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THE EXISTENCE OF SPECIAL ORTHONORMAL FRAMES

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On an n-dimensional Riemann manifold M, the Laplace operator Δ on functions can be written (locally) in the form

$$\Delta = \sum_{i=1}^{n} X_i \circ X_i \ ,$$

where $\{X_1, \cdots, X_n\}$ is a (local) frame, if and only if the frame is orthonormal and $\operatorname{div} X_i = 0$, $i = 1, \cdots, n$. In Theorem 2.1, we formulate a condition relating the existence of special orthonormal frames to the Riemann curvature tensor. In Theorem 3.6, we show that the stronger condition: X_i is a Killing vector field, $i = 1, \cdots, n$, which implies $\operatorname{div} X_i = 0$, requires that M be Riemannian locally symmetric. It is further shown that most simply connected irreducible Riemannian symmetric spaces cannot have orthonormal frames consisting of Killing vector fields and that the spheres S^n have such frames if and only if $n = 1$, 3, or 7.

1. Introduction and motivation

Let M be an n-dimensional Riemannian manifold of class C^∞ with metric tensor g. The Riemannian connection ∇ is characterized by the conditions

$$X \cdot g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \tag{1}$$

for all vector fields X, Y, and Z

(the connection is a metric connection) and

$$[X, Y] = \nabla_X Y - \nabla_Y X \tag{2}$$

for all vector fields X and Y

(the connection is torsionless).

Let $\{X_1, \cdots, X_n\}$ be an orthonormal frame on an open set U of M, that is,

$$g(X_i, X_j) = \delta_{ij} \ , \qquad i, j = 1, \cdots, n \ .$$

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The Christoffel symbols with respect to this frame are defined by

$$\nabla_{X_j} X_k = \Gamma^i_{jk} X_i\,, \qquad j, k = 1, \cdots, n\,.$$

(The usual summation convention will be assumed, except in the case of contractions.) Then (1) and (2) give

$$(3) \qquad \Gamma^i_{jk} + \Gamma^k_{ji} = 0 \qquad \text{(skew symmetry)}$$

(in particular, $\Gamma^i_{ji} = 0$) and

$$(4) \qquad [X_j, X_k] \equiv c^i_{jk} X_i = (\Gamma^i_{jk} - \Gamma^i_{kj}) X_i\,.$$

From (3) and (4), we obtain

$$(4') \qquad 2\Gamma^i_{jk} = c^i_{jk} - c^k_{ji} + c^j_{ik}\,.$$

In terms of this orthonormal frame, the Laplace operator Δ on functions is given by

$$(5) \qquad \Delta = \sum_{i=1}^{n} (X_i \circ X_i + (\operatorname{div} X_i) X_i)\,,$$

where

$$\operatorname{div} X_i = \sum_{j=1}^{n} \Gamma^j_{ji}\,, \qquad i = 1, \cdots, n.$$

(The classical choice of sign has been adopted in the definition of Δ.)

The question which motivates this paper is the following: given $m \in M$, when is it possible to find an orthonormal frame $\{X_1, \cdots, X_n\}$ in some open neighborhood U of m such that

$$(6) \qquad \Delta = \sum_{i=1}^{n} X_i \circ X_i$$

on U?

Definition 1.1. A Riemannian manifold M is said to have the *divergence property* if, in some neighborhood of each point of M, there exists an orthonormal frame $\{X_1, \cdots, X_n\}$ such that any one of the following equivalent conditions holds:

(a) $\operatorname{div} X_i = 0, \qquad i = 1, \cdots, n;$

(b) equation (6) holds;

(c) $\displaystyle\sum_{j=1}^{n} \Gamma^j_{ji} = 0\,, \qquad i = 1, \cdots, n\,.$

Such a frame will be called *divergence-free*.

We remark that it is not possible to use fewer than n vector fields in (6) nor to use vector fields which are not orthonormal. The vector fields

$$Y_\mu = a^i_\mu X_i \,, \qquad \mu = 1, \cdots, n \,,$$

satisfy

$$\Delta = \sum_{\mu=1}^{n} Y_\mu \circ Y_\mu = \sum_{i=1}^{n} (X_i \circ X_i + (\operatorname{div} X_i) X_i)$$

if and only if

$$\sum_{\mu=1}^{n} a^j_\mu a^i_\mu = \delta^{ji} \,, \qquad j, i = 1, \cdots, n \,, \tag{7}$$

and

$$\sum_{\mu=1}^{n} a^j_\mu (X_j \cdot a^i_\mu) = \operatorname{div} X_i \,, \qquad i = 1, \cdots, n \,.$$

Since (7) implies that the matrix $(a^j_\mu) \in O(n; \boldsymbol{R}) \subset Gl(n; \boldsymbol{R})$, it is not possible to have any $Y_\mu = 0$, and the n vector fields $\{Y_1, \cdots, Y_n\}$ will then also be orthonormal.

We note also that the requirement $\operatorname{div} X_i = 0$, $i = 1, \cdots, n$, may equally well be stated as $\delta\omega^i = 0$, $i = 1, \cdots, n$, where $\{\omega^1, \cdots, \omega^n\}$ is the dual frame, and δ is the metric transpose of the exterior derivative d.

All manifolds of dimension 1 are locally flat and have the divergence property (trivially).

Proposition 1.2. *If* $\dim M = 2$, *then* M *has the divergence property if and only if* M *is locally flat.*

Proof. By skew-symmetry, condition (c) of Definition 1.1 reduces to $\Gamma^2_{21} = 0$ and $\Gamma^1_{12} = 0$; that is, it is necessary and sufficient that all Christoffel symbols vanish for an orthonormal frame to be divergence-free. The known fact that orthonormal frames with vanishing Christoffel symbols exist if and only if the curvature vanishes can also be obtained as a special case of Theorem 2.1 below.

