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CRITICAL EXPONENT AND DISPLACEMENT OF NEGATIVELY CURVED FREE GROUPS

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Abstract

We study the action of the fundamental group Γ of a negatively curved 3-manifold M on the universal cover $\widetilde{M}$ of M. In particular we consider the ergodicity properties of the action and the distances by which points of $\widetilde{M}$ are displaced by elements of Γ. First we prove a displacement estimate for a general n-dimensional manifold with negatively pinched curvature and free fundamental group. This estimate is given in terms of the critical exponent D of the Poincaré series for Γ. For the case in which $n = 3$, assuming that Γ is free of rank $k \geq 2$, that the limit set of Γ has positive 2-dimensional Hausdorff measure, that $D = 2$ and that the Poincaré series diverges at the exponent 2, we prove a displacement estimate for Γ which is identical to the one given by the $\log(2k-1)$ theorem [1] for the constant-curvature case.

1. Introduction

In the following M is a *complete* Riemannian n-manifold with *finitely generated* fundamental group Γ. We will assume that the sectional curvature $\mathcal{K}$ satisfies $-b^2 \leq \mathcal{K} \leq -a^2$ for some $0 < a \leq b$. A manifold which satisfies curvature bounds of this type will be said to have *negatively pinched curvature*. The Riemannian universal cover of M is denoted by $\widetilde{M}$, and M is identified with $\widetilde{M}/\Gamma$. The following additional notations and terminologies will also be used:

- D is the critical exponent of the Poincaré series

$$\sum_{\gamma\in\Gamma} \exp(-s\,\mathrm{dist}(x, \gamma x))$$

of Γ. This means that for every $x \in \widetilde{M}$, the series diverges when $s < D$ and converges when $s > D$.

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- Γ is said to be *divergent* if the Poincaré series diverges at $s = D$.
- $\mathcal{S}$ denotes a arbitrary, fixed free generating set of Γ, with convention that the inverses are not included.
- $\Lambda(\Gamma)$ is the limit set of Γ, which is the unique minimal closed Γ-invariant subset of S_∞.
- $\mathfrak{D}$ denotes the Hausdorff dimension of $\Lambda(\Gamma)$ (with respect to the Busemann metric, the Gromov metric or the shadow metric: See §2).
- $\mathfrak{M}^d$ denotes the d-dimensional Hausdorff measure on S_∞ with respect to the Busemann metric.
- For each $x \in \widetilde{M}$ and $\gamma \in \Gamma$, we refer to $\mathrm{dist}(x, \gamma x)$ as the *displacement* of x under γ.

Theorem 1.1. *Suppose that $-1 \leq \mathcal{K} \leq -a^2$ and that Γ is free. Then*

$$\sum_{\gamma \in \mathcal{S}} \frac{1}{1 + \exp(D \,\mathrm{dist}(x, \gamma x))} \leq \frac{1}{2}.$$

Theorem 1.2. *Let $M = \widetilde{M}/\Gamma$ be a 3-manifold with $-1 \leq \mathcal{K} \leq -a^2$. Suppose that Γ is free, that $\mathfrak{M}^2(\Lambda(\Gamma)) > 0$, that $D = 2$ and that Γ is divergent. Then the displacement satisfies*

$$\sum_{\gamma \in \mathcal{S}} \frac{1}{1 + \exp(\mathrm{dist}(x, \gamma x))} \leq \frac{1}{2}.$$

As corollary to the above Theorem 1.1, we have the following

Corollary 1.3. *Suppose that M is a rank-1 locally symmetric space normalized so that $-1 \leq \mathcal{K} \leq -a^2$, and that Γ is free. Then,*

$$\sum_{\gamma \in \mathcal{S}} \frac{1}{1 + \exp(\mathfrak{D} \,\mathrm{dist}(x, \gamma x))} \leq \frac{1}{2}.$$

The study of the displacement function of fundamental group Γ is crucial in understanding the geometry of the manifold M. In *constant curvature* spaces, the current best estimate is due to the work of Culler and Shalen [4] where they have obtained an $\log 3$ lower bound for the displacement of rank 2 free group. Later the same estimate has been generalized to rank k in [1] where the $\log(2k-1)$ theorem is obtained.

In this paper we prove the estimate given by Theorem 1.1. The crucial difference from the constant curvature case is the involvement of the critical exponent in the estimate. Hence it provides an relationship between displacement and Hausdorff dimension $\mathfrak{D}$. When the critical exponent D is < 1 we have a estimate which is stronger than the $\log(2k-1)$ estimate even for constant curvature spaces (see Corollary 3.7). Theorem 1.2 is closely related to the question of rigidity for Γ with $D = 2$ (see Theorem 3.5). In [8] a topological condition on M has been established for which Γ is divergent at D.

In Section 2 will study invariant densities on S_∞ and in particular a decomposition theorem of Γ with respect to the conformal invariant density is proved. Section 3 is devoted to proving the displacement function estimates and additional corollaries.

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2. The boundary S_∞

In some situations we will take the dimension of M to be 3, otherwise we will assume M is n-dimensional in general.

Let $y \in \widetilde{M}$ and $\zeta \in S_\infty$ be given. The following notations will be assumed throughout.

- S_y denotes the unit sphere in the tangent space at y.
- Φ_y denotes the natural homeomorphism between S_∞ and S_y.
- $c_y^\zeta(t)$ denotes the geodesic ray connecting y and ζ.

2.1 Metrics on S_∞

Fix a point $x \in \widetilde{M}$. Let $\delta > 0$ be a positive real number. Let ξ, ζ in S_∞ be given.

