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NONNEGATIVE RICCI CURVATURE, SMALL LINEAR DIAMETER GROWTH AND FINITE GENERATION OF FUNDAMENTAL GROUPS

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1. Introduction

In 1968, Milnor conjectured that a complete noncompact manifold, M^n, with nonnegative Ricci curvature has a finitely generated fundamental group [11]. This was proven for a manifold with nonnegative sectional curvature by Cheeger and Gromoll [6]. However, it remains an open problem even for manifolds with strictly positive Ricci curvature.

The conjecture is of particular interest because, if it is true, then by work of Cheeger-Gromoll, Milnor and Gromov, the fundamental group is almost nilpotent [7], [11], [8]. On the other hand, given any finitely generated torsion free nilpotent group, Wei has constructed an example of a manifold with positive Ricci curvature that has the given group as a fundamental group [15].

Schoen and Yau have proven the conjecture in dimension 3 for manifolds with strictly positive Ricci curvature [13]. In fact they have proven that such a manifold is diffeomorphic to Euclidean space.

Anderson and Li have each proven that if a manifold with nonnegative Ricci curvature has Euclidean volume growth, then the fundamental group is actually finite [2], [10]. Anderson uses volume comparison arguements while Li uses the heat equation to prove this theorem.

Abresch and Gromoll have proven that manifolds with small diameter growth, $(o(r^{1/n}))$, nonnegative Ricci curvature and sectional curvature bounded away from negative infinity have finite topological type [1, Thm. A]. Thus the fundamental group is finitely generated in such

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manifolds. Their theorem is proven using an inequality refered to as the Excess Theorem, [1, Prop. 2.3], which is a crucial ingredient in this paper as well.

Abresch and Gromoll have also proven that a manifold with nonnegative Ricci curvature has linear almost intrinsic diameter growth [1, Prop 1.1]. That is, given any $\varepsilon > 0$, there exists an explicit constant, $C(\varepsilon, n)$, such that given any p and q in $\partial B_{x_0}(r)$ joined by a curve in the annulus, $Ann_{x_0}(r - \varepsilon r, r + \varepsilon r)$, if σ is the shortest such curve then $L(c) \leq C(\varepsilon, n)r$. This proposition is proven using volume comparison arguements.

In this paper we prove that a manifold with small linear diameter growth has a finitely generated fundamental group.

Theorem 1. *There exists a universal constant,*

$$S_n = \frac{n}{n-1}\frac{1}{4}\frac{1}{3^n}\left(\frac{n-2}{n-1}\right)^{n-1}, \tag{1}$$

such that if M^n is complete and noncompact with nonnegative Ricci curvature and has small linear diameter growth,

$$\limsup_{r\to\infty} \frac{diam(\partial B_p(r))}{r} < 4S_n,$$

then it has a finitely generated fundamental group.

In fact if the manifold has small linear growth in the ray density function then it also has a finitely generated fundamental group. (Note 8).

We also prove a more general theorem, Theorem 11. If M^n has nonnegative Ricci curvature and an infinitely generated fundamental group then it has a tangent cone at infinity which is not polar. Currently it is not known whether a manifold with nonnegative Ricci curvature can have a tangent cone at infinity which is not polar [4].

The definitions of tangent cone and polar length spaces are reviewed in Section 5 (Defn 9, Defn 10) along with the precise statement of Theorem 11. The precise statement involves the universal constant, S_n, defined in (1).

Note that if M^n has Euclidean volume growth then by Cheeger and Colding, [4], its tangent cones at infinity have poles, and thus M^n has a finitely generated fundamental group. However, in this case, Anderson and Li have each proven that the fundamental group is actually finite. [2, Cor 1.5], [10].

On the other hand, if M^n has linear volume growth then by [14], its diameter growth is sublinear and its tangent cone at infinity is $[0, \infty)$ or $(-\infty, \infty)$, thus we have the following corollary of either theorem.

