}$$

Lemma 7.3. *There exists a constant $c_{10}(\varepsilon) > 0$ such that*

$$\frac{|\nabla_i u|^2}{u^2} - \frac{1}{u}\frac{\partial u}{\partial t} \leq \frac{c_{10}(\varepsilon)}{t}, \qquad 0 \leq t < +\infty. \tag{70}$$

Proof. Denote $u_t = \partial u/\partial t$. Then

$$u_t = \Delta u + P(t)u - \frac{1}{\sqrt{\varepsilon}}u^2,$$

$$-\frac{u_t}{u} = -\frac{\Delta u}{u} - P(t) + \frac{1}{\sqrt{\varepsilon}}u \leq -\frac{\Delta u}{u} + \frac{1}{\sqrt{\varepsilon}}u,$$

$$\frac{|\nabla_i u|^2}{u^2} - \frac{u_t}{u} \leq \frac{|\nabla_i u|^2}{u^2} - \frac{\Delta u}{u} + \frac{1}{\sqrt{\varepsilon}}u, \qquad 0 \leq t < +\infty. \tag{71}$$

The lemma now follows from Lemma 7.1 and (69).

Now we state the Harnack inequality for the controlling function u.

Lemma 7.4. *For any $x, y \in M$ and $0 < t_1 < t_2 < +\infty$, we have*

$$u(x,t_1) \leq u(y,t_2)\left(\frac{t_2}{t_1}\right)^{c_{10}} \exp\left[\frac{\gamma_0^2(x,y)}{4(t_2 - t_1)}\right],$$

where we use $\gamma_t(x, y)$ *to denote the distance between* x *and* y *with respect to the metric* $g_{ij}(t)$.

Proof. Suppose $\gamma(S)\colon [0,1] \to M$ is a geodesic with respect to the metric $g_{ij}(0)$ such that

$$\begin{gathered} \dot{\gamma}(S) \equiv \gamma_0(x,y), \qquad 0 \le S \le 1, \\ \gamma(0) = y, \quad \gamma(1) = x. \end{gathered}$$

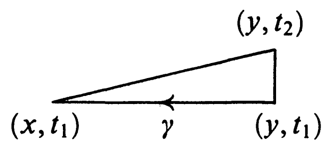
Define

$$\varphi(s) = f(\gamma(s), t_1 s + t_2(1-s)), \qquad 0 \le s \le 1,$$

where $f(x,t) = \log u(x,t)$. Then we have

$$\begin{aligned} f(x,t_1) - f(y,t_2) = \varphi(1) - \varphi(0) &= \int_0^1 \varphi'(s)\, ds \\ &= \int_0^1 [\nabla_i^0 f \cdot \nabla_i^0 \gamma - (t_2 - t_1) f_t]\, ds \\ &\le \int_0^1 [|\nabla_i^0 \gamma| \cdot |\nabla_i^0 f| - (t_2 - t_1) f_t]\, ds, \end{aligned}$$

where ∇^0 denotes the covariant derivative with respect to $g_{ij}(0)$.



Since

$$|\nabla_i^0 \gamma| = \gamma_0(x,y), \qquad |\nabla_i^0 f| \le |\nabla_i f|,$$

$$f(x,t_1) - f(y,t_2) \le \int_0^1 \left[\gamma_0(x,y)|\nabla_i f| - (t_2 - t_1)\frac{u_t}{u}\right] ds, \tag{72}$$

from Lemma 7.3 it follows that

$$-\frac{u_t}{u} \le \frac{c_{10}}{t} - |\nabla_i f|^2, \tag{73}$$

so that

$$\begin{aligned} f(x,t_1) - f(y,t_2) &\le \int_0^1 \left[\gamma_0(x,y)|\nabla_i f| - (t_2 - t_1)|\nabla_i f|^2 + \frac{c_{10}}{t}(t_2 - t_1)\right] ds \\ &\le \int_0^1 \left[\frac{\gamma_0^2(x,y)}{4(t_2 - t_1)} + \frac{c_{10}}{t}(t_2 - t_1)\right] ds \\ &= \frac{\gamma_0^2(x,y)}{4(t_2 - t_1)} + c_{10}\int_{t_1}^{t_2} \frac{dt}{t} = \frac{\gamma_0^2(x,y)}{4(t_2 - t_1)} + c_{10}\log\frac{t_2}{t_1}. \end{aligned}$$

Therefore we have

$$u(x,t_1) \leq u(y,t_2)\left(\frac{t_2}{t_1}\right)^{c_{10}} \exp\left\{\frac{\gamma_0^2(x,y)}{4(t_2-t_1)}\right\}.$$

As soon as we prove the Harnack Inequality for the function $u(x,t)$, we can control $u(x,t)$ better than we did in Lemma 7.1; of course we need extra conditions on the initial data $u(x,0)$. As a first step we prove the following lemma.