In [7], Gromov defined a metric on S_∞ as follows. Let $y, z \in \widetilde{M}$ be given. Let us consider arbitrary continuous curve $c(t)$ in $\widetilde{M}$ with initial point and end point denoted by $c(t_0) = y$ and $c(t_1) = z$ respectively. Define a nonnegative real-valued function $\mathcal{G}_x$ on $\widetilde{M} \times \widetilde{M}$ by

$$\mathcal{G}_x(y,z) := \inf_{\text{all } c} \left(\int_{[t_0,t_1]} \exp(-\delta \operatorname{dist}(x, c(t)))\, \mathrm{d}t \right).$$

In particular, Gromov showed that there exists $\delta(b) > 0$ depending only on the lower pinching constant b such that for any δ with $0 < \delta \leq \delta(b)$, the function $\mathcal{G}_x$ extends continuously to $S_\infty \times S_\infty$ and defines a distance. Every element of Γ extends to S_∞ as a Lipschitz map with respect to $\mathcal{G}_x$.

There are also many other equivalent metrics on S_∞. In particular, Kaimanovich [9] has studied the following metrics.

K_x **metric** : Let B_ζ denote the Busemann function based at x_0, i.e.,

$$B_\zeta := \lim_{\tau \to \infty} \operatorname{dist}(x, c_{x_0}^{\zeta}(\tau)) - \tau.$$

Changing the base point x_0 will change B_ζ only by adding a constant. Hence the function defined by $B_\zeta(x,y) := B_\zeta(x) - B_\zeta(y)$, for $x, y \in \widetilde{M}$, is independent of the base point. The function $B_\zeta(x,y)$ is called the Busemann cocycle. Define a real-valued function $\beta_x : S_\infty \times S_\infty \longrightarrow \mathbb{R}$ by $\beta_x(\xi,\zeta) := B_\xi(x,y) + B_\zeta(x,y)$ where y is a point on the geodesic connecting ξ and ζ. It is clear from the definition of the Busemann function that $\beta_x(\xi,\zeta)$ is defined independently of the choice of y. Geometrically, $\beta_x(\xi,\zeta)$ is the length of the segment on the geodesic connecting ξ and ζ cut out by the horospheres centered at ξ and ζ passing through the point x. The K_x metric is then defined by

$$K_x(\xi,\zeta) := \exp\left(-\frac{1}{2}\delta\beta_x(\xi,\zeta)\right).$$

L_x **metric:** Let $\alpha_x(\xi,\zeta)$ denote the distance between x and the geodesic connecting ξ and ζ. The function $L_x : S_\infty \times S_\infty \longrightarrow \mathbb{R}$ is then defined by

$$L_x(\xi,\zeta) := \exp(-\delta\alpha_x(\xi,\zeta)).$$

d_x **metric:** Define a function $l_x : S_\infty \times S_\infty \longrightarrow \mathbb{R}$ by

$$l_x(\xi, \zeta) := \sup \left\{ \tau \mid \mathrm{dist}\left(c^x_\xi(\tau), c^x_\zeta(\tau)\right) = 1 \right\}.$$

Geometrically, a neighborhood about ξ in S_∞ with respect to the topology induced by l_x is the shadow cast by the intersection of 1-ball about $c^x_\xi(\tau)$ and τ-sphere about x. The d_x metric is then defined by

$$d_x(\xi, \zeta) := \exp(-\delta l_x(\xi, \zeta)).$$

Proposition 2.1 ([9]). *There exists a positive number $\delta(a, b) > 0$ depending only on the pinching constants a and b such that for every $0 < \delta \le \delta(a, b)$ the metrics K_x, L_x, d_x are equivalent to Gromov's metric.*

From now on we will fix a $\delta > 0$ having the property stated in Proposition 2.1.

For completeness we include the following comparison theorem by Toponogov [10]. Let $\widetilde{M_a}$ and $\widetilde{M_b}$ denote the simply connected complete constant curvature manifolds with $\mathcal{K} = -a$ and $\mathcal{K} = -b$ respectively. Denote a geodesic triangle with $\theta = \angle A$ and $r = \mathrm{dist}(A, B)$, $t = \mathrm{dist}(A, C)$, $s = \mathrm{dist}(B, C)$ by $\triangle ABC$.

Theorem 2.2 (Toponogov's Comparison). *Keeping the above notation, let $\triangle ABC$ be a geodesic triangle in $\widetilde{M}$. Then we have*

(i) $$\cos\theta \sinh ar \sinh at \ge \cosh ar \cosh at - \cosh as,$$

(ii) $$\cos\theta \sinh br \sinh bt \le \cosh br \cosh bt - \cosh bs.$$

2.2 Γ-Invariant Density on S_∞

First, let us recall a simple uniqueness result.

We will say that two Borel measures on S_∞ are in the same Γ-class if the Radon-Nikodym derivative of $\gamma^*\nu_1$ with respect to ν_1 is equal to the Radon-Nikodym derivative of $\gamma^*\nu_2$ with respect to ν_2.

Proposition 2.3. *Let Γ be nonelementary and discrete. Suppose that Γ acts ergodically on S_∞ with respect to a measure ν defined on S_∞. Then every measure of S_∞ in the same measure class as ν is a constant multiple of ν.*

Proof. Denote the measure class of ν by $[\nu]$. Let $\mu \in [\nu]$ then $\sigma := \frac{1}{2}(\mu+\nu)$ is also in $[\nu]$. Since both ν and μ are absolutely continuous with respect to σ, their Radon-Nikodym derivatives with respect to σ defines a ν-measurable Γ-invariant function on S_∞. Then, the result follows from ergodicity of Γ, which implies these functions are equal to a non-zero constant almost ν-everywhere. q.e.d.

A subset A of $\widetilde{M}$ is called *uniformly discrete* if there exists a positive number ϵ such that for any two distinct point $x, y \in A$ we have $\operatorname{dist}(x, y) > \epsilon$.

