

## HYPERSURFACES OF COMPLEX PROJECTIVE SPACE WITH CONSTANT SCALAR CURVATURE

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### 1. Introduction

In his dissertation, B. Smyth [3] classified the complex hypersurfaces of the simply connected complex space forms which are complete and Einsteinian.<sup>1</sup> In particular, he proved the following theorem:

*Let  $M$  be a complete complex hypersurface of the complex projective space  $P_{n+1}(C)$  of dimension  $n + 1$  for  $n \geq 2$ . If  $M$  is an Einstein space with respect to the metric induced from the Fubini-Study metric of  $P_{n+1}(C)$ , then  $M$  is either a complex hyperplane  $P_n(C)$  or a complex quadric in  $P_{n+1}(C)$ .*

The purpose of this note is to point out that the theorem of Smyth combined with the theorem of Riemann-Roch-Hirzebruch yields the following:

*Let  $M$  be a complete complex hypersurface of  $P_{n+1}(C)$ . If  $M$  has constant scalar curvature with respect to the induced metric, then  $M$  is either a complex hyperplane  $P_n(C)$  or a complex quadric in  $P_{n+1}(C)$ .*

### 2. Kähler manifolds with constant scalar curvature

Let  $M$  be a Kähler manifold with metric  $ds^2 = 2 \sum_{\alpha, \beta} g_{\alpha\beta} dz^\alpha d\bar{z}^\beta$  and the fundamental 2-form  $\Phi = \frac{2}{i} \sum_{\alpha, \beta} g_{\alpha\beta} dz^\alpha d\bar{z}^\beta$ . The first Chern class  $c_1(M)$  of  $M$  is represented by the closed 2-form

$$\gamma_1 = \frac{1}{2\pi i} \sum_{\alpha, \beta} R_{\alpha\beta} dz^\alpha d\bar{z}^\beta,$$

where  $R_{\alpha\beta}$  denotes the Ricci tensor. We denote by  $[\Phi]$  and  $[\gamma_1]$  the cohomology classes represented by  $\Phi$  and  $\gamma_1$ , respectively, so that  $c_1(M) = [\gamma_1]$ .

If  $M$  is an Einstein space, then its scalar curvature  $2 \sum_{\alpha, \beta} g^{\alpha\beta} R_{\alpha\beta}$  is constant and  $[\gamma_1] = k[\Phi]$  for some constant  $k$ . Conversely, we have

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<sup>1</sup> In [1] Chern showed that even the corresponding local result is true. Takahashi [4] obtained a partial generalization of the result of Smyth by showing that if a hypersurface in a space of constant holomorphic sectional curvature has parallel Ricci tensor, then it is Einsteinian and symmetric.

**Lemma.** *If  $M$  is a compact Kähler manifold such that its scalar curvature is constant and  $[\gamma_1] = k[\Phi]$ , then  $M$  is an Einstein space.*

In fact, from the harmonic integral theory we see easily (by local calculation) that  $\gamma_1$  is harmonic if and only if the scalar curvature is constant. If  $\gamma_1$  is harmonic and  $[\gamma_1] = k[\Phi]$ , then  $\gamma_1 = k\Phi$ . This proves the lemma.

### 3. The first Chern class of a hypersurface

Let  $h$  be the generator of  $H^2(P_{n+1}(C); \mathbf{Z})$  corresponding to the divisor class of a hyperplane  $P_n(C)$  so that the first Chern class  $c_1(P_{n+1}(C))$  of  $P_{n+1}(C)$  is given by

$$c_1(P_{n+1}(C)) = (n + 2)h .$$

Let  $M$  be a complete complex hypersurface in  $P_{n+1}(C)$ . By the well known theorem of Chow  $M$  is algebraic. Let  $d$  be the degree of  $M$ . Then (cf. Hirzebruch [2, p. 159]) the first Chern class  $c_1(M)$  of  $M$  is given by

$$c_1(M) = (n - d + 2)\bar{h} ,$$

where  $\bar{h}$  is the image of  $h$  under the natural homomorphism  $H^*(P_{n+1}(C); \mathbf{Z}) \rightarrow H^*(M; \mathbf{Z})$  induced by the imbedding  $M \rightarrow P_{n+1}(C)$ .

Let  $\Psi$  be the fundamental 2-form of  $P_{n+1}(C)$ . Since  $\dim H^*(P_{n+1}(C); \mathbf{R}) = 1$ , it follows that  $[\Psi] = ah$ , where  $a$  is a constant. Since the fundamental 2-form  $\Phi$  of  $M$  is the restriction of  $\Psi$  to  $M$ , we have  $[\Phi] = a\bar{h}$ . This, together with  $[\gamma_1] = c_1(M) = (n - d + 2)h$ , implies that  $[\gamma_1] = k[\Phi]$  for some constant  $k$ . Our assertion in §1 now follows from the lemma.

**Remark.** The same reasoning as above, together with the result on p. 159 of Hirzebruch [2], gives the following: Let  $M_n$  be a complete intersection of  $r$  non-singular hypersurfaces in  $P_{n+1}(C)$ . If  $M$  has constant scalar curvature, then  $M$  is an Einstein space.

### Bibliography

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