Lemma 7.5. *Under the assumptions of the Main Theorem stated in* §1, *for any* $0 < T < +\infty$, *we can find a constant* $C(T) > 0$ *such that*

$$0 < u(x,t) \leq \frac{C(T)}{[1+\gamma_0(x_0,x)]^{2+\delta}} \quad \text{on } M \times [0,T].$$

Proof. In (7) we assumed

$$R(x,0) \leq u(x,0) \leq 2R(x,0),$$

and condition (B) in the Main Theorem implies that

$$0 < u(x,0) \leq 2R(x,0) \leq \frac{\tilde{c}_2}{[1+\gamma_0(x_0,x)]^{2+\delta}}. \tag{74}$$

Suppose $\zeta(x) \in C_0^\infty(\mathbf{R})$ is the cut-off function defined in (102) of §4, and let

$$\psi(x) = \int_M \zeta\left(\frac{\gamma_0(x,y)}{64\sqrt{k_0}}\right) \frac{dy}{[1+\gamma_0(x_0,y)]^{2+\delta}} \Big/ \int_M \zeta\left(\frac{\gamma_0(x_0,y)}{64\sqrt{k_0}}\right) dy.$$

Then, similar to the proof of Lemma 4.2, we know that $\psi(x) \in C^\infty(M)$, $\psi(x) > 0$, and we can find constants $\tilde{c}_3, \tilde{c}_4, \tilde{c}_5 > 0$ such that, for $\forall x \in M$,

$$\frac{\tilde{c}_3}{[1+\gamma_0(x_0,x)]^{2+\delta}} \leq \psi(x) \leq \frac{\tilde{c}_4}{[1+\gamma_0(x_0,x)]^{2+\delta}},$$

$$|\nabla_i^0 \psi(x)| \leq \tilde{c}_5 \psi(x), \qquad |\nabla_i^0 \nabla_j^0 \psi(x)| \leq \tilde{c}_5 \psi(x).$$

Then, similar to the proof of Lemma 4.3, we can show that

$$\Delta_t \psi(x) \leq \tilde{c}_6(T)\psi(x), \qquad 0 \leq t \leq T,\ x \in M.$$

Now define

$$\varphi(x,t) = \frac{\tilde{c}_2}{\tilde{c}_3} e^{\tilde{c}_6 t} \psi(x) \quad \text{on } M \times [0,T].$$

Then from (74) we have

$$0 < u(x,0) \leq \varphi(x,0),$$

$$\frac{\partial \varphi}{\partial t} = \tilde{c}_6 \varphi \geq \Delta_t \varphi \quad \text{on } M \times [0,T].$$

Thus we get

$$(75)\qquad \begin{aligned} &u(x,0) \le \varphi(x,0),\\ &\frac{\partial \varphi}{\partial t} \ge \Delta\varphi \quad \text{on } M\times[0,T],\\ &\varphi(x,t) \le \frac{\tilde{c}_2\tilde{c}_4\tilde{c}_3^{-1}}{[1+\gamma_0(x_0,x)]^{2+\delta}} e^{\tilde{c}_6 T} \quad \text{on } M\times[0,T]. \end{aligned}$$

By (18) we have

$$(76)\qquad \begin{aligned} &\frac{\partial u}{\partial t} \le \Delta u + P(t)u \le \Delta u + \tilde{c}_7 u,\\ &\frac{\partial}{\partial t}(e^{-\tilde{c}_7 t}u) \le \Delta(e^{-\tilde{c}_7 t}u). \end{aligned}$$

Define

$$w(x,t) = \varphi(x,t) - e^{-\tilde{c}_7 t}u(x,t).$$

Then from (75) and (76) it follows that

$$\begin{aligned} &\frac{\partial w}{\partial t} \ge \Delta w \quad \text{on } M\times[0,T],\\ &w(x,0) \ge 0 \quad \text{on } M. \end{aligned}$$

By using Lemmas 7.1 and 4.5 we know respectively that

$$w(x,t) \ge -u(x,t) \ge -\tilde{c}_8 \quad \text{on } M\times[0,T],$$

and that

$$w(x,t) \ge 0 \quad \text{on } M\times[0,T],$$

so that

$$u(x,t) \le e^{\tilde{c}_7 t}\varphi(x,t) \le e^{\tilde{c}_7 T}\varphi(x,t) \quad \text{on } M\times[0,T].$$

Thus we get

$$u(x,t) \le \frac{\tilde{c}_9(T)}{[1+\gamma_0(x_0,x)]^{2+\delta}} \quad \text{on } M\times[0,T]$$

from (75), and $u(x,t) > 0$ from Lemma 7.1. Hence the proof of Lemma 7.5 is complete.