Corollary 2. *If M^n is complete with nonnegative Ricci curvature and linear volume growth then it has a finitely generated fundamental group.*

The proofs of both Theorem 1 and Theorem 11 are based on two main lemmas.

The Halfway Lemma, (Lemma 5), concerns complete Riemannian manifolds with infinitely generated fundamental groups. It does not require the Ricci curvature condition and makes a special selection of generators and representative loops, such that the loops are minimal halfway around. It is stated and proven in Section 2.

The Uniform Cut Lemma, (Lemma 7), gives a uniform estimate on special cut points which are the halfway points of noncontractible geodesic loops in a manifold with nonnegative Ricci curvature. The proof applies the Excess Theorem of Abresch and Gromoll. It appears in Section 3.

Section 4 contains the proof of Theorem 1 and Section 5 contains the proof and background material for Theorem 11. Both of these sections require the results of Sections 1 and 2 but are independant of each other.

The author would like to thank Professor Cheeger for suggesting that Theorem 1 be strenthened to its current form and for his assistance during the revision process. Background material can be found in [3] and [9].

2. The Halfway Lemma

In this section our manifold, N^n, is a complete Riemannian manifold but does not have a bound on its Ricci curvature. It may or may not be noncompact. Recall that a group is infinitely generated if it does not have a presentation with a finite set of generators. If $\pi_1(N)$ is not finitely generated, N^n must be noncompact.

Definition 3. If G is a group, we say that $\{g_1, g_2, ...\}$ is a ordered set of *independant generators* of G if each g_i can not be expressed as a word in the previous generators and their inverses, $g_1, g_1^{-1}, ...g_{i-1}, g_{i-1}^{-1}$.

Definition 4. Given $g \in \pi_1(N)$, we say γ is a *minimal representative geodesic loop* of g if $\gamma = \pi \circ \tilde{\gamma}$, where $\tilde{\gamma}$ is minimal from $\tilde{x_0}$ to $g\tilde{x_0}$

in $\tilde{N}$. Note that $L(\gamma) = d_{\tilde{N}}(\tilde{x_0}, g\tilde{x_0})$.

We now state the Halfway Lemma.

Lemma 5. (Halfway Lemma). *Let $x_0 \in N^n$ where N^n is a complete Riemannian manifold with a fundamental group $\pi_1(N, x_0)$. Then there exists an ordered set of independant generators $\{g_1, g_2, g_3 ...\}$ of $\pi_1(N, x_0)$ with minimal representative geodesic loops, γ_k, of length d_k such that*

$$d_N(\gamma_k(0), \gamma_k(d_k/2)) = d_k/2. \tag{2}$$

If $\pi_1(N)$ is infinitely generated we have a sequence of such generators.

Definition 6. Given $x_0 \in N$, we will call an ordered set of generators of $\pi_1(N, x_0)$ as constructed in Lemma 5, a set of *halfway generators* based at x_0 of $\pi_1(N)$.

Proof. In order to prove the Halfway Lemma, we need to choose a sequence of generators whose representative curves don't have redundant extra looping. Let $G = \pi_1(N^n, x_0)$.

We first define the sequence of generators $\{g_1, g_2, g_3 ...\}$. Define $g_1 \in G$ such that it minimizes distance,

$$d_{\tilde{N}}(\tilde{x_0}, g_1\tilde{x_0}) \le d_{\tilde{N}}(\tilde{x_0}, g\tilde{x_0}) \qquad \forall g \in G.$$

Let $G_i \in G$ be the elements of G generated by $g_1, ... g_i$ and their inverses. So $G_1 = \{e, g_1, g_1^{-1}, g_1^2, ...\}$. Define each $g_k \in G$ iteratively such that each minimizes distance among all elements in $G \setminus G_{k-1}$,

$$d_{\tilde{N}}(\tilde{x_0}, g_k\tilde{x_0}) \le d_{\tilde{N}}(\tilde{x_0}, g\tilde{x_0}) \qquad \forall g \in G \setminus G_{k-1}.$$

Note that $G \setminus G_{k-1}$ is nonempty for all k when we have an infinitely generated fundamental group.