Definition 2.4. Let $V \subset \widetilde{M}$ be a uniformly discrete subset. Let $h_s(x, y) : \widetilde{M} \times \widetilde{M} \longrightarrow \mathbb{R}^+$ be a family of positive continuous functions indexed by $s \in \mathbb{R}$ such that $K_s(x, y, \zeta) := \lim_{z\to\zeta} \frac{h_s(x,z)}{h_s(y,z)}$ exists and defines a family of continuous functions on $\widetilde{M} \times \widetilde{M} \times S_\infty$. Suppose that there exists a number α for which the series $Z_{x,s,V} := \sum_{v\in V} h_s(x, v)$ is convergent when $s > \alpha$ and divergent when $s \leq \alpha$ for any $x \in \widetilde{M}$. Then we call $Z_{s,V}$ an α-series of V. If we also have $h_s(\gamma x, \gamma y) = h_s(x, y)$ for every isometry γ of $\widetilde{M}$, then $Z_{s,V}$ will be called an invariant α- series of V.

For a given α-series $Z_{s,V}$ of V we will define a Borel measure $\nu_{x,Z,V,s}$ on V by $\sum_{v\in V} h_s(x, v)\delta_v$.

Proposition 2.5. *Let x be a point of $\widetilde{M}$ and $W \subset \widetilde{M}$ infinite uniformly discrete subset. Let $Z_{s,W}$ be an α-series of W. Then for every $V \subset W$ there exists a sequence (s_i) of real numbers larger than α such that:*

(a) $\lim_{i\to\infty} s_i = \alpha$.

(b) *The sequence $(Z^{-1}_{x,s_i,W}\nu_{x,Z,V,s_i})_i$ is a weakly convergent sequence of Borel measures of $\widetilde{M} \cup S_\infty$ with mass at most 1, and the limit Borel measure denoted by $\nu_{x,Z,V}$ is supported on S_∞.*

(c) *$\nu_{x,Z,W}$ is a probability measure.*

(d) *$[\nu_{y,Z,V}]_{y\in\widetilde{M}}$ is a density of S_∞ with Radon-Nikodym derivative at $\zeta \in S_\infty$ given by $K_\alpha(x, y, \zeta)$.*

Proof. Take a sequence (s'_i) which converges to α from above. For all i, the Borel measure $Z^{-1}_{x,s'_i,W}\nu_{x,Z,V,s'_i}$ has mass at most 1 . Hence there exists a subsequence (s'_{i_j}) of (s'_i) such that $Z^{-1}_{x,s'_{i_j},W}\nu_{x,Z,V,s'_{i_j}}$ converges weakly to a Borel measure $\nu_{x,Z,V}$ with mass at most 1. If $V = W$ then the measures in the above sequence are probability measures, hence $\nu_{x,Z,W}$ is a probability measure. If $B \subset \widetilde{M}$ is any compact subset then $B \cap V$ is finite. Since $Z_{x,s'_{i_j},W} \longrightarrow \infty$, we must have $\nu_{x,Z,V}(B \cap V) = 0$. Hence, $\nu_{x,Z,V}$ has support contained in S_∞. Finally, for $y \in \widetilde{M}$ and $\zeta \in S_\infty$ we have

$$\lim_{z \to \zeta} \frac{h_{s'_{i_j}}(x,z)}{h_{s'_{i_j}}(y,z)} = K_{s'_{i_j}}(x,y,\zeta),$$

so the Radon-Nikodym derivative of $\nu_{x,Z,V}$ is given by the limit of $K_{s'_{i_j}}(x,y,\zeta)$ as $s'_{i_j} \longrightarrow \alpha$, which is $K_\alpha(x,y,\zeta)$. q.e.d.

Proposition 2.6. *Let W be an infinite uniformly discrete subset of $\widetilde{M}$. Let $\mathcal{V}$ be a collection of subsets of W with $W \in \mathcal{V}$. Suppose we have an invariant α-series $Z_{s,W}$ of W. Then there exists a family $(\nu_V)_{V \in \mathcal{V}}$ of Borel measures for $\widetilde{M} \cup S_\infty$, indexed by the collection $\mathcal{V}$ and satisfying the following conditions:*

(1) $\nu_W(S_\infty) = 1$.

(2) *For any finite collection $(V_j)_{1 \le j \le n}$ of disjoint sets in $\mathcal{V}$ with $V := \coprod_j V_j \in \mathcal{V}$, we have $\nu_V = \sum_j \nu_{V_j}$.*

(3) *For any $V \in \mathcal{V}$ and γ isometry of $\widetilde{M}$ with $\gamma V \in \mathcal{V}$ we have $\gamma^*\nu_{\gamma V} = \nu_V$.*

(4) *The support of ν_V is contained in $\overline{V} \cap S_\infty$.*

Proof. From Proposition 2.5 (c), we have (1). For (2), let χ_{V_j} be the characteristic function of V_j. Then $\nu_{V_j} = \nu_V \chi_{V_j}$, which gives $\sum_j \nu_{V_j} = \sum_j \nu_V \chi_{V_j} = \nu_V$. Let γ be a isometry of $\widetilde{M}$. Since, $Z_{s,W}$ is an invariant α-series, we have $\gamma^*\nu_{\gamma x,Z,\gamma V,s} = \nu_{x,Z,V,s}$; therefore by Proposition 2.5 we have $\gamma^*\nu_{\gamma x,\gamma V} = \nu_{x,V}$. This gives us (3). Finally, (4) follows easily from Proposition 2.5 (b). q.e.d.

Next we will prove a result which is a generalization of the Culler-Shalen paradoxical decomposition theorem to this abstract setting.

Theorem 2.7. *Let Γ be a finitely generated, free discrete group of isometries of $\widetilde{M}$ with generating set Ω. Let $x \in \widetilde{M}$ be given. Suppose* *we have an invariant α-series $Z_{s,W}$ with $W = \Gamma x$. Set $\Psi := \Omega \coprod \Omega^{-1}$. Then there exist a Γ-invariant density $[\mu_y]_{y\in\widetilde{M}}$ on S_∞, and a family $[\nu_\psi]_{\psi\in\Psi}$ of Borel measures on S_∞ with:*

(1) $\mu_x(S_\infty) = 1$.