Example 1.3. Let U be an open set in $\boldsymbol{R}^3$, and let $f(x, y, z)$ and $h(x, y, z)$ be functions of class C^∞ on U with

$$\frac{\partial f}{\partial y} + \frac{\partial h}{\partial z} = 0 \,.$$

Define the metric tensor g on U by taking

$$X_1 = \frac{\partial}{\partial x} - f\frac{\partial}{\partial y} - h\frac{\partial}{\partial z} \,, \quad X_2 = \frac{\partial}{\partial y} \,, \quad X_3 = \frac{\partial}{\partial z}$$

to be an orthonormal frame. It is easily (but tediously) verified that this orthonormal frame is divergence-free, that the curvature is not covariant constant (in general), and that this example cannot include the case of constant curvature unless U is a flat Riemmannian manifold.

The rather computational Example 1.3 demonstrates that there exist non-trivial manifolds having the divergence property. It also shows that we are not presently able to give an effective criterion (other than that given in Corollary 2.2 below) for determining when a manifold has the divergence property. This paper will present chiefly results concerning manifolds having the following stronger property.

Definition 1.4. A Riemannian manifold M is said to have the *Killing property* if, in some neighborhood of each point of M, there exists an orthonormal frame $\{X_1, \cdots, X_n\}$ such that each X_i, $i = 1, \cdots, n$, is a Killing vector field (local infinitesimal isometry). Such a frame will be called a *Killing frame.*

Since [13, p. 50] a linear combination, with constant coefficients, of Killing vector fields is again a Killing vector field, a manifold has the Killing property if and only if it is always possible to find frames consisting of Killing vector fields such that $g(X_i, X_j) =$ constant for each choice of i and j.

The normality condition of Definition 1.4 implies that the isometries are "translations", that is, the streamlines of the isometries are geodesics, since a necessary and sufficient condition for this is that the Killing vector field have constant length (cf. [8, p. 349], or [13, p. 50], or take $X = Y$ in Proposition 3.1 below).

The necessary and sufficient condition that a vector field X be a Killing vector field is that the Lie derivative, with respect to X, of the metric tensor g vanish. By (1) and (2), this condition is equivalent to

$$g(\nabla_Y X, Z) + g(Y, \nabla_Z X) = 0 \tag{8}$$

for all vector fields Y and Z .

Thus, an orthonormal frame $\{X_1, \cdots, X_n\}$ is a Killing frame if and only if

$$g(\nabla_{X_j} X_k, X_i) + g(X_j, \nabla_{X_i} X_k) = 0 ,$$

or

$$\Gamma^i_{jk} + \Gamma^j_{ik} = 0 , \qquad i, j, k = 1, \cdots, n . \tag{9}$$

When (9) is combined with (3), we find that the Christoffel symbols for a Killing frame must be skew-symmetric in any pair of indices. In particular, $\Gamma^j_{ji} = 0$, so any manifold which has the Killing property also has the divergence property.

Manifolds of dimension 1 are locally flat, and so have the Killing property (trivially). If $\dim M = 2$, then M has the Killing property if and only if M is locally flat. This follows from Proposition 1.2 and the fact that the Killing property implies the divergence property, or from the classical result that a surface cannot carry even one infinitesimal translation unless it is locally flat.

Example 1.5. The spheres S^n, considered as Riemannian manifolds imbedded in $\boldsymbol{R}^{n+1}$ in the usual way, have the Killing property for $n = 1, 3, 7$; in fact, there is a global Killing frame. The construction depends essentially on the existence of a multiplication in $\boldsymbol{R}^2$ (complex numbers), $\boldsymbol{R}^4$ (quaternions), and $\boldsymbol{R}^8$ (Cayley numbers) (cf. [4, p. 141]). Explicitly, writing points in $\boldsymbol{R}^8$ as column vectors and identifying the tangent spaces to S^7 with hyperplanes, we can define vector fields X_i, $i = 1, \cdots, 7$, by

p	$X_1(p)$	$X_2(p)$	$X_3(p)$	$X_4(p)$	$X_5(p)$	$X_6(p)$	$X_7(p)$
x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8
x^2	$-x^1$	$-x^4$	x^3	$-x^6$	x^5	$-x^8$	x^7
x^3	x^4	$-x^1$	$-x^2$	$-x^7$	x^8	x^5	$-x^6$
x^4	$-x^3$	x^2	$-x^1$	x^8	x^7	$-x^6$	$-x^5$
x^5	x^6	x^7	$-x^8$	$-x^1$	$-x^2$	$-x^3$	x^4
x^6	$-x^5$	$-x^8$	$-x^7$	x^2	$-x^1$	x^4	x^3
x^7	x^8	$-x^5$	x^6	x^3	$-x^4$	$-x^1$	$-x^2$
x^8	$-x^7$	x^6	x^5	$-x^4$	$-x^3$	x^2	$-x^1$

Since $p \cdot X_i(p) = 0$ and $X_i(p) \cdot X_j(p) = \delta_{ij}$, this gives a global orthonormal frame on S^7, which is also a Killing frame since

$$\exp tX_i \colon p \to (\exp tX_i)p = p \cos t + X_i(p) \sin t$$

is an isometry of S^7 onto itself for each $t \in \boldsymbol{R}$.

If we imbed $\boldsymbol{R}^4$ in $\boldsymbol{R}^8$ as the subset $x^5 = x^6 = x^7 = x^8 = 0$, the restrictions of X_1, X_2, X_3 above yield a Killing frame on S^3.