Thus if there exists $h \in G$ with

$$d_{\tilde{N}}(\tilde{x_0}, h\tilde{x_0}) < d_{\tilde{N}}(\tilde{x_0}, g_k\tilde{x_0}) \tag{3}$$

then $h \in G_{k-1}$.

Given a curve $C : [0, d] \to N$, let $C(t_1 \to t_2)$ represent the segment of C running from t_1 to t_2. We allow $t_2 < t_1$.

Suppose that there is a $k \in \mathbf{N}$ such that

$$d_N(\gamma_k(0), \gamma_k(d_k/2)) < d_k/2. \tag{4}$$

Let $T = d_N(\gamma_k(0), \gamma_k(d_k/2))$. Then there exists a geodesic σ from $\gamma_k(d_k/2) = \sigma(0)$ to $x_0 = \sigma(T)$. This geodesic segment cannot overlap $\gamma_k(d_k/2 \to d_k)$ because it has a shorter length, $T < d_k/2$.

Then there exists elements $h_1 = [\sigma(0 \to T) \circ \gamma_k(0 \to T)] \in \pi_1(N)$ and $h_2 = [\gamma_k(T \to d_k) \circ \sigma(T \to 0)] \in \pi_1(N)$. Furthermore, since σ meets γ at an angle, the lift is not a minimal geodesic, so

$$d_{\tilde{N}}(\tilde{x_0}, h_1\tilde{x_0}) \leq L(\sigma(0 \to T) \circ \gamma_k(0 \to T)) = T + d_k/2 < d_k$$

and

$$d_{\tilde{N}}(\tilde{x_0}, h_2\tilde{x_0}) \leq L(\gamma_k(T \to d_k) \circ \sigma(T \to 0)) = T + (d_k/2) < d_k.$$

Thus, by (3), h_1 and h_2 are in G_{k-1}. So $g_k = h_2 \circ h_1 \in G_k$ contradicting our choice of $g_k \in G \setminus G_{k-1}$. So our assumption in (4) is false. q.e.d.

3. The Uniform Cut Lemma

In this section, M^n is a complete manifold with nonnegative Ricci curvature of dimension, $n \geq 3$. It may or may not be compact. Recall that when $n = 2$, Ricci curvature is just sectional curvature and Theorems 1 and 11 follow from [6].

The Uniform Cut Lemma describes special cut points which are the halfway points, $\gamma(D/2)$, of geodesic loops, γ, from a base point, x_0. Recall that no geodesic from x_0 through a cut point of x_0 is minimal after passing through that cut point. So if $x \in \partial B_{x_0}(RD)$ where $R > 1/2$, then any minimal geodesic from x to x_0 does not hit the cut point $\gamma(D/2)$. Thus

$$d_M(x, \gamma(D/2)) > (R - 1/2)D.$$

The Uniform Cut Lemma gives a uniform and scale invariant estimate for this inequality.

Lemma 7 (Uniform Cut Lemma). *Let M^n be a complete with nonnegative Ricci curvature and dimension $n \geq 3$. Let γ be a noncontractible geodesic loop based at a point, $x_0 \in M^n$, of length, $L(\gamma) = D$, such that the following conditions hold:*

i) If σ based at x_0 is a loop homotopic to γ then $L(\sigma) \geq D$

ii) The loop γ is minimal on $[0, D/2]$ and is also minimal on $[D/2, D]$. Then there is a universal constant S_n, defined in (1), such that if $x \in \partial B_{x_0}(RD)$ where $R \geq (1/2 + S_n)$ then

$$d_M(x, \gamma(D/2)) \geq (R - 1/2)D + 2S_nD.$$

Note that the minimal representative geodesic loops of the halfway generators constructed in the Halfway Lemma satisfy the hypothesis of the Uniform Cut Lemma.

Proof. We first prove the lemma for $R = (1/2 + S_n)$.