(2) $\mu_x = \sum_{\psi\in\Psi} \nu_\psi$.

(3) $\int_{S_\infty} K(x, \psi^{-1}x, \xi)\mathrm{d}\nu_{\psi^{-1}}(\xi) = 1 - \int_{S_\infty} \mathrm{d}\nu_\psi$.

Proof. Let us write every element $\gamma \in \Gamma$ as a reduced word $\psi_1 \cdots \psi_n$ with $\{\psi_j\} \subset \Psi$. Then we have the decomposition of Γ as $\Gamma = \{1\} \coprod \coprod_{\psi\in\Psi} I_\psi$, where I_ψ is the set of nontrivial elements in Γ with inital letter ψ. By the fact that Γ act freely on $\widetilde{M}$ we have $W = \Gamma x = \{x\} \coprod \coprod_{\psi\in\Psi} V_\psi$ where $V_\psi = \{\gamma x : \gamma \in I_\psi\}$. Let $\mathcal{V}$ denote the collection consisting of all sets of the form $\coprod_{\psi\in\Psi'} V_\psi$ or $\{x\} \coprod \coprod_{\psi\in\Psi'} V_\psi$ for $\Psi' \subset \Psi$. Applying Proposition 2.6 with W and $\mathcal{V}$ so defined, we get a family of Borel measures $(\mu_{y,V_\psi})_{y\in\widetilde{M}}$ for each $\psi \in \Psi$. By Proposition 2.6 (1), $\mu_{x,W}$ is a probability measure on S_∞, which gives (1). Define $\nu_\psi := \mu_{x,V_\psi}$ for each $\psi \in \Psi$. By the above decomposition of W, we have $\mu_{x,W} = \mu_{x,x} + \sum_{\psi\in\Psi} \nu_\psi$. But $\mu_{x,x} = 0$, which gives (2). Since $W = V_\psi \coprod \psi V_{\psi^{-1}}$, we have $\psi V_{\psi^{-1}} \in \mathcal{V}$. Then by (3) of Proposition 2.6, we get

$$\mu_{\psi^{-1}x, V_{\psi^{-1}}} = \psi^* \mu_{x, W - V_\psi} = \psi^*(\mu_x - \nu_\psi).$$

By Proposition 2.5 (d), we have

$$\mathrm{d}\mu_{\psi^{-1}x, V_{\psi^{-1}}} = K(\psi^{-1}x, x, \xi)\mathrm{d}\mu_{x, V_{\psi^{-1}}}.$$

From this, we get

$$\int_{S_\infty} K(\psi^{-1}x, x, \xi)\mathrm{d}\nu_{\psi^{-1}} = \int_{S_\infty} \mathrm{d}(\psi^*(\mu_{x,W} - \nu_\psi)) = 1 - \int_{S_\infty} \mathrm{d}\nu_\psi.$$

The last equality gives us (3), which concludes the proof. q.e.d.

Next we will construct two α-series of $W = \Gamma x$. The first construction is based on the work of Ancona, uses the λ-Green's function, which gives rise to the Poisson kernel density (λ-harmonic measure) on S_∞. The second construction, utilizes the Poincaré series, gives rise to the Patterson-Sullivan measure [15], which is a D-conformal density.

Harmonic density. Let λ_1 and $\widetilde{\lambda}_1$ denote the infinum of the spectrum of Δ on $M = \widetilde{M}/\Gamma$, and of $\widetilde{\Delta}$ on $\widetilde{M}$, respectively. Recall that for a noncompact open manifold, the infinum of the spectrum is characterized as

$$\lambda_1 := \inf_{f \in C_o^\infty, f \neq 0} \left(\frac{\int |\nabla f|^2}{\int f^2} \right)$$

where C_o^∞ is the space of smooth functions on M with compact support. Note that we always have $\lambda_1 \leq \widetilde{\lambda}_1$.

The λ_1-harmonic functions have been studied by Ancona in [2].

Proposition 2.8 (Ancona). *For each $s < \lambda_1$, the elliptic operator $\widetilde{\Delta} + sI$ has a Green function $G_s(x,y)$, and there exists a function $f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that $\sum_{\gamma\in\Gamma} \hat{G}_s(x,\gamma y)$ converges for $s < \lambda_1$ and diverges for $s \geq \lambda_1$, where $\hat{G}_s(x,\gamma y) := \exp(f(\operatorname{dist}(y,\gamma y)))G_s(x,\gamma y)$. Furthermore, $\mathfrak{P}_s(x,y,\zeta) := \lim_{z\to\zeta} \frac{G_s(x,z)}{G_s(y,z)}$ defines the Poisson kernel of $\widetilde{\Delta} + sI$ at $\zeta \in S_\infty$.*

Theorem 2.9. *Let x be any point of $\widetilde{M}$. With notation as in Proposition 2.8, the series $\sum_{v\in V} \hat{G}_s(x,v)$ is a λ_1-series for $W = \Gamma x$. Furthermore, there is a family of Borel measures $[\omega_y^1]_{y\in\widetilde{M}}$ on S_∞ such that:*

(i) *For all $x, y \in \widetilde{M}$, Radon-Nikodym derivative $\mathrm{d}\omega_y^1/\mathrm{d}\omega_x^1$ at any point $\zeta \in S_\infty$ is equal to $\mathfrak{P}_{\lambda_1}(x,y,\zeta)$.*

(ii) *ω_x^1 is of mass 1.*

Proof. The first assertion follows from Proposition 2.5 with $\alpha = \lambda_1$. The second assertion then follows from Proposition 2.8. q.e.d.