Lemma 7.6. *Under the assumptions of the Main Theorem, suppose* $\varepsilon_1 > 0$ *and* $u(x,0)$ *satisfies*

$$\int_M u(x,0)^{n/2-\varepsilon_1}\, dv_0 \le c_2 < +\infty.$$

If the constant $\varepsilon > 0$ *in* (29) *is small enough, then we can find a constant* $c_3 = c_3(\varepsilon_1, \varepsilon, c_2) > 0$ *such that*

$$u(x,t) \le c_3\left(\frac{1}{t+1}\right)^{1+\varepsilon_1/2}, \qquad 0 \le t < +\infty.$$

Proof. Suppose dv_t is the volume element of the metric $g_{ij}(t)$. Then

$$dv_t = \sqrt{\det(g_{ij}(t))}\, dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n. \tag{77}$$

It is easy to show that

$$\frac{\partial}{\partial t} dv_t = -R\, dv_t. \tag{78}$$

We have

$$\begin{aligned}
&\frac{\partial}{\partial t}\int_M u(x,t)^{n/2-\varepsilon_1}\, dv \\
&\quad = \left(\frac{n}{2}-\varepsilon_1\right)\int_M u^{n/2-\varepsilon_1-1}\frac{\partial u}{\partial t}\, dv + \int_M u^{n/2-\varepsilon_1}\frac{\partial}{\partial t}\, dv \\
&\quad = -\int_M u^{n/2-\varepsilon_1} R\, dv \\
&\qquad + \left(\frac{n}{2}-\varepsilon_1\right)\int_M u^{n/2-\varepsilon_1-1}\left[\Delta u + P(t)u - \frac{1}{\sqrt{\varepsilon}}u^2\right] dv \\
&\quad \le \left(\frac{n}{2}-\varepsilon_1\right)\int_M u^{n/2-\varepsilon_1-1}[\Delta u + P(t)u]\, dv \\
&\quad = \left(\frac{n}{2}-\varepsilon_1\right) P(t)\int_M u^{n/2-\varepsilon_1}\, dv \\
&\qquad - \left(\frac{n}{2}-\varepsilon_1\right)\left(\frac{n}{2}-\varepsilon_1-1\right)\int_M u^{n/2-\varepsilon_1-2}|\nabla_i u|^2\, dv \\
&\quad \le \left(\frac{n}{2}-\varepsilon_1\right) P(t)\int_M u^{n/2-\varepsilon_1}\, dv.
\end{aligned}$$

Because of Lemmas 7.2 and 7.5 we can integrate by parts on the whole complete manifold M. Thus we get

$$\frac{\partial}{\partial t}\int_M u^{n/2-\varepsilon_1}\, dv \le \frac{n}{2}P(t)\int_M u^{n/2-\varepsilon_1}\, dv, \tag{79}$$

which implies

$$\int_{M_t} u^{n/2-\varepsilon_1}\, dv \le e^{(n/2)\int_0^t P(t)\,dt}\int_{M_0} u^{n/2-\varepsilon_1}\, dv_0,$$

$$\int_{M_t} u^{n/2-\varepsilon_1}\, dv \le c_2 e^{\int_0^t (n/2)P(t)\,dt}, \qquad 0 \le t < +\infty. \tag{80}$$

Moreover, by definition we have

$$P(t) = \left(2+\frac{1}{\sqrt{\varepsilon}}\right)\left[\frac{\varepsilon}{t+1} + \frac{C(\varepsilon)}{(t+1)^2}\right],$$

$$\int_0^t P(t)\, dt = (2\varepsilon+\sqrt{\varepsilon})\log(t+1) + \left(2+\frac{1}{\sqrt{\varepsilon}}\right)C(\varepsilon)\left[1-\frac{1}{t+1}\right].$$

If ε is small enough, then

$$\frac{n}{2}\int_0^t P(t)\,dt \le \tilde{C}(\varepsilon) + n\sqrt{\varepsilon}\log(t+1),$$

and, in consequence of (80),

$$\int_{M_t} u^{n/2-\varepsilon_1}\,dv \le c_4(\varepsilon)(t+1)^{n\sqrt{\varepsilon}}, \qquad 0 \le t < +\infty. \tag{81}$$

For fixed t, we can find a point $x \in M$ such that

$$\frac{1}{2}\sup_{M_t} u \le u(x,t) \le \sup_{M_t} u, \tag{82}$$

where $\sup_{M_t} = \sup_{y\in M} u(y,t)$.