Assume on the contrary, that there is a point $x \in M^n$ such that $d_M(x, x_0) = (1/2 + S_n)D$ and $d_M(x, \gamma(D/2)) = H < 3S_nD$. Let $C : [0, H] \longmapsto M^n$ be a minimal geodesic from $\gamma(D/2)$ to x.

Let $\tilde{M}$ be the universal cover of M, let $\tilde{x_0} \in \tilde{M}$ be a lift of x_0, and let $g \in \pi_1(M, x_0)$ be the element represented by the given loop, γ. By the first condition on γ, its lift, $\tilde{\gamma}$, is a minimal geodesic running from $\tilde{x_0}$ to $g\tilde{x_0}$. Thus

$$d_{\tilde{M}}(\tilde{x_0}, g\tilde{x_0}) = D.$$

We can lift the joined curves, $C(0 \to H) \circ \gamma(0 \to D/2)$, to a curve in the universal cover, $\tilde{C} \circ \tilde{\gamma}$, which runs from $\tilde{x_0}$ through $\tilde{\gamma}(D/2)$ to a point $\tilde{x} \in \tilde{M}$. Note that $L(\tilde{C}) = L(C) = H$.

We can examine the triangle formed by $\tilde{x_0}$, $g\tilde{x_0}$ and $\tilde{x}$ using the Excess Theorem of Abresch and Gromoll [1]. For easy reference we introduce their variables r_0 and r_1.

By our assumption on x,

$$r_0 = d_{\tilde{M}}(\tilde{x}, \tilde{x_0}) \geq d_M(x, x_0) = (1/2 + S_n)D. \tag{5}$$

Furthermore,

$$r_1 = d_{\tilde{M}}(g_k\tilde{x}, \tilde{x_0}) \geq d_M(x, x_0) = (1/2 + S_n)D.$$

The excess of $\tilde{x}$ relative to $\tilde{x_0}$ and $g\tilde{x_0}$ satisfies

$$e(\tilde{x}) := r_0 + r_1 - d(\tilde{x_0}, g\tilde{x_0}) \geq 2(1/2 + S_n)D - D = 2S_nD. \tag{6}$$

On the other hand, by the Excess Theorem [1, Prop 2.3], we can estimate the excess from above in terms of the distance, l, from $\tilde{x}$ to the minimal geodesic, $\tilde{\gamma}$. In particular for $n \geq 3$, *Ricci* ≥ 0, they have proven that

$$e(\tilde{x}) \leq 2\left(\frac{n-1}{n-2}\right)\left(\frac{1}{2}\, C_3\, l^n\right)^{1/(n-1)} \tag{7}$$

where

$$C_3 = \frac{n-1}{n}\left(\frac{1}{r_0 - l} + \frac{1}{r_1 - l}\right) \tag{8}$$

if $l < \min\{r_0, r_1\}$.

We now need to estimate l from above. Suppose that the closest point on $\tilde{\gamma}$ to $\tilde{x}$ occurs at a point $\tilde{\gamma}(t_0)$. Then

$$\begin{aligned} l = d_{\tilde{M}}(\gamma(t_0), \tilde{x}) &\leq d_{\tilde{M}}(\tilde{\gamma}(D/2), \tilde{x}) \\ &\leq L(C) = HD < 3S_n D. \end{aligned}$$

Since $S_n < 1/20$ for $n \geq 3$,

$$r_0 - l \geq (1/2 + S_n)D - 3S_n D > D/4.$$

and, similarly,

$$r_1 - l > D/4.$$

In particular, $l < \min\{r_0, r_1\}$.