λ-dimensional Hausdorff measure. It is not straightforward to define an "area" measure on S_∞ in the variable curvature case. However, there are useful ways of doing this, which involve making appropriate choices of a metric on S_∞ and considering the corresponding Hausdorff measure.

For reasons that will become clear shortly, we will be working with the K_x metric on S_∞. Let C be a subset of S_∞. The λ-dimensional Hausdorff measure $\mathfrak{M}_x^\lambda(C)$ of C on the metric space(S_∞, K_x) is defined as $\lim_{r\to 0} \mathfrak{M}_r^\lambda(C)$ where

$$\mathfrak{M}_r^\lambda := \inf\Big\{ \sum_j r_j^{\lambda/\delta} \,|\, C \subset \cup_j B(\xi_j, r_j);\, \xi \in C,\, r_j \leq r \Big\}.$$

Here $B(\xi_j, r_j)$ is an r_j-ball about ξ_j with respect to the metric K_x. Observe that for any $x \in \widetilde{M}$ and any $\gamma \in \Gamma$, we have $\gamma^*\mathfrak{M}_x^\lambda = \mathfrak{M}_{\gamma^* x}^\lambda$; this follows from the straightforward identity $B_{\gamma\zeta}(\gamma x, \gamma y) = B_\zeta(x, y)$.

(Note that in the case of $\mathbb{H}^3$, if we take a round metric about a point z_0 in $\mathbb{H}^3$ and denote its extension to $\partial\mathbb{H}^3$ by ρ_0, then $\mathfrak{M}_{z_0}^2$ is the area metric corresponding to the spherical metric A_{z_0}.)

A family of finite Borel measures $[\nu_y]_{y\in\widetilde{M}}$ will be called a *λ-conformal density under the action of* Γ if for every $x \in \widetilde{M}$ and every $\gamma \in \Gamma$ we have $\gamma^*\nu_y = \nu_{\gamma^* y}$, and the Radon-Nikodym derivative $\frac{\mathrm{d}\nu_y}{\mathrm{d}\gamma^*\nu_y}(\zeta)$ at any point $\zeta \in S_\infty$ is equal to $\exp(-\lambda B_\zeta(\gamma^{-1}y, y))$. (This is to be interpreted as being vacuously true if, for example, the measures in the family are all identically zero.)

First we recall a fundamental fact about conformal density, which was originally proved by Sullivan in the hyperbolic case and generalized to the pinched negatively curved spaces by Yue [16]. It relates the divergence of Γ at the critical exponent D with ergodicity of the D-conformal density under the action of Γ.

Proposition 2.10 (Sullivan). *Let Γ be a nonelementary, discrete, torsion-free and divergent at D. Suppose $[\nu]$ is a D-conformal density under the action of Γ, then Γ is ergodic with respect to $[\nu]$.*

Proposition 2.11. *Let Γ be a discrete group of isometries of $\widetilde{M}$. Suppose $\mathfrak{M}_x^\lambda$ is a finite measure. Then $\mathfrak{M}_x^\lambda$ is a λ-conformal density under the action of Γ.*

Proof. Let $\gamma \in \Gamma$ and ξ, $\zeta \in S_\infty$ be given. By definition we have $\beta_x(\gamma\xi, \gamma\zeta) = B_{\gamma\xi}(x, \gamma y) + B_{\gamma\zeta}(x, \gamma y)$ where γy is a point on the geodesic connecting $\gamma\xi$ and $\gamma\zeta$. Since

$$\begin{aligned} B_{\gamma\xi}(x, \gamma y) &= B_{\gamma\xi}(x) - B_{\gamma\xi}(\gamma y) \\ &= B_\xi(\gamma^{-1}x) - B_\xi(y) \end{aligned}$$

we have

$$\begin{aligned} \beta_x(\gamma\xi, \gamma\zeta) &= [B_\xi(\gamma^{-1}x) - B_\xi(y) + B_\xi(x) - B_\xi(x)] \\ &\quad + [B_\zeta(\gamma^{-1}x) - B_\zeta(y) + B_\zeta(x) - B_\zeta(x)] \\ &= B_\xi(\gamma^{-1}x, x) + B_\zeta(\gamma^{-1}x, x) + \beta_x(\xi, \zeta). \end{aligned}$$

Hence, as $\xi \longrightarrow \zeta$ we have $B_\xi(\gamma^{-1}x, x) + B_\zeta(\gamma^{-1}x, x) \longrightarrow 2B_\zeta(\gamma^{-1}x, x)$.

Therefore

$$\lim_{\xi\to\zeta}\frac{K_x(\gamma\xi,\gamma\zeta)}{K_x(\xi,\zeta)}=\exp(-\delta B_\zeta(\gamma^{-1}x,x)).$$

Finally, using the fact that $\gamma^*\mathfrak{M}_x^\lambda=\mathfrak{M}_{\gamma^*x}^\lambda$ and the above transformation property of K_x we have the desired result :

$$\frac{\mathrm{d}\mathfrak{M}_x^\lambda}{\mathrm{d}\gamma^*\mathfrak{M}_x^\lambda}(\zeta)=\exp(-\lambda B_\zeta(\gamma^{-1}x,x)).$$

q.e.d.

Proposition 2.12. *Suppose* $\dim\widetilde{M}=3$. *Let* A_x *be the normalized area measure on* S_x. *Then, for any Borel subset* $C\subset S_x$ *we have*

$$\Phi_x^*\mathfrak{M}_x^{2b}(C)\le A_x(C).$$

Hence, if $\mathfrak{M}_x^{2b}(\Lambda(\Gamma))>0$ *then by normalization,* $\mathfrak{M}_x^{2b}$ *defines a probability measure on* $\Lambda(\Gamma)$ *which is bounded by* A_x.