Counterexample 1.6. Eisenhart [1, p. 212] stated that the trajectories of two infinitesimal translations meet at constant angles; that is, if X and Y are Killing vector fields of constant length, then $g(X, Y) = \text{constant}$. This statement was deleted in [2, p. 240], and was questioned in 1952 by Nijenhuis (cf. [8, p. 351, footnote]). A counterexample is found on S^7 by taking $X = [X_1, X_2]$ and $Y = X_3$ in Example 1.5. Then X, being the bracket product of Killing vector fields, is itself a Killing vector field. It is easily verified that the components of $X(p) = [X_1, X_2](p)$, as a point of $\boldsymbol{R}^8$, are $2(x^4, x^3, -x^2, -x^1, x^8, x^7, -x^6, -x^5)$, from which it follows that X has constant length 2 but does not meet $Y = X_3$ at a constant angle on S^7. These considerations show that it is impossible to extend Theorem 1.6 [13, p. 50] to translations as claimed by Yano [13, p. 51].

2. Conditions in terms of curvature

The torsionless metric connection on M will always be denoted by ∇. Let $\tilde{\nabla}$ be any other connection on an open set $U \subset M$. For any vector field X on U,

$$E_X = \nabla_X - \tilde{\nabla}_X \tag{10}$$

is a derivation of the algebra of tensor fields on U. Since the derivations E_X depend linearly (with respect to variable coefficients) on X, they define a tensor field E of type $(1, 2)$ on U. Conversely, any such tensor field E defines a connection $\tilde{\nabla}$ by (10). With respect to an arbitrary frame $\{X_1, \cdots, X_n\}$, the tensor E is expressed in terms of components by

$$E_{X_j}X_k = \eta^i_{jk}X_i ,$$

where

$$\eta^i_{jk} = \Gamma^i_{jk} - \tilde{\Gamma}^i_{jk} . \tag{10'}$$

The curvature transformations $\tilde{R}(X, Y)$ of $\tilde{\nabla}$ are given by

$$\begin{aligned}\tilde{R}(X, Y) &\equiv \tilde{\nabla}_X \circ \tilde{\nabla}_Y - \tilde{\nabla}_Y \circ \tilde{\nabla}_X - \tilde{\nabla}_{[X,Y]} \equiv [\tilde{\nabla}_X, \tilde{\nabla}_Y] - \tilde{\nabla}_{[X,Y]} \\ &= R(X, Y) - [\nabla_X, E_Y] + [\nabla_Y, E_X] + [E_X, E_Y] + E_{[X,Y]} ;\end{aligned}$$

so the connection $\tilde{\nabla}$ will be locally flat if and only if

$$R(X, Y) = [\nabla_X, E_Y] - [\nabla_Y, E_X] - [E_X, E_Y] - E_{[X,Y]} , \tag{11}$$

or, in terms of components,

$$R^i_{jkl} = \nabla_k\eta^i_{lj} - \nabla_l\eta^i_{kj} + \eta^i_{ls}\eta^s_{kj} - \eta^i_{ks}\eta^s_{lj} . \tag{11'}$$

The connection $\tilde{\nabla}$ is a metric connection if and only if (1) is satisfied, and therefore if and only if each E_X is skew-symmetric, that is,

$$g(E_XY, Z) + g(Y, E_XZ) = 0 \tag{12}$$

for all vector fields X, Y, and Z ,

or, in terms of components,

$$g_{si}\eta^s_{jk} + g_{ks}\eta^s_{ji} = 0 , \tag{12'}$$

which becomes

$$(12'') \qquad \eta^i_{jk} + \eta^k_{ji} = 0$$

in the case of an orthonormal frame.

Given a flat metric connection $\tilde{\nabla}$ on U, it is not hard to show that, in some neighborhood of each point m of U, there is an orthonormal frame with respect to which the Christoffel symbols $\tilde{\Gamma}^i_{jk}$ of $\tilde{\nabla}$ vanish, i.e., for which

$$(13) \qquad \eta^i_{jk} = \Gamma^i_{jk}\,, \qquad i, j, k = 1, \cdots, n\,.$$

In fact, the conditions (11′) are the integrability conditions for the system

$$(14) \qquad X_j \cdot v^\mu_k = v^\mu_s(\Gamma^s_{jk} - \eta^s_{jk})\,, \qquad \mu, j, k = 1, \cdots, n\,,$$

whose solutions define a new frame $\{Y_1, \cdots, Y_n\}$ by $X_k = v^\mu_k Y_\mu$, for which the Christoffel symbols of $\tilde{\nabla}$ vanish. Moreover, if the given frame $\{X_i\}$ is orthonormal and (12) holds, and if the initial values of the solutions v^μ_k are chosen so that the matrix $(v^\mu_k)_m$ is orthogonal, then the solution matrix (v^μ_k) is orthogonal, so that $\{Y_\mu\}$ is an orthonormal frame. This follows from

$$\begin{aligned} X_j \cdot \left(\sum_{k=1}^n v^\mu_k v^\nu_k\right) &= \sum_{k=1}^n (X_j \cdot v^\mu_k) v^\nu_k + \sum_{k=1}^n v^\mu_k (X_j \cdot v^\nu_k) \\ &= \sum_{k=1}^n v^\mu_s(\Gamma^s_{jk} - \eta^s_{jk}) v^\nu_k + \sum_{s=1}^n v^\mu_s v^\nu_k (\Gamma^k_{js} - \eta^k_{js}) = 0 \end{aligned}$$

by (3) and (12″).

Thus we have proved the following in one direction.

Theorem 2.1. *Given a tensor field E of type* (1, 2) *satisfying the skew-symmetry condition* (12) *on an open set U, then* (11) *is a necessary and sufficient condition in order that in some neighborhood of each point of U there exist an orthonormal frame satisfying* (13).

Conversely, if we have an orthonormal frame satisfying (13), then (11) is true for this frame by direct computation, and therefore for all frames since (11) is tensorial.