Substituting this into the Abresch and Gromoll's estimate (7), (8), we have

$$\begin{aligned} e(\tilde{x_k}) &< 2\left(\frac{n-1}{n-2}\right)\left(\frac{1}{2}\left(\frac{n-1}{n}\right)\left(\frac{2}{D/4}\right)(3S_n D)^n\right)^{1/(n-1)} \\ &\leq 2D\left(\frac{n-1}{n-2}\right)\left(4\left(\frac{n-1}{n}\right)(3S_n)^n\right)^{1/(n-1)}. \end{aligned}$$

Combining this with (6) and cancelling D, we get

$$2S_n < 2\left(\frac{n-1}{n-2}\right)\left(4\left(\frac{n-1}{n}\right)(3S_n)^n\right)^{1/(n-1)}. \tag{9}$$

Cancelling 2 and exponentiating, we have

$$S_n^{n-1} = 4\left(\frac{n-1}{n-2}\right)^{n-1}\frac{3^n(n-1)}{n}S_n^n,$$

and

$$S_n > \frac{n}{n-1}\frac{1}{4}\frac{1}{3^n}\left(\frac{n-2}{n-1}\right)^{n-1}.$$

This contradicts the definition of S_n in (1) and we've proven the theorem for $R = (1/2 + S_n)$.

If $R \geq (1/2+S_n)$, let $x \in \partial B_{x_0}(RD)$ and let $y \in \partial B_{x_0}((1/2+S_n)D)$ be a point on a minimal geodesic from x to $\gamma(D/2)$. Then, by the above case,

$$\begin{aligned} d_M(x,\gamma(D/2)) &= d_M(x,y)+d_M(y,\gamma(D/2)) \\ &\geq (RD-(1/2+S_n)D)+3S_nD \\ &= (R-1/2)D+2S_nD. \end{aligned}$$

q.e.d.

4. The Small Linear Diameter Growth Theorem

In this section we prove Theorem 1.

Proof of Theorem 1. Assume that M^n has an infinitely generated fundamental group $\pi_1(M^n,x_0)$. Then by the Halfway Lemma, (Lemma 5), there is a sequence of halfway generators, g_k, whose minimal representative geodesic loops based at x_0, γ_k, satisfy the hypothesis of the Uniform Cut Lemma (Lemma 7). Let $d_k = L(\gamma_k)$. Note that d_k diverges to infinity.

By the Uniform Cut Lemma, given any $x_k \in \partial B_{x_0}((1/2+S_n)d_k)$ we have

$$d_M(x_k,\gamma(d_k/2)) \geq 3S_nd_k.$$

Thus the point $y_k \in \partial B_{x_0}((1/2)d_k)$ on the minimal geodesic from x_k to x_0, satisfies,

$$\begin{aligned} d_M(y_k,\gamma_k(d_k/2)) &\geq d_M(x_k,\gamma_k(d_k/2)) - d(x_k,y_k) \\ &\geq (3S_nd_k)-(S_nd_k) = 2S_nd_k. \end{aligned}$$

This allows us to estimate the diameter growth,

$$\begin{aligned} \limsup_{r\to\infty} \frac{diam(\partial B_{x_0}(r))}{r} &\geq \limsup_{k\to\infty} \frac{d(y_k,\gamma_k(d_k/2))}{(d_k/2)} \\ &\geq \limsup_{k\to\infty} \frac{2S_nd_k}{d_k/2} = 4S_n. \end{aligned}$$

This contradicts the small linear diameter growth of M^n. q.e.d.

Note 8. Note that we could choose $x_k = \gamma((1/2+S_n)d_k)$ where γ is any ray based at x_0. Then $y_k = \gamma((1/2)d_k)$. Thus we really only require that the ray density function,

$$D(R) = \sup_{x\in\partial B_{x_0}(R)} \inf_{rays\ \gamma,\ \gamma(0)=x_0} d(x,\gamma(R)), \tag{10}$$

is bounded to get a contradiction.

That is, if

$$\limsup_{R\to\infty} \frac{D(R)}{R} < 4S_n, \tag{11}$$

then M^n has a finitely generated fundamnetal group.

5. The Pole Group Theorem

In this section we state and prove Theorem 11. We begin with some background material.