Proof. Let ξ and ζ in S_∞. Then we have

$$B_x(\xi,\zeta)=\lim_{\tau\to\infty}(2\tau-\mathrm{dist}(y,c_x^\xi(\tau))-\mathrm{dist}(y,c_x^\zeta(\tau)))$$

where y is any arbitrary point on the geodesic connecting ξ and ζ. By letting $y\longrightarrow\xi$ we get $\beta_x(\xi,\zeta)=\lim_{\tau\to\infty}(2\tau-\mathrm{dist}(c_x^\xi(\tau),c_x^\zeta(\tau))$.

Let $\mathcal{B}_K(\zeta,r)$ denote the ball of radius r about ζ with respect to the K_x-metric. Let $\xi\in\mathcal{B}_K(\zeta,r)$ be an element with $\exp(-b\beta_x(\xi,\zeta))=r^{2b/\delta}$ and minimal $\angle x\xi\zeta$. By the above formula for $\beta_x(\xi,\zeta)$ we then have $K_x(\xi,\zeta)=\lim_{\tau\to\infty}\exp(\frac{\delta}{2}(\mathrm{dist}(c_x^\xi(\tau),c_x^\zeta(\tau))-\tau))$. Let $s(\tau)$ denote $\mathrm{dist}(c_x^\xi(\tau),c_x^\zeta(\tau))$ and $\theta:=\angle x\xi\zeta$. Then using inequality (i) of Proposition 2.2 we have:

$$\cosh^2 b\tau-\cosh bs(\tau)\ge\cos\theta\sinh^2 b\tau.$$

Substituting $1+\sinh^2 b\tau$ for $\cosh^2 b\tau$ we get

$$\sinh^2 b\tau(1-\cos\theta)+1\ge\cosh bs(\tau).$$

By using the equality $\sinh^2\frac{bs(\tau)}{2}=\frac{1}{2}(\cosh bs(\tau)-1)$ we then obtain

$$\sinh^2 b\tau(1-\cos\theta)\ge 2\sinh^2\frac{bs(\tau)}{2},$$

or

$$\left\{\frac{\exp(bs(\tau)/2)-\exp(-bs(\tau)/2)}{\exp(b\tau)-\exp^(-b\tau)}\right\}^2 \leq \frac{1}{2}(1-\cos\theta).$$

Hence for large τ we have

$$\lim_{\tau\to\infty}\{\exp(bs(\tau)/2-b\tau)\}^2 \leq \frac{1}{2}(1-\cos\theta).$$

By using the last equation we get $r^{2b/\delta} \leq A_x(\Phi_x^{-1}(\mathcal{B}_K(\zeta,r)))$.

Denote by $C' := \Phi_x(C)$. Let $\mathcal{B} := \cup_j \mathcal{B}_K(\zeta_j, r_j)$ be a cover of C', with $\zeta_j \in C'$ and $r_j \leq r$. Then

$$\mathfrak{M}_r^{2b}(C') = \inf_{\mathcal{B}} \sum_j r_j^{2b/\delta} \leq \inf_{\mathcal{B}} \sum_j A_x(\Phi_x^{-1}(\mathcal{B}_K(\zeta_j, r_j)).$$

Let $V_j := \Phi_x^{-1}\mathcal{B}_K(\zeta_j, r_j)$. Then $C \subset \cup_j V_j$ is a cover of C. Hence we have $\inf_{\cup V_j} \sum A_x(V_j) = A_x(C)$ by regularity of A_x. Therefore $\mathfrak{M}_r^{2b}(C') \leq A_x(C)$. Letting $r \to 0$ we get the desired result. q.e.d.

Corollary 2.13. *Suppose* $\mathfrak{M}_x^{2b}(\Lambda(\Gamma)) > 0$. *Then* $[\mathfrak{M}_y^{2b}]_{y\in\widetilde{M}}$ *can be normalized so as to define a* $2b$*-conformal density under the action of* Γ *whose total mass is* 1.

Proof. The corollary follows from Proposition 2.11 and Proposition 2.12. q.e.d.

The Patterson-Sullivan construction. Let $x \in \widetilde{M}$ and $s > 0$ be given. Denote the Poincaré series for a infinite uniformly discrete subset $W \subset \widetilde{M}$ by $Z_W(x,s)$, i.e., set

$$Z_W(x,s) := \sum_{v\in W} \exp(-s\,\mathrm{dist}(x,v)).$$

In particular, W can be the orbit Γx of x. We will use the notation $Z_\Gamma(x)$ to denote the Poincare series for $W = \Gamma x$ in this case.

Proposition 2.14. *There is a unique number* $0 \leq D \leq (n-1)b$, *such that for* $s > D$, Z_W *converges and* Z_W *diverges for* $s < D$.

If $Z_W(x,D)$ is divergent, then we say that Γ is divergent.

Let $V \subset W$. First consider the case that $Z_V(x,D)$ is divergent. Following Patterson-Sullivan, we will construct a family of Borel measures. Define:

$$\mu_{V,x,s} = \frac{1}{Z_V(x,s)} \sum_{v\in V} \exp(-s\,\mathrm{dist}(x,v))\delta_v \ ; \ s > D.$$

Since $Z_V(x,D)=\infty$, we have as $s\longrightarrow D$ through a suitable sequence, $\mu_{V,x,s}$ converges weakly to a limit probability measure, which we denote by $\mu_{V,x}$. This limit measure has its support contained in S_∞.

From the definition of $\mu_{V,x,s}$ we have

$$\frac{\mathrm{d}\mu_{V,x,s}}{\mathrm{d}\mu_{V,y,s}}(v)=\exp(s(\operatorname{dist}(y,v)-\operatorname{dist}(x,v))).$$

As $v\longrightarrow\zeta$, we have $\exp(s(\operatorname{dist}(y,v)-\operatorname{dist}(x,v)))\longrightarrow\exp(-sB_\zeta(x,y))$. Hence

$$\frac{\mathrm{d}\mu_{V,x,s}}{\mathrm{d}\mu_{V,y,s}}(v)\longrightarrow\frac{\mathrm{d}\mu_{V,x}}{\mathrm{d}\mu_{V,y}}(\zeta)=\exp(-DB_\zeta(x,y)).$$

Therefore $[\mu_{V,x}]$ is a D-conformal density.