Practical application of Theorem 2.1 depends on being able to recognize the desired properties of the Christoffel symbols Γ^i_{jk} for a special orthonormal frame in terms of the components η^i_{jk} of E relative to an arbitrary frame. Inspection of the transformation laws for a tensor of type (1, 2) gives easily

Corollary 2.2. *A necessary and sufficient condition that a Riemannian manifold have the divergence property is that, for arbitrary frames, the Riemann curvature tensor can be written (locally) in the form* (11′) *where the tensor η^i_{jk} satisfies not only the skew-symmetry condition* (12′) *but also*

$$\sum_{j=1}^n \eta^j_{jk} = 0\,, \qquad k = 1, \cdots, n\,.$$

Corollary 2.3. *A necessary and sufficient condition that a Riemannian* *manifold have the Killing property is that, for arbitrary orthonormal frames, the Riemann curvature tensor can be written (locally) in the form* (11′) *where the tensor* η^i_{jk} *satisfies not only the skew-symmetry condition* (12″) *but also*

$$\eta^i_{jk} + \eta^j_{ik} = 0\,, \qquad i, j, k = 1, \cdots, n\,. \tag{15}$$

The condition

$$\eta^i_{jk} + \eta^i_{kj} = 0 \tag{16}$$

for arbitrary frames is a necessary and sufficient condition that ∇ and $\tilde{\nabla}$ have the same geodesics [5, p. 146]. Since (16) and (12″) imply (15), while (15) and (12″) imply (16), we have also

Corollary 2.4. *A necessary and sufficient condition that a Riemannian manifold have the Killing property is that there exist (locally) a flat metric connection (with torsion, in general) having the same geodesics as the Riemannian connection.*

Example 2.5. Any compact connected Lie group G has the Killing property if the metric is the bi-invariant metric induced by the Killing form of its Lie algebra $\mathfrak{g}$. (Cf. [3, p. 92, pp. 125–6, pp. 188–9]: the canonical flat connection, making left-invariant vector fields covariant constant, has the same geodesics at $e \in G$ as the Riemannian connection, namely, the one-parameter subgroups.) It will follow from Proposition 3.7 that, for this example, the tensor η^i_{jk} in (11′) is covariant constant so that (11′) reduces to a condition considered by Eisenhart [1, p. 137] but without requiring that the connection be metric nor that the η^i_{jk} satisfy (12′). Right-invariant vector fields on G can be used similarly to give Killing frames on G.

The condition that a special orthonormal frame consist of infinitesimal conformal motions (rather than of infinitesimal isometries) is also invariant, but does not give anything new.

Proposition 2.6. *A "conformal" frame on a Riemannian manifold must be a Killing frame.*

Proof. The appropriate conditions (for orthonormal frames) are (12″) and

$$\eta^i_{jk} + \eta^j_{ik} = \delta^i_j \lambda_k \tag{17}$$

for suitable λ_k determining a 1-form $\lambda = \lambda_k \omega^k$. However, these conditions imply $\lambda = 0$, as is seen by alternate applications of (17) and (12″):

$$\begin{aligned}
\eta^i_{jk} &= -\eta^j_{ik} + \delta^i_j\lambda_k = \eta^k_{ij} + \delta^i_j\lambda_k = -\eta^i_{kj} + \delta^k_i\lambda_j + \delta^i_j\lambda_k \\
&= \eta^j_{ki} + \delta^k_i\lambda_j + \delta^i_j\lambda_k = -\eta^k_{ji} + \delta^j_k\lambda_i + \delta^k_i\lambda_j + \delta^i_j\lambda_k \\
&= \eta^i_{jk} + \delta^j_k\lambda_i + \delta^k_i\lambda_j + \delta^i_j\lambda_k\,,
\end{aligned}$$

which implies

$$\delta^j_k\lambda_i + \delta^k_i\lambda_j + \delta^i_j\lambda_k = 0\,, \quad \text{or} \quad \lambda_j = 0\,.$$

3. Consequences of the Killing property

If, for any vector field X defined on U, the derivation A_X of the algebra of tensor fields on U is defined by

$$A_X \equiv L_X - \nabla_X ,$$

then [5, p. 235]

$$\nabla_Y X = -A_X Y \qquad \text{for all vector fields } Y, \tag{18}$$

since the connection ∇ is torsionless. The condition (8) that X be a Killing vector field is therefore equivalent to the condition

$$g(A_X Y, Z) + g(Y, A_X Z) = 0 \qquad \text{for all vector fields } Y \text{ and } Z; \tag{19}$$

i.e., that A_X be skew-symmetric with respect to the metric tensor g. Moreover [5, p. 235], [7, p. 110], [6, p. 535]

$$\nabla_Y A_X = [\nabla_Y, A_X] = R(X, Y) \qquad \text{for all vector fields } Y, \tag{20}$$

and

$$\begin{aligned}(\nabla_X R)(Y, Z) &= -(A_X R)(Y, Z)\\ &\equiv -[A_X, R(Y, Z)] + R(A_X Y, Z) + R(Y, A_X Z)\end{aligned} \qquad \text{for all vector fields } Y \text{ and } Z, \tag{21}$$

whenever X is a Killing vector field while, if X and Y are both Killing vector fields, then the Killing vector field $[X, Y]$ satisfies

$$A_{[X,Y]} = [A_X, A_Y] - R(X, Y) . \tag{22}$$

Proposition 3.1. *Suppose that X and Y are Killing vector fields. Then*

$$A_X Y + A_Y X = 0 \tag{23}$$

if and only if

$$g(X, Y) = \textit{constant} . \tag{24}$$

Proof. By (1), (18), and (19), we have

$$\begin{aligned}Z \cdot g(X, Y) &= g(\nabla_Z X, Y) + g(X, \nabla_Z Y)\\ &= -g(A_X Z, Y) - g(X, A_Y Z) = g(Z, A_X Y + A_Y X)\end{aligned}$$

for arbitrary vector fields Z.