Let $\tau = 2t$ and $t \ge 1$. If $\gamma_0^2(x,y) \le t$, then from Lemma 7.4 we have $u(x,t) \le c_5 u(y,\tau)$. Thus we get

$$\begin{aligned}\int_{M_\tau} u^{n/2-\varepsilon_1}\,dv &\ge \int_{\gamma_0^2(x,y)\le t} u^{n/2-\varepsilon_1}(y,\tau)\,dv_\tau(y)\\ &\ge c_6\int_{\gamma_0^2(x,y)\le t} u(x,t)^{n/2-\varepsilon_1}\,dv_\tau(y)\\ &\ge c_6 u(x,t)^{n/2-\varepsilon_1}\int_{\gamma_0^2(x,y)\le t} dv_\tau(y).\end{aligned}$$

Let $A = \{y \in M \mid \gamma_0^2(x,y) \le t\}$. Then, using (4),

$$\int_{M_\tau} u^{n/2-\varepsilon_1}\,dv \ge c_6 u(x,t)^{n/2-\varepsilon_1}\int_A dv_\tau, \tag{83}$$

$$\begin{aligned}\frac{\partial}{\partial t}\int_A dv_t = \int_A \frac{\partial}{\partial t}dv_t &= -\int_A R\,dv_t\\ &\ge -\int_A\left[\frac{\varepsilon}{(t+1)} + \frac{C(\varepsilon)}{(t+1)^2}\right]dv_t.\end{aligned}$$

Therefore

$$\frac{\partial}{\partial t}\int_A dv_t \ge -\left[\frac{\varepsilon}{t+1} + \frac{C(\varepsilon)}{(t+1)^2}\right]\int_A dv_t, \quad 0 \le t < +\infty, \tag{84}$$

$$\int_A dv_t \ge \left(\int_A dv_0\right)\exp\left\{-\int_0^t\left[\frac{\varepsilon}{t+1} + \frac{C(\varepsilon)}{(t+1)^2}\right]dt\right\},$$

$$\int_A dv_t \ge \frac{c_7}{(t+1)^\varepsilon}\int_A dv_0, \qquad 0 \le t < +\infty. \tag{85}$$

By condition (A) in the Main Theorem, we have

$$\int_A dv_0 = \int_{\gamma_0^2(x,y)\le t} dv_0(y) \ge C\gamma_0^n \ge C(t+1)^{n/2}.$$

Since we assume $t \geq 1$,

$$\int_A dv_0 \geq C(t+1)^{n/2}, \tag{86}$$

which together with (85) implies

$$\int_A dv_\tau \geq c_8(t+1)^{n/2}(\tau+1)^{-\varepsilon}.$$

Since $\tau = 2t$,

$$\int_A dv_\tau \geq c_9(t+1)^{n/2-\varepsilon}.$$

Substituting this into (83) yields

$$\int_{M_\tau} u^{n/2-\varepsilon_1}\, dv \geq c_{10}(t+1)^{n/2-\varepsilon} u(x,t)^{n/2-\varepsilon_1}. \tag{87}$$

On the other hand, from (81) it follows that

$$\int_{M_\tau} u^{n/2-\varepsilon_1}\, dv \leq c_4(\tau+1)^{n\sqrt{\varepsilon}}.$$

Using $\tau = 2t$ we have

$$\int_{M_\tau} u^{n/2-\varepsilon_1}\, dv \leq \tilde{c}_4(t+1)^{n\sqrt{\varepsilon}}, \tag{88}$$

which together with (87) gives

$$c_{10}(t+1)^{n/2-\varepsilon} u(x,t)^{n/2-\varepsilon_1} \leq \tilde{c}_4(t+1)^{n\sqrt{\varepsilon}},$$

$$u(x,t) \leq c_{11}\left(\frac{1}{t+1}\right)^{[n/2-\varepsilon_1]^{-1}[n/2-\varepsilon-n\sqrt{\varepsilon}]}.$$

Moreover, from (82) we get

$$\sup_{x\in M} u(x,t) \leq c_{12}\left(\frac{1}{t+1}\right)^{[n/2-\varepsilon_1]^{-1}[n/2-\varepsilon-n\sqrt{\varepsilon}]}, \qquad 0 \leq t < +\infty,$$

if $\varepsilon > 0$ is small enough, then

$$\left(\frac{n}{2}-\varepsilon_1\right)^{-1}\cdot\left(\frac{n}{2}-\varepsilon-n\sqrt{\varepsilon}\right) \geq 1+\frac{\varepsilon_1}{2},$$

and therefore

$$u(x,t) \leq C(\varepsilon_1,\varepsilon)\left(\frac{1}{t+1}\right)^{1+\varepsilon_1/2}, \qquad 0 \leq t < +\infty, \tag{89}$$

which completes the proof of Lemma 7.6.

Now we state the main result of this section.