Given a complete noncompact manifold, M^n, with nonnegative Ricci curvature, we can define tangent cones at infinity by taking any pointed sequence of rescalings of the manifold. A subsequence of such a sequence must converge in the Gromov-Hausdorff topology to a pointed length space, (X, x_0), by Gromov's Compactness Theorem. [9]

Definition 9. A pointed length space, (X, x_0), is called a *tangent cone at infinity* of M if there exists a point $p \in M$ and a sequence r_k of positive real numbers diverging to infinity such that for all $R > 0$,

$$d_{GH}\big((B_R(x_0) \in X, x_0, d_X), (B_R(p) \in M, p, d_M/r_k)\big) \to 0,$$

as $r_k \to \infty$. Here d_M is the length space distance function on M induced by the Riemannian metric g_M and d_{GH} is the Gromov-Hausdorff distance.

Note that (X, x_0) need not be unique. See [12] and [4, 8.37]. Furthermore, (X, x_0) need not be a metric cone unless the manifold has Euclidean volume growth [4].

Definition 10 [4, Sect. 4]. A length space, X, has a *pole* at a point $x \in X$ if for all y not equal to x there exists a curve $\gamma : [0, \infty) \to X$ such that $\gamma(0) = x$, $d_X(\gamma(t), \gamma(s)) = |s-t|$ for all $s, t \geq 0$, and $\gamma(d(x, y)) = y$.

There is no known example of a manifold with nonnegative Ricci curvature with a tangent cone at infinity which does not have a pole at its base point [4]. In order to find an example of such a manifold, intuitively one would need to construct a sequence of cut points on the manifold which remain *uniformly cut* even after rescaling. By Lemmas 5 and 7, such cut points exist if the manifold has an infinitely generated fundamental group. This is the intuition behind Theorem 11 and Theorem 1.

Theorem 11 (Pole Group Theorem). *If a complete noncompact manifold, M^n, with nonnegative Ricci curvature has a fundamental group which is not finitely generated, then it has a tangent cone at infinity, (Y, y_0), which does not have a pole at its base point.*

In fact, if (Z, z_0) is a length space with

$$d_{GH}((B_{z_0}(1), z_0, d_Z), (B_{y_0}(1), y_0, d_Y)) < S_n/4. \tag{12}$$

where S_n was defined in (1), then Z does not have a pole at z_0.

Proof. First we choose a special sequence of rescalings of M^n and a corresponding tangent cone at infinity. Let x_0 be any base point in M. By the Halfway Lemma, there is a sequence of halfway generators, g_k, of lengths, d_k, corresponding to a base point x_0. Take a subsequence of this sequence such that $M_k = (M^n, x_0, d_M/d_k)$ converges to a tangent cone $Y = (Y, y_0, d_Y)$. [9]

By the Halfway Lemma and the Uniform Cut Lemma, we know that for all $k \in \mathbf{N}$, for all $r \geq 1/2 + S_n$ and for all $x \in \partial B_{x_0}(rd_k)$ then

$$d_M(x, \gamma_k(d_k/2)) \geq (r - 1/2 + 2S_n)d_k. \tag{13}$$

We want to find a "cut point" in Y and in Z. By Definition 9, taking $R = 1$ there exists $N \in \mathbf{N}$ such that for all $k \geq N$,

$$d_{GH}((B_{y_0}(1) \subset Y, y_0, d_Y), (B_{x_0}(1) \subset M_k, x_0, d_{M_k})) < S_n/12.$$

Thus, by (12), for all $k \geq N$, the length space Z satisfies

$$d_{GH}((B_{z_0}(1) \subset Z, z_0, d_Z), (B_{x_0}(1) \subset M_k, x_0, d_{M_k})) < \varepsilon_n = S_n/3. \tag{14}$$

So there is a map $F_k : B_{x_0}(1) \subset M_k \longmapsto B_{z_0}(1) \subset Z$, with $F_k(x_0) = z_0$ that is ε_n almost distance preserving,

$$|d_{M_k}(x_1, x_2) - d_Z(F_k(x_1), F_k(x_2))| < \varepsilon_n \qquad \forall x_1, x_2 \in X, \tag{15}$$

and ε_n almost onto.

$$\forall z \in B_{z_0}(1) \subset Z, \quad \exists x_z \in B_{x_0}(1) \subset M_k \ s.t. \ d_Z(F_k(x_z), x) < \varepsilon_n. \tag{16}$$

The map F_k can also be thought of as a map defined on $B_{x_0}(d_k) \in M^n$.