The above construction was based on the assumption that $Z_V(x,D)=\infty$. If $Z_V(x,D)<\infty$ we can use the following lemma proved by Patterson [12] and presented in [4].

Lemma 2.15. *There exits a real-valued function $\alpha(t)$, such that the perturbed Poincaré series $\widetilde{Z}_V(x,s):=\sum_{v\in V}\exp(-\alpha(\operatorname{dist}(x,v)))$ is finite for $s>D$ and infinite when $0\le s\le D$.*

Proof. Let $s_k:=\theta_kD$ for some increasing sequence of positive numbers $\theta_k\longrightarrow 1$. Let $\{R_k\}_{k\ge1}$ be a monotone increasing sequence of positive numbers with $R_k\longrightarrow\infty$ and $\operatorname{dist}(x,v)\le R_k$ for all $v\in V_k$. We also choose $V_k\subset V$ so that

$$\sum_{v\in V_k}\exp(-s_k\operatorname{dist}(x,v))\ge k.$$

Note that $0<s_k<D$, so $Z_V(x,s_k)<\infty$. Let $\beta(t)$ denote a continuous increasing function with $\beta(R_k)=\theta_k$ and $\beta(t)<1$ for $t\ge0$. Then the desired adjustment function α is defined by $\alpha(t):=\int_0^t\beta(\tau)\mathrm{d}\tau\le\theta_kt$ for $0\le t\le R_k$. To see this, note that

$$\widetilde{Z}(x,D)\ge\sum_{v\in V_k}\exp(-D\alpha(\operatorname{dist}(x,v)))\ge\sum_{v\in V_k}\exp(Ds_k\operatorname{dist}(x,v))\ge k$$

for every $k\ge1$. So we have $\widetilde{Z}(x,D)=\infty$. q.e.d.

Note that, $\widetilde{Z}(x,D)=\infty$ for one x implies $\widetilde{Z}(x,D)$ is divergent for all x, by the fact that $\widetilde{Z}(x',s)\le\exp(s\operatorname{dist}(x,x'))\widetilde{Z}(x,s)$, for all $x',x\in\widetilde{M}$.

Proposition 2.16. *For a given adjustment function α, we have:*

$$\exp(\alpha(\mathrm{dist}(x',v))-\alpha(\mathrm{dist}(x,v)))\longrightarrow\exp(B_\xi(x',x))$$

as $v\to\xi$. Hence the above construction with $\widetilde{Z}_V(x,s)$ in place of $\widetilde{Z}_V(x,s)$ still defines a conformal density.

Proof. Let $\{v_k\}$ be a sequence in V converges to $\xi\in S_\infty$. Then $\mathrm{dist}(x,v_k)\longrightarrow\infty$ and

$$\lim_{k\to\infty}\exp(\mathrm{dist}(y,v_k)-\mathrm{dist}(x,v_k))=\exp(B_\xi(y,x)).$$

Since $\alpha'(t)=\beta(t)$ approaches to 1 as $t\longrightarrow\infty$, we have

$$\lim_{k\to\infty}\frac{\alpha(\mathrm{dist}(y,v_k))-\alpha(\mathrm{dist}(x,v_k))}{\mathrm{dist}(y,v_k)-\mathrm{dist}(x,v_k)}=1,$$

which gives the desired result:

$$\lim_{k\to\infty}\exp(\alpha(\mathrm{dist}(y,v_k))-\alpha(\mathrm{dist}(x,v_k)))=\exp(B_\xi(y,x)).$$

q.e.d.

Proposition 2.17. *The series $\widetilde{Z}_V(x,s)$ defined by Lemma 2.15 is an invariant D-series for V with $D\in[0,(n-1)b]$, and its Radon-Nikodym derivative at $\zeta\in S_\infty$ is given by $\exp(-DB_\zeta(x',x))$.*

Proof. The proposition follows from Proposition 2.14, Lemma 2.15 and Proposition 2.16. q.e.d.

Let us define a function $\Theta:\widetilde{M}\times\widetilde{M}\times S_\infty\longrightarrow\mathbb{R}^+$ by $\Theta(x,y,\xi):=\exp(-B_\xi(x,y))$.

Theorem 2.18. *Let Γ be a finitely generated, free discrete group of isometries of $\widetilde{M}$ with free generating set Ω. Let $x\in\widetilde{M}$. Set $\Psi:=\Omega\coprod\Omega^{-1}$. Then there exist a D-conformal density $[\mu_y]^D_{y\in\widetilde{M}}$ under the action of Γ, and a family $[\nu_\psi]_{\psi\in\Psi}$ of Borel measures on S_∞ with:*

(1) $\mu_x(S_\infty)=1$.

(2) $\mu_x=\sum_{\psi\in\Psi}\nu_\psi$.

(3) $\int_{S_\infty}\Theta^D(x,\psi^{-1}x,\xi)\mathrm{d}\nu_{\psi^{-1}}(\xi)=1-\int_{S_\infty}\mathrm{d}\nu_\psi$.

Proof. This follows from Proposition 2.17 and Theorem 2.7. q.e.d.