Theorem 7.7. *Under the same hypotheses as in the Main Theorem, if we let*

$$\varepsilon_1 = \frac{n\delta}{8(2+\delta)} > 0,$$

then there exists a constant $c_3 > 0$ depending only on $n, \varepsilon, \delta, c_1$, and c_2 such that if $\varepsilon > 0$ in condition (B) *of the main theorem is small enough, we have*

$$R(x,t) \leq c_3 \left(\frac{1}{t+1}\right)^{1+\varepsilon_1}, \qquad 0 \leq t < +\infty. \tag{90}$$

Proof. By condition (B) of the Main Theorem, we have

$$0 < R(x,0) \leq \frac{c_2}{\gamma_0(x_0,x)^{2+\delta}} \quad \forall x \in M.$$

Thus

$$\int_M R(x,0)^{n/2-2\varepsilon_1}\, dv_0 \leq \tilde{C} < +\infty, \tag{91}$$

which together with (7) implies

$$\int_M u(x,0)^{n/2-2\varepsilon_1}\, dv_0 \leq C < +\infty. \tag{92}$$

Moreover, by Corollary 6.5 there exists a fixed $\eta > 0$ such that

$$R(x,t) \leq \frac{C(n)\varepsilon^\eta}{t+1} + \frac{C(n,\varepsilon)}{(t+1)^2}, \qquad 0 \leq t < +\infty. \tag{93}$$

From Lemma 7.1 it follows that

$$0 < u(x,t) \leq \frac{4\sqrt{C(n)}\varepsilon^{\eta/2}}{t+1} + \frac{\tilde{c}_1}{(t+1)^2}, \qquad 0 \leq t < +\infty. \tag{94}$$

Now, if $\varepsilon > 0$ is small enough, by (92), (94) and Lemma 7.6 we know that

$$u(x,t) \leq c_3 \left(\frac{1}{t+1}\right)^{1+\varepsilon_1}, \qquad 0 \leq t < +\infty. \tag{95}$$

Since from Lemma 7.1

$$0 < R(x,t) \leq u(x,t), \qquad 0 \leq t < +\infty,$$

we have

$$0 < R(x,t) \leq c_3 \left(\frac{1}{t+1}\right)^{1+\varepsilon_1}, \qquad 0 \leq t < +\infty, \tag{96}$$

which completes the proof of the theorem.

8. Higher derivatives of the curvature tensor

In this section we are going to control the higher derivatives of the curvature tensor R_{ijkl}.

Theorem 8.1. *Under the same hypotheses as in the Main Theorem, if we let*

$$\varepsilon_1 = \frac{n\delta}{8(2+\delta)} > 0,$$

then for any fixed $\eta > 0$ and any integer $m \geq 0$, there exist constants $c_m(\eta) > 0$ such that

$$(1) \qquad |\nabla^m R_{ijkl}|^2 \leq c_m(\eta)\left(\frac{1}{t+1}\right)^{2+\varepsilon_1}, \qquad \eta \leq t < +\infty.$$

Proof. From Theorem 7.7 and Corollary 5.8 we know respectively that

$$0 < R \leq C\left(\frac{1}{t+1}\right)^{1+\varepsilon_1}, \qquad 0 \leq t < +\infty,$$

and that

$$|\mathring{\mathrm{R}}\mathrm{m}|^2 \leq \varepsilon R^2, \qquad 0 \leq t < +\infty.$$

Thus

$$|\mathrm{Rm}|^2 = |\mathring{\mathrm{R}}\mathrm{m}|^2 + \frac{2}{n(n-1)}R^2 \leq \left(\varepsilon + \frac{2}{n(n-1)}\right)R^2,$$

and therefore

$$|\mathrm{Rm}|^2 \leq C\left(\varepsilon + \frac{2}{n(n-1)}\right)\left(\frac{1}{t+1}\right)^{2+2\varepsilon_1}, \qquad 0 \leq t < +\infty,$$

$$(2) \qquad |R_{ijkl}|^2 \leq c_0\left(\frac{1}{t+1}\right)^{2+\varepsilon_1}, \qquad 0 \leq t < +\infty.$$

Hence in the case $m = 0$ the theorem is true. Now we prove the theorem by induction. Suppose we already have the following:

$$(3) \qquad |\nabla^S R_{ijkl}|^2 \leq C_s(\eta)\left(\frac{1}{t+1}\right)^{2+\varepsilon_1}, \qquad \eta \leq t < +\infty,$$

for $s = 0, 1, 2, \cdots, m$.