Note that F_k maps the halfway points, $\gamma_k(d_k/2)$, into an annulus,

$$F_k(\gamma_k(d_k/2)) \subset Ann_{z_0}(1/2 - \varepsilon_n, 1/2 + \varepsilon_n).$$

This annulus is precompact by the Gromov's Compactness Theorem [9]. Thus, there is a subsequence of $\{F_k(\gamma_k(d_k/2))\}$ that converges to a point

$$z_1 \in Cl(Ann_{z_0}(1/2-\varepsilon_n, 1/2+\varepsilon_n)). \tag{17}$$

In particular, we can choose a $k \geq N$ such that

$$d_Z(F_k(\gamma_k(d_k/2)), z_1) < \varepsilon_n. \tag{18}$$

We will show that this z_1 is our "cut point".

We claim that z_1 has no ray based at z_0 passing through it. Suppose, on the contrary, that it does. Then there is a curve, $C : [0,1) \longmapsto Z$ such that $c(0) = 0$, $c(t_1) = z_1$ and $d_Y(c(t), c(s)) = |s-t| \quad \forall\, t,\, s \in [0,1)$.

Let $z_2 = C(1/2+h)$ where $h \in [3S_n, 1/2)$. By (17), $t_1 \geq 1/2 - \varepsilon_n$, so

$$d_Z(z_2, z_1) = (1/2+h) - t_1 \leq (1/2+h) - (1/2-\varepsilon_n) = h + \varepsilon_n. \tag{19}$$

By (16), there exist $x_2 \in B_{x_0}(1) \subset M_k$ which is mapped almost onto z_2,

$$d_Z(F_k(x_2), z_2) < \varepsilon_n. \tag{20}$$

By (15), the triangle inequality and (20), we know that

$$\begin{aligned} d_M(x_0, x_2) &= d_{M_k}(x_0, x_2) d_k > (d_Z(F_k(x_0), F_k(x_2)) - \varepsilon_n) d_k \\ &\geq (d_Z(z_0, z_2) - d_Z(z_2, F_k(x_2)) - \varepsilon_n) d_k \\ &> ((1/2+h) - 2\varepsilon_n) d_k. \end{aligned}$$

Recall that $h \geq 3S_n$ and $\varepsilon_n = S_n/3$, so $x_2 \in \partial B_{x_0}(r d_k)$ with $r \geq 1/2 + S_n$. Thus by the uniform cut property, (13),

$$d_M(x_2, \gamma_k(d_k/2)) \geq (r - 1/2 + 2S_n) d_k \geq (h - 2\varepsilon_n + 2S_n) d_k. \tag{21}$$

However, F_k is ε_n almost distance preserving, (15). So applying the triangle inequality, (20), (19), and (18), we have

$$\begin{aligned} d_M(x_2, \gamma_k(d_k/2)) &= d_{M_k}(x_2, \gamma_k(d_k/2)) d_k \\ &< (d_Z(F_k(x_2), F_k(\gamma_k(d_k/2))) + \varepsilon_n) d_k \\ &\leq (d_Z(F_k(x_2), z_2) + d_Z(z_2, z_1) \\ &\quad + d_Z(z_1, F_k(\gamma_k(d_k/2))) + \varepsilon_n) d_k \\ &< (\varepsilon_n + (h + \varepsilon_n) + \varepsilon_n + \varepsilon_n) d_k. \end{aligned}$$

Combining this equation with (21), we get

$$(h + \varepsilon_n + 3\varepsilon_n)d_k > (h - 2\varepsilon_n + 2S_n)d_k.$$

So $\varepsilon_n > S_n/3$, contradicting (14). q.e.d.

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