Now assume that M has the Killing property and that $\{X_1, \cdots, X_n\}$ is a Killing frame. By Proposition 3.1 (or (9) and (3)), we have

$$A_{X_j}X_k + A_{X_k}X_j = 0\,, \qquad j, k = 1, \cdots, n\,. \tag{25}$$

The tensor E of § 2, corresponding to our Killing frame, is determined by

$$E_Y X_k = \nabla_Y X_k\,, \qquad k = 1, \cdots, n\,, \tag{26}$$

for all vector fields Y,

which expresses (13), and satisfies the skew-symmetry conditions (cf. (12) and (16))

$$g(E_X Y, Z) + g(Y, E_X Z) = 0\,,$$
$$E_X Y + E_Y X = 0\,,$$

which will be used without explicit reference. We have also, from (18), (26), and skew-symmetry,

$$E_{X_j} = A_{X_j}\,, \qquad j = 1, \cdots, n\,. \tag{27}$$

From (2) and (26) we have

$$[X_k, X_l] = \nabla_{X_k}X_l - \nabla_{X_l}X_k = 2E_{X_k}X_l\,; \tag{28}$$

so

$$E_{[X_k, X_l]}X_j = -E_{X_j}[X_k, X_l] = -2E_{X_j}E_{X_k}X_l\,. \tag{29}$$

Equations (11), (27), and (20) give

$$\begin{aligned} R(X_k, X_l) &= \nabla_{X_k}E_{X_l} - \nabla_{X_l}E_{X_k} - [E_{X_k}, E_{X_l}] - E_{[X_k, X_l]} \\ &= R(X_l, X_k) - R(X_k, X_l) - [E_{X_k}, E_{X_l}] - E_{[X_k, X_l]}\,; \end{aligned}$$

so

$$3R(X_k, X_l) = -[E_{X_k}, E_{X_l}] - E_{[X_k, X_l]}\,,$$

and

$$3R(X_k, X_l)X_j = -E_{X_k}E_{X_l}X_j + E_{X_l}E_{X_k}X_j + 2E_{X_j}E_{X_k}X_l\,, \tag{30}$$

or

$$3R^i_{jkl} = -\eta^i_{ks}\eta^s_{lj} + \eta^i_{ls}\eta^s_{kj} + 2\eta^i_{js}\eta^s_{kl}\,. \tag{30'}$$

Proposition 3.2. *If M has the Killing property, then all sectional curvatures are non-negative.*

Proof. From (30)

$$g(X_k, R(X_k, X_l)X_l) = g(X_k, E_{X_l}E_{X_k}X_l) \\ = g(E_{X_k}X_l, E_{X_k}X_l) \geq 0 \, .$$

Since any frame obtained from the given Killing frame by a constant orthogonal matrix is again a Killing frame, this argument covers all sectional curvatures.

Lemma 3.3. *The quantity $X_l \cdot g(X_i, E_{X_k}X_j)$ is skew-symmetric in any pair of indices.*

Proof. We compute

$$\begin{aligned} 3X_l \cdot g(X_i, E_{X_k}X_j) &= 3g(\nabla_{X_l}X_i, E_{X_k}X_j) + 3g(X_i, \nabla_{X_l}E_{X_k}X_j) \\ &= 3g(E_{X_l}X_i, E_{X_k}X_j) + g(X_i, 3R(X_k, X_l)X_j) + 3g(X_i, E_{X_k}E_{X_l}X_j) \\ &= 2g(X_i, [E_{X_k}, E_{X_l}]X_j + E_{X_j}E_{X_k}X_l) \, , \end{aligned}$$

where the first step follows from (1), the second from (26), (27), and (20), and the third step from (30). The last expression is clearly skew-symmetric in k and l. Furthermore, the quantity $g(X_i, E_{X_k}X_j)$ is skew-symmetric in any pair of indices.

Lemma 3.4. *Each $R(X_k, X_l)X_j$ is a Killing vector field.*

Proof. By (2), (18), (22), (27), (28), and (30), we have

$$\begin{aligned} [X_j, [X_k, X_l]] &= \nabla_{X_j}[X_k, X_l] - \nabla_{[X_k, X_l]}X_j \\ &= -A_{[X_k, X_l]}X_j + A_{X_j}[X_k, X_l] \\ &= -[E_{X_k}, E_{X_l}]X_j + R(X_k, X_l)X_j + 2E_{X_j}E_{X_k}X_l \\ &= 4R(X_k, X_l)X_j \, . \end{aligned}$$

Lemma 3.5. *Each Killing vector field $Y = R(X_k, X_l)X_j$ satisfies (24) for $X = X_h$, $h = 1, \cdots, n$, and therefore also*

$$A_{R(X_k, X_l)X_j}X_h = -A_{X_h}R(X_k, X_l)X_j = -E_{X_h}R(X_k, X_l)X_j \, . \tag{31}$$

Proof. It suffices to prove

$$X_h \cdot g(X_i, E_{X_l}E_{X_k}X_j) = -X_h \cdot g(X_i, E_{X_j}E_{X_k}X_l) \, , \tag{32}$$

since this will give, by (30),

$$\begin{aligned} &X_h \cdot g(X_i, 3R(X_k, X_l)X_j) \\ &\quad = X_h \cdot g(X_i, -E_{X_k}E_{X_l}X_j + E_{X_l}E_{X_k}X_j + 2E_{X_j}E_{X_k}X_l) = 0 \end{aligned}$$