Now suppose $s = m + 1$. From Lemma 3.2 we have

$$\frac{\partial}{\partial t}|\nabla^m R_{ijkl}|^2 = \Delta|\nabla^m R_{ijkl}|^2 - 2|\nabla^{m+1}R_{ijkl}|^2 + \sum_{i+j=m} \nabla^i \mathrm{Rm} * \nabla^j \mathrm{Rm} * \nabla^m \mathrm{Rm}.$$

Let $\alpha = 2 + \varepsilon_1$ and let a be a constant to be determined later. Then

$$\begin{aligned}(4)\qquad &\frac{\partial}{\partial t}[a + (t+1)^\alpha |\nabla^m R_{ijkl}|^2] \\ &= \Delta[a + (t+1)^\alpha|\nabla^m R_{ijkl}|^2] - 2(t+1)^\alpha |\nabla^{m+1} R_{ijkl}|^2 \\ &\quad + (t+1)^\alpha \sum_{i+j=m} \nabla^i \mathrm{Rm} * \nabla^j \mathrm{Rm} * \nabla^m \mathrm{Rm} \\ &\quad + \alpha(t+1)^{\alpha-1} |\nabla^m R_{ijkl}|^2.\end{aligned}$$

Let

$$\varphi(x,t) = a + (t+1)^\alpha |\nabla^m R_{ijkl}|^2.$$

Using the induction hypothesis

$$|\nabla^m R_{ijkl}|^2 \le C_m(\eta)\left(\frac{1}{t+1}\right)^\alpha, \qquad \eta \le t < +\infty,$$

we have

$$(5)\qquad a \le \varphi(x,t) \le a + C_m(\eta), \qquad \eta \le t < +\infty,$$

$$\begin{aligned}(6)\qquad \frac{\partial \varphi}{\partial t} &= \Delta\varphi - 2(t+1)^\alpha |\nabla^{m+1} R_{ijkl}|^2 + \alpha(t+1)^{\alpha-1}|\nabla^m R_{ijkl}|^2 \\ &\quad + (t+1)^\alpha \sum_{i+j=m} \nabla^i \mathrm{Rm} * \nabla^j \mathrm{Rm} * \nabla^m \mathrm{Rm},\end{aligned}$$

$$\alpha(t+1)^{\alpha-1}|\nabla^m R_{ijkl}|^2 \le \frac{\alpha C_m(\eta)}{(t+1)}, \qquad \eta \le t < +\infty,$$

$$(t+1)^\alpha \sum_{i+j=m} \nabla^i \mathrm{Rm} * \nabla^j \mathrm{Rm} * \nabla^m \mathrm{Rm} \le C\left(\frac{1}{t+1}\right)^{\alpha/2}, \qquad \eta \le t < +\infty.$$

Thus from (6) we get

$$(7)\qquad \frac{\partial\varphi}{\partial t} \le \Delta\varphi - 2(t+1)^\alpha |\nabla^{m+1} R_{ijkl}|^2 + \frac{C}{(t+1)}, \qquad \eta \le t < +\infty,$$

where $0 < C < +\infty$ is some constant. From Lemma 3.2 it follows that

$$\begin{aligned}(8)\qquad \frac{\partial}{\partial t}|\nabla^{m+1} R_{ijkl}|^2 &= \Delta|\nabla^{m+1} R_{ijkl}|^2 - 2|\nabla^{m+2} R_{ijkl}|^2 \\ &\quad + \sum_{i+j=m+1} \nabla^i \mathrm{Rm} * \nabla^j \mathrm{Rm} * \nabla^{m+1} \mathrm{Rm}.\end{aligned}$$

Using the induction hypothesis (3) again we have

$$\begin{aligned}\sum_{i+j=m+1} \nabla^i \mathrm{Rm} * \nabla^j \mathrm{Rm} * \nabla^{m+1} \mathrm{Rm} &\le C\left(\frac{1}{t+1}\right)^{\alpha/2} |\nabla^{m+1} R_{ijkl}|^2 \\ &\quad + C\left(\frac{1}{t+1}\right)^{3\alpha/2}, \qquad \eta \le t < +\infty.\end{aligned}$$

Substituting this into (8) yields

$$\frac{\partial}{\partial t}|\nabla^{m+1}R_{ijkl}|^2 \le \Delta|\nabla^{m+1}R_{ijkl}|^2 - 2|\nabla^{m+2}R_{ijkl}|^2 + C\left(\frac{1}{t+1}\right)^{\alpha/2}|\nabla^{m+1}R_{ijkl}|^2 + C\left(\frac{1}{t+1}\right)^{3\alpha/2}, \quad \eta \le t < +\infty,$$

$$\begin{aligned}\frac{\partial}{\partial t}[(t+1)^\alpha|\nabla^{m+1}R_{ijkl}|^2] \le \Delta[(t+1)^\alpha|\nabla^{m+1}R_{ijkl}|^2] &- 2(t+1)^\alpha|\nabla^{m+2}R_{ijkl}|^2 \\ &+ C(t+1)^{\alpha/2}|\nabla^{m+1}R_{ijkl}|^2 \\ &+ C\left(\frac{1}{t+1}\right)^{\alpha/2} + \alpha(t+1)^{\alpha-1}|\nabla^{m+1}R_{ijkl}|^2.\end{aligned}$$