for $h = 1, \cdots, n$. We prove (32) by using

$$[X_k, X_i] = (\Gamma^s_{ki} - \Gamma^s_{ik})X_s = 2\Gamma^s_{ki}X_s$$

and the skew-symmetry of

$$X_i \cdot \Gamma^j_{lh} = X_i \cdot g(X_j, E_{X_l} X_h)$$

as shown in Lemma 3.3. Then

$$\begin{aligned} X_k \cdot (X_j \cdot \Gamma^i_{lh}) &= -X_k \cdot (X_i \cdot \Gamma^j_{lh}) \\ &= -X_i \cdot (X_k \cdot \Gamma^j_{lh}) - 2(X_s \cdot \Gamma^j_{lh})\Gamma^s_{ki} \\ &= X_i \cdot (X_j \cdot \Gamma^k_{lh}) - 2(X_s \cdot \Gamma^j_{lh})\Gamma^s_{ki} , \end{aligned}$$

and

$$\begin{aligned} X_k \cdot (X_j \cdot \Gamma^i_{lh}) &= X_j \cdot (X_k \cdot \Gamma^i_{lh}) + 2(X_s \cdot \Gamma^i_{lh})\Gamma^s_{kj} \\ &= -X_i \cdot (X_j \cdot \Gamma^k_{lh}) - 2(X_s \cdot \Gamma^k_{lh})\Gamma^s_{ji} + 2(X_s \cdot \Gamma^i_{lh})\Gamma^s_{kj} . \end{aligned}$$

These give

$$\begin{aligned} &2X_i \cdot (X_j \cdot \Gamma^k_{lh}) \\ &\quad = 2(X_s \cdot \Gamma^j_{lh})\Gamma^s_{ki} + 2(X_s \cdot \Gamma^k_{lh})\Gamma^s_{ij} + 2(X_s \cdot \Gamma^i_{lh})\Gamma^s_{kj} . \end{aligned}$$

Since $X_i \cdot (X_j \cdot \Gamma^k_{lh})$ is skew-symmetric in l and j, we must have

$$\begin{aligned} &(X_s \cdot \Gamma^k_{lh})\Gamma^s_{ij} + (X_s \cdot \Gamma^i_{lh})\Gamma^s_{kj} \\ &\qquad = -(X_s \cdot \Gamma^k_{jh})\Gamma^s_{il} - (X_s \cdot \Gamma^i_{jh})\Gamma^s_{kl} , \end{aligned}$$

which is equivalent to

$$X_h \cdot (\Gamma^i_{ls}\Gamma^s_{kj}) = -X_h \cdot (\Gamma^i_{js}\Gamma^s_{kl}) . \tag{32'}$$

The above proof also shows that each $R(X_k, X_l)X_j$ is an infinitesimal translation. The same holds also for the Killing vector fields $[X_k, X_l] = 2E_{X_k}X_l$ since

$$g(E_{X_k}X_l, E_{X_k}X_l) = g(X_k, R(X_k, X_l)X_l) = constant ,$$

although these fields do not have constant coefficients in general.

Theorem 3.6. *If M has the Killing property, then M is Riemannian locally symmetric, that is, the Riemannian curvature is covariant constant.*

Proof. From Lemma 3.5, we have

$$\nabla_{X_h}(R(X_k, X_l)X_j) = -A_{R(X_k,X_l)X_j}X_h = E_{X_h}R(X_k, X_l)X_j .$$

Thus

$$\begin{aligned}(\nabla_{X_h}R)(X_k, X_l)X_j \\ &= \nabla_{X_h}(R(X_k, X_l)X_j) - R(\nabla_{X_h}X_k, X_l)X_j \\ &\quad - R(X_k, \nabla_{X_h}X_l)X_j - R(X_k, X_l)\nabla_{X_h}X_j \\ &= E_{X_h}R(X_k, X_l)X_j - R(E_{X_h}X_k, X_l)X_j \\ &\quad - R(X_k, E_{X_h}X_l)X_j - R(X_k, X_l)E_{X_h}X_j ,\end{aligned}$$

or

$$\begin{aligned}(\nabla_{X_h}R)(X_k, X_l) &= [E_{X_h}, R(X_k, X_l)] - R(E_{X_h}X_k, X_l) - R(X_k, E_{X_h}X_l) \\ &= -(\nabla_{X_h}R)(X_k, X_l)\end{aligned}$$

by (21) and (27).

Proposition 3.7. *For a Killing frame $\{X_1, \cdots, X_n\}$, the following conditions (for arbitrary i, j, k, l) are equivalent:*

(a) *the coefficients $c^i_{jk} \equiv g(X_i, [X_j, X_k])$ in (4) are constant;*

(b) *the Christoffel symbols $\Gamma^i_{jk} = g(X_i, \nabla_{X_j}X_k)$ are constant;*

(c) $R(X_k, X_l)X_j = E_{X_j}E_{X_k}X_l$;

(d) *the tensor η^i_{jk} in (11′) is covariant constant.*

Proof. The equivalence of (a) and (b) follows from (4) and (4′). As noted in the proof of Lemma 3.4, we have

$$4R(X_k, X_l)X_j = -A_{[X_k,X_l]}X_j + A_{X_j}[X_k, X_l] ,$$

which will equal $2A_{X_j}[X_k, X_l] = 4E_{X_j}E_{X_k}X_l$ as required in condition (c) if and only if

$$-A_{[X_k,X_l]}X_j = A_{X_j}[X_k, X_l] ,$$

which is equivalent to (a) by Proposition 3.1. Finally we compute, in the Killing frame, using (13) and (27),

$$\begin{aligned}\nabla_l\eta^i_{kj} &= X_l\cdot\eta^i_{kj} + \Gamma^i_{ls}\eta^s_{kj} - \eta^i_{sj}\Gamma^s_{lk} - \eta^i_{ks}\Gamma^s_{lj} \\ &= X_l\cdot g(X_i, E_{X_k}X_j) + g(X_i, E_{X_l}E_{X_k}X_j + E_{X_j}E_{X_l}X_k - E_{X_k}E_{X_l}X_j) \\ &= -\frac{1}{2}X_l\cdot g(X_i, E_{X_k}X_j) = -\frac{1}{4}X_l\cdot g(X_i, [X_k, X_j]) ,\end{aligned}$$

where the next to the last step follows from the computation used in the proof of Lemma 3.3. This shows that (a) and (d) are equivalent.