Let $\psi(x,t) = (t+1)^\alpha|\nabla^{m+1}R_{ijkl}|^2$. Then

$$\tag{9} \begin{aligned}\frac{\partial\psi}{\partial t} &\le \Delta\psi - 2(t+1)^\alpha|\nabla^{m+2}R_{ijkl}|^2 + C(t+1)^{\alpha-1}|\nabla^{m+1}R_{ijkl}|^2 \\ &\quad + C\left(\frac{1}{t+1}\right)^{\alpha/2}, \qquad \eta \le t < +\infty, \\ \frac{\partial\psi}{\partial t} &\le \Delta\psi - 2(t+1)^\alpha|\nabla^{m+2}R_{ijkl}|^2 + \frac{C}{(t+1)}\psi + C\left(\frac{1}{t+1}\right)^{\alpha/2} \\ &\qquad \eta \le t < +\infty.\end{aligned}$$

Now define $F(x,t) = \varphi(x,t)\psi(x,t)$. Then

$$\tag{10} F(x,t) = (t+1)^\alpha|\nabla^{m+1}R_{ijkl}|^2[a + (t+1)^\alpha|\nabla^m R_{ijkl}|^2].$$

Combining (6) and (9) gives

$$\tag{11} \begin{aligned}\frac{\partial F}{\partial t} &\le \Delta F - 2\nabla_p\varphi\cdot\nabla_p\psi - 2(t+1)^{2\alpha}|\nabla^{m+1}R_{ijkl}|^4 \\ &\quad + \frac{C}{(t+1)}\psi - 2(t+1)^\alpha\varphi|\nabla^{m+2}R_{ijkl}|^2 + \frac{C}{(t+1)}\varphi\psi \\ &\quad + C\left(\frac{1}{t+1}\right)^{\alpha/2}, \qquad \eta \le t < +\infty.\end{aligned}$$

Using (5) we have

$$\frac{C}{(t+1)}\psi = \frac{C}{(t+1)\varphi}\varphi\psi \le \frac{C}{a}\cdot\frac{F}{(t+1)}, \qquad \eta \le t < +\infty.$$

From (11) and the definitions of $\varphi(x,t)$ and $\psi(x,t)$ we get

$$
\begin{aligned}
\frac{\partial F}{\partial t} \le \Delta F &- 2\nabla_p\varphi\cdot\nabla_p\psi - 2(t+1)^{2\alpha}|\nabla^{m+1}R_{ijkl}|^4 \\
&- 2(t+1)^{\alpha}\varphi|\nabla^{m+2}R_{ijkl}|^2 + \frac{C(a,\eta)}{(t+1)}F \\
&+ C(a,\eta)\left(\frac{1}{t+1}\right)^{\alpha/2}, \qquad \eta \le t < +\infty,
\end{aligned} \tag{12}
$$

$$
\begin{aligned}
-2\nabla_p\varphi\cdot\nabla_p\psi &= -2[(t+1)^{\alpha}\nabla_p|\nabla^m R_{ijkl}|^2][(t+1)^{\alpha}\nabla_p|\nabla^{m+1}R_{ijkl}|^2] \\
&= -2(t+1)^{2\alpha}\nabla_p|\nabla^m R_{ijkl}|^2\cdot\nabla_p|\nabla^{m+1}R_{ijkl}|^2 \\
&= -8(t+1)^{2\alpha}\nabla^m\mathrm{Rm} * \nabla^{m+1}\mathrm{Rm} * \nabla^{m+1}\mathrm{Rm} * \nabla^{m+2}\mathrm{Rm} \\
&\le (t+1)^{2\alpha}|\nabla^{m+1}R_{ijkl}|^4 \\
&\quad + 16(t+1)^{2\alpha}|\nabla^m R_{ijkl}|^2|\nabla^{m+2}R_{ijkl}|^2.
\end{aligned}
$$

Using the induction hypothesis (3) we have

$$
(t+1)^{\alpha}|\nabla^m R_{ijkl}|^2 \le C_m(\eta), \qquad \eta \le t < +\infty.
$$

Thus

$$
\begin{aligned}
-2\nabla_p\varphi\cdot\nabla_p\psi \le\ & (t+1)^{2\alpha}|\nabla^{m+1}R_{ijkl}|^4 \\
&+ 16C_m(\eta)(t+1)^{\alpha}|\nabla^{m+2}R_{ijkl}|^2, \qquad \eta \le t < +\infty.
\end{aligned}
$$

Substituting this into (12) yields

$$
\begin{aligned}
\frac{\partial F}{\partial t} \le \Delta F &- (t+1)^{2\alpha}|\nabla^{m+1}R_{ijkl}|^4 + 16C_m(\eta)(t+1)^{\alpha}|\nabla^{m+2}R_{ijkl}|^2 \\
&- 2(t+1)^{\alpha}\varphi|\nabla^{m+2}R_{ijkl}|^2 + \frac{C(a,\eta)}{(t+1)}F \\
&+ C(a,\eta)\left(\frac{1}{t+1}\right)^{\alpha/2}, \qquad \eta \le t < +\infty,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial F}{\partial t} \le \Delta F &- \psi^2 + [16C_m(\eta) - 2\varphi](t+1)^{\alpha}|\nabla^{m+2}R_{ijkl}|^2 \\
&+ \frac{C(a,\eta)}{(t+1)}F + C(a,\eta)\left(\frac{1}{t+1}\right)^{\alpha/2}, \quad \eta \le t < +\infty.
\end{aligned}
$$

But $a \le \varphi \le a + C_m(\eta)$; if we choose $a \ge 8C_m(\eta)$, then

$$
16C_m(\eta) - 2\varphi \le 0, \qquad \eta \le t < +\infty,
$$

$$
\begin{aligned}
\frac{\partial F}{\partial t} \le \Delta F &- \frac{F^2}{[a + C_m(\eta)]^2} + \frac{C(a,\eta)}{(t+1)}F \\
&+ C(a,\eta)\left(\frac{1}{t+1}\right)^{\alpha/2}, \qquad \eta \le t < +\infty.
\end{aligned} \tag{13}
$$

Obviously we can choose η such that $0 < \eta \le \frac{1}{2}T_0$, where T_0 is the constant in Corollary 5.18. From Corollary 5.18 it follows that

$$F(x,\eta) \le \tilde{C}(\eta) \quad \forall x \in M. \tag{14}$$

Furthermore, from (13), (14) and Theorem 4.12 we have

$$F(x,t) \le C(\eta), \qquad \eta \le t < +\infty,$$
$$[a + (t+1)^\alpha |\nabla^m R_{ijkl}|^2](t+1)^\alpha |\nabla^{m+1} R_{ijkl}|^2 \le C(\eta), \qquad \eta \le t < +\infty.$$

Thus

$$(t+1)^\alpha |\nabla^{m+1} R_{ijkl}|^2 \le \frac{C(\eta)}{a}, \qquad \eta \le t < +\infty.$$

Since $\alpha = 2 + \varepsilon_1$, we get

$$|\nabla^{m+1} R_{ijkl}|^2 \le C_{m+1}(\eta) \left(\frac{1}{t+1}\right)^{2+\varepsilon_1}, \qquad \eta \le t < +\infty. \tag{15}$$

Hence the theorem is also true in the case $s = m + 1$.

Proof of the Main Theorem. By the evolution equation

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij},$$

$R_{ij} > 0$ for all time $0 \le t < +\infty$, and

$$0 < R_{ij} < R g_{ij}, \qquad 0 \le t < +\infty;$$

therefore

$$0 > \frac{\partial}{\partial t} g_{ij} > -2R g_{ij}, \qquad 0 \le t < +\infty, \tag{16}$$

$$g_{ij}(x,0) \ge g_{ij}(x,t) \ge g_{ij}(x,0) e^{-2\int_0^t R(x,t)\,dt}, \qquad 0 \le t < +\infty. \tag{17}$$

From Theorem 7.7 we have

$$0 < R(x,t) \le C \left(\frac{1}{t+1}\right)^{1+\varepsilon_1}, \qquad 0 \le t < +\infty,\ \varepsilon_1 > 0,$$

which implies

$$0 < \int_0^\infty R(x,t)\,dt \le \tilde{C} < +\infty. \tag{18}$$

Therefore combining (16), (17) and (18) yields

$$\begin{aligned} &g_{ij}(x,0) \ge g_{ij}(x,t) \ge e^{-2\tilde{C}} g_{ij}(x,0), \\ &\frac{\partial}{\partial t} g_{ij}(x,t) < 0, \qquad 0 \le t < +\infty. \end{aligned} \tag{19}$$

Thus there exists a metric $g_{ij}(x,\infty) > 0$ such that

$$g_{ij}(x,t) \xrightarrow{C^0} g_{ij}(x,\infty) \quad \text{as } t \to \infty. \tag{20}$$

Since the curvature tensor actually is the second derivative of the metric, from Theorem 8.1 we know that

$$R_{ijkl}(x,\infty) \equiv 0, \qquad x \in M. \tag{21}$$

Therefore we still have

$$g_{ij}(t) \xrightarrow{C^\infty} g_{ij}(\infty) \quad \text{as time } t \to +\infty, \tag{22}$$

and hence we complete the proof of the Main Theorem.

Acknowledgement. The author would like to thank Professors R. Hamilton and S. T. Yau for their help and encouragement.

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