For use in § 5, we note that the next to the last step above can be altered to give

$$\nabla_l\eta^1_{kj} = \frac{1}{3}g(X_i, E_{X_l}E_{X_k}X_j + E_{X_j}E_{X_l}X_k - E_{X_k}E_{X_l}X_j) . \tag{33}$$

Condition (a) is clearly satisfied in Example 2.5. By Lemma 3.3, condition (b) is satisfied if $\dim M < 4$. *Ad hoc* proofs (omitted) show that this is also true for $\dim M \leq 6$. The computation in Counterexample 1.6 shows that (a) is not satisfied in the case $M = S^7$.

4. The Killing property and symmetric spaces

By Theorem 3.6, a Riemannian manifold having the Killing property must be locally symmetric. Thus, each point of a connected Riemannian manifold having the Killing property has an open neighborhood which is isometric to an open neighborhood in a simply connected Riemannian symmetric space M. Then M also has the Killing property and, moreover, has global Killing frames. In fact, a local Killing frame exists on M because of the given local isometry, and can be extended uniquely to give a global Killing frame. The extension of each Killing vector field to a global Killing vector field is possible since the symmetry implies completeness; the extension remains orthonormal since the Riemannian structure on M is subordinate to a real analytic Riemannian structure (cf. [12, p. 240], [3, p. 187]).

The Lie algebra $\mathfrak{g}$ of the group G of isometries of M is isomorphic to the Lie algebra of global Killing vector fields on M, where $X \in \mathfrak{g}$ corresponds to the Killing vector field X^* of the 1-parameter family of isometries induced by $\exp tX \in G$ acting on M.

For any simply connected Riemannian symmetric space M, there exists [12, p. 243] a Riemannian product decomposition

$$M = M_0 \times M_1 \times \cdots \times M_m , \tag{34}$$

unique up to a permutation of the M_μ, $\mu > 0$, in which M_0 is a Euclidean space and each M_μ, $\mu > 0$, is a simply connected irreducible Riemannian symmetric space. By Proposition 3.2, no factor M_μ, $\mu > 0$, can be of noncompact type if M has the Killing property. The decomposition comes from a decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_m ,$$

direct sum of ideals, and implies [12, p. 243] that any $a \in G$ is the composition of isometries of M_μ, $\mu = 0, 1, \cdots, m$, onto itself, or of M_μ onto M_ν, $\mu\nu > 0$, $\mu \neq \nu$, if M_μ and M_ν are isometric.

Theorem 4.1. *A simply connected Riemannian space M has the Killing property if and only if each factor, in any Riemannian product decomposition* (34), *has the Killing property.*

Proof. (Our original proof has been considerably shortened, following suggestions privately communicated by Joseph A. Wolf.) It is clear that the product of Riemannian manifolds having the Killing property will have the

Killing property with respect to the product Riemannian structure. Conversely, let $\{X_1^*, \cdots, X_n^*\}$ be a Killing frame on M, corresponding to $\{X_1, \cdots, X_n\} \in \mathfrak{g}$, and let X_{i_μ} be the component of X_i lying in $\mathfrak{g}_\mu$, $\mu = 0, 1, \cdots, m$. Then $X_{i\mu}^*$ is tangent to each copy of M_μ in the product decomposition of M, and is induced by the 1-parameter family $\exp tX_{i_\mu}$ which acts trivially on all components except M_μ. Since the metric g on M is the product of the metrics g_μ on M_μ, it follows that $g(X_i^*, X_j^*) =$ constant on M implies $g_\mu(X_{i\mu}^*, X_{j\mu}^*) =$ constant on M_μ. Since the set $\{X_{1\mu}^*, \cdots, X_{n\mu}^*\}$ must generate the tangent space to M_μ at any point, it follows that M_μ has the Killing property, by the remark following Definition 1.4.

The above theorem reduces the local classification of Riemannian manifolds having the Killing property to determining which manifolds appearing in a decomposition (34) have the Killing property.

Certainly a Euclidean space, being flat, has the Killing property. The same is true for a compact simple simply connected Lie group, by Example 2.5, and for the spheres S^1, S^3, and S^7 as noted in Example 1.5.

Many of the remaining possibilities are eliminated on the stronger ground that these manifolds cannot carry even one Killing vector field of constant length. These results follow from the work of Wolf [9], [10], [11], [12], and the discussion below also follows the outline kindly supplied to us by Professor Wolf.

Irreducible components of noncompact type are eliminated, *a fortiori*, by Theorem 1 of [11]. Components of compact type are eliminated if rank G $=$ rank K, where K is the isotropy subgroup of G at some point $m \in M$, since this implies ([10], or [12, p. 255]) that M has positive Euler characteristic and therefore cannot carry any non-vanishing vector field. Also most components of compact type with rank $G >$ rank K, non-trivial K, are eliminated [10] because there are only a finite number of isometries of constant displacement, so that there is no non-trivial 1-parameter family of isometries of constant displacement.

The survivors of this last screening are the odd-dimensional spheres $S^{2n-1} = SO(2n)/SO(2n-1)$ and the spaces $M = SU(2n)/Sp(n)$, $n > 1$. The spheres S^n with $n \neq 1, 3, 7$ do not have the Killing property ([9], or § 5 below). In the remaining case, Killing vector fields of constant length exist [10] but we have not yet determined whether any of these spaces have the Killing property.

5. Spaces of constant curvature

Although the question of which spaces of constant (positive) curvature cannot have the Killing property has been settled in § 4, we give here an independent (local) proof which does not depend on the theory of symmetric spaces, but on tensorial identities. We have

$$3\nabla_l \eta^i_{kj} = \eta^i_{ls}\eta^s_{kj} - \eta^i_{ks}\eta^s_{lj} + \eta^i_{js}\eta^s_{lk} = -3\nabla_k \eta^i_{lj} , \tag{35}$$

which follows from (33) and the skew-symmetry of η, and

$$\nabla_l \eta^i_{kj} = R^i_{jkl} - \eta^i_{js}\eta^s_{kl} , \tag{36}$$

which follows from (30′), (35), and skew-smmetry. These can be used to derive the important identity

$$R^i_{shk}\eta^s_{lj} - \eta^i_{sj}R^s_{lhk} - \eta^i_{ls}R^s_{jhk} = (\nabla_j\eta^i_{ls})\eta^s_{hk} . \tag{37}$$

In fact, we have

$$\begin{aligned}
9(R^i_{shk}\eta^s_{lj} - \eta^i_{sj}R^s_{lhk} - \eta^i_{ls}R^s_{jhk}) &= 9\nabla_h(\nabla_k\eta^i_{lj}) - 9\nabla_k(\nabla_h\eta^i_{lj}) \\
&= 3\nabla_h(\eta^i_{ks}\eta^s_{lj} - \eta^i_{ls}\eta^s_{kj} + \eta^i_{js}\eta^s_{kl}) - 3\nabla_k(\eta^i_{hs}\eta^s_{lj} - \eta^i_{ls}\eta^s_{hj} + \eta^i_{js}\eta^s_{hl}) \\
&= 3(R^i_{skh}\eta^s_{lj} - \eta^i_{ls}R^s_{jkh} + \eta^i_{js}R^s_{lkh}) \\
&\quad -4(\eta^i_{st}\eta^s_{lj} - \eta^i_{ls}\eta^s_{jt} + \eta^i_{js}\eta^s_{lt})\eta^t_{kh} \\
&= -3(R^i_{shk}\eta^s_{lj} - \eta^i_{ls}R^s_{jhk} - \eta^i_{sj}R^s_{lhk}) + 12(\nabla_j\eta^i_{lt})\eta^t_{hk} .
\end{aligned}$$

The first step uses the definition of curvature, the second uses (35); then we work out the covariant derivatives of the products, using (36) for those terms in which ∇_h acts on a factor η involving the index k, and (35) for all other terms. The last step uses (35) and skew-symmetry.

The identity (37), with the left-hand side interpreted as $(\nabla_h\nabla_k - \nabla_k\nabla_h)\eta^i_{lj}$, can be used to give a tensorial proof of Theorem 3.6 by the same devices used in the proofs of (32) and Lemma 3.5.

Proposition 5.1. *A space M of constant positive curvature can have the Killing property only if* $\dim M = 3$ *or* 7.

Proof. In what follows, the summation convention will be assumed only for indices designated by the letters r, s, or t. We assume that

$$R^i_{jkl} = K(\delta^i_k\delta_{jl} - \delta^i_l\delta_{jk}) , \qquad \text{where } K \neq 0 , \tag{38}$$

and that M has the Killing property. When (38) is combined with (30′) for $k = i$, we obtain

$$\eta^i_{js}\eta^s_{il} = \eta^j_{is}\eta^s_{li} = K(\delta^i_i\delta_{jl} - \delta^i_l\delta_{ji}) . \tag{39}$$

When (38) is combined with (37), we obtain

$$\begin{aligned}
&(\nabla_j\eta^i_{ls})\eta^s_{hk} \\
&\quad = K(\delta^i_h\eta^k_{lj} - \delta^i_k\eta^h_{lj} - \eta^i_{hj}\delta_{lk} + \eta^i_{kj}\delta_{lh} - \eta^i_{lh}\delta_{jk} + \eta^i_{lk}\delta_{jh}) .
\end{aligned} \tag{40}$$

We take $h = t$, $k = r$, multiply (40) by η^r_{pt}, and sum on r and t. The left-hand side gives, by (39),

$$\begin{aligned}
(\nabla_j\eta^i_{ls})\eta^s_{tr}\eta^r_{pt} &= (\nabla_j\eta^i_{ls})K(\delta^t_t\delta_{sp} - \delta^t_p\delta_{st}) \\
&= K(n-1)(\nabla_j\eta^i_{lp}) .
\end{aligned}$$

The right-hand side gives

$$K(\delta^i_t\eta^r_{lj} - \delta^i_r\eta^t_{lj} - \eta^i_{tj}\delta_{lr} + \eta^i_{rj}\delta_{lt} - \eta^i_{lt}\delta_{jr} + \eta^i_{lr}\delta_{jt})\eta^r_{pt}$$
$$= -6K(\nabla_j\eta^i_{pl}) = 6K(\nabla_j\eta^i_{lp})$$

by (35). Since $K = 0$ is excluded by hypothesis, the equality of these expressions requires either $n = 7$, or

$$\nabla_j\eta^i_{lp} = 0\,, \qquad i, j, l, p = 1, \cdots, n\,. \tag{41}$$

If $\dim M > 3$, condition (41) cannot be satisfied under the assumption (38). For, in this case, we can choose $i = h \neq j \neq k \neq l$ in (40). Then (41) would imply $0 = K\eta^k_{lj}$, $1 \leq j \neq k \neq l \leq n$, but the vanishing of the η^k_{lj} implies that M is locally flat, by (11′). If $\dim M < 3$, the Killing property implies that M is locally flat, as noted in § 1.

Added in proof. It can be shown that if $X_1, \cdots, X_p$, $p < \dim M$, are orthonormal Killing vector fields, then $(\nabla_X R)(Y, Z)W$ vanishes whenever X, Y, Z, W lie in the p-dimensional distribution generated by $\{X_1, \cdots, X_p\}$. This generalizes Theorem 3.6.

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