In the case where the curvature is a constant k, the density of Patterson-Sullivan conformal measure coincides with the Poisson kernel. Let $\mathcal{P}_k$ denote the corresponding Poisson kernel on $\widetilde{M}_k$. Then, $\mathcal{P}_k : \widetilde{M} \times \widetilde{M} \times S_\infty \longrightarrow \mathbb{R}$ is given by

$$\mathcal{P}_k(x, y, \xi) = (\cosh k \operatorname{dist}(x, y) - \sinh k \operatorname{dist}(x, y) \cos \angle yx\xi)^{-1}.$$

Proposition 2.19. *Let $(x, y, \xi) \in \widetilde{M} \times \widetilde{M} \times S_\infty$ and $\lambda \geq 0$ be given. Then we have*

$$\text{(I)} \qquad \Theta^{a\lambda}(x, y, \xi) \geq \mathcal{P}_a^\lambda(x, y, \xi)$$

$$\text{(II)} \qquad \Theta^{b\lambda}(x, y, \xi) \leq \mathcal{P}_b^\lambda(x, y, \xi)$$

Proof. Since

$$\exp(-a\lambda B_\xi(x, y)) = \lim_{v \to \xi} \frac{\cosh a\lambda \operatorname{dist}(y, v)}{\cosh a\lambda \operatorname{dist}(x, v)}$$

and by Proposition 2.2 (i) we have

$$\frac{\cosh a\lambda \operatorname{dist}(y, v)}{\cosh a\lambda \operatorname{dist}(x, v)}$$
$$\geq (\cosh a\lambda \operatorname{dist}(x, y) - \tanh a\lambda \operatorname{dist}(y, v) \sinh a\lambda \operatorname{dist}(x, y) \cos \angle xyv)^{-1}.$$

Hence, by letting $v \longrightarrow \xi$ we have $\exp(-a\lambda B_\xi(x, y)) \leq \mathcal{P}_a^\lambda(x, y, \xi)$, which proves (I). The proof for (II) is similar, but we use Proposition 2.2 (ii) in place of 2.2 (i). q.e.d.

3. Displacement

In this section, we will use the previous results to study the displacement function of Γ. We set $b = 1$ (i.e $-1 \leq \mathcal{K} \leq -a^2$) throughout this section.

3.1 Useful lemmas

Lemma 3.1. *Let a point $x \in \widetilde{M}$ and an isometry γ of $\widetilde{M}$ be given. Let α and β be nonnegative numbers with $\alpha \leq 1/2$ and $\beta \leq 1$. Let $\lambda \neq 0$ be a positive number. Suppose that there exists a Borel measure ν on S_∞ such that:*

(i) $\nu(S_\infty) \leq \alpha$.

(ii) $\int_{S_\infty} \Theta(x, \gamma^{-1}x, \xi)^\lambda \mathrm{d}\nu \geq \beta$.

Then

$$\mathrm{dist}(x, \gamma x) \geq \frac{1}{\lambda} \log \frac{\beta}{\alpha}.$$

Proof. By Proposition 2.19 we have

$$\begin{aligned}
\beta &\leq \int_{S_\infty} \Theta(x, \gamma^{-1}x, \xi)^\lambda \mathrm{d}\nu \leq \int_{S_\infty} \mathcal{P}_1^\lambda(x, \gamma^{-1}x, \xi)\mathrm{d}\nu \\
&\leq \sup_{S_\infty} \mathcal{P}_1^\lambda \nu(S_\infty) \\
&\leq \left(\frac{1}{\cosh(\mathrm{dist}(\gamma x, x)) - \sinh(\mathrm{dist}(\gamma x, x))} \right)^\lambda \alpha.
\end{aligned}$$

Using the last inequality and the definitions of $\cosh z$ and $\sinh z$, we have

$$\frac{\beta}{\alpha} \leq \exp(\lambda \,\mathrm{dist}(\gamma x, x)),$$

which gives

$$\mathrm{dist}(x, \gamma x) \geq \frac{1}{\lambda} \log \frac{\beta}{\alpha}.$$

q.e.d.

Lemma 3.2. *Let* $\dim M = 3$*. Fix a point* $x \in \widetilde{M}$*. Let us use the notation of Lemma* 3.1*. Suppose that* ν *satisfies:*

(i) $\nu \leq \mathfrak{M}_x^2$.

(ii) $\nu(S_\infty) \leq \alpha$.

(iii) $\int_{S_\infty} \Theta(x, \gamma^{-1}x, \xi)^2 \mathrm{d}\nu \geq \beta$.

Then we have

$$\mathrm{dist}(x, \gamma x) \geq \frac{1}{2} \log \frac{\beta(1-\alpha)}{\alpha(1-\beta)}.$$

Proof. We define a function $f : \mathbb{R} \times [0, \pi] \longrightarrow \mathbb{R}$ by

$$f(z, \phi) := \left(\frac{1}{c(z\rho) - s(z\rho) \cos \phi} \right)^2$$

where $\rho := \operatorname{dist}(x, \gamma x)$, $c(z\rho) := \cosh z\rho$ and $s(z\rho) := \sinh z\rho$. Set $\Theta := \Theta(x, \gamma^{-1}x, \xi)$.

Let A denote the normalized area measure on S_x. Giving S_x the spherical coordinates (θ, ϕ), we have $\mathrm{d}A = \frac{1}{4\pi} \sin \phi \mathrm{d}\phi \mathrm{d}\theta$.

Let $\phi' := \cos^{-1}(1 - 2\alpha)$, and let $C \subset S_x$ denote the spherical cap given by $\phi \leq \phi'$, so that $A(C) = \alpha$. Then by hypothesis (ii) we have $\nu(S_\infty) \leq A(C)$.

By Proposition 2.12, we have that $\Phi_x^* \mathfrak{M}_x^2 \leq A$, which gives $\Phi_x^* \nu \leq A$ by hypothesis (i). Set $\nu' := \Phi_x^* \nu$ and ${\Theta'}^2 := \Theta \circ \Phi_x$. Then, by Proposition 2.19 (II) we have

$$\int_{S_x} {\Theta'}^2 \mathrm{d}\nu' \leq \int_{S_x} \mathcal{P}_1^2(x, y, \phi) \mathrm{d}\nu'.$$

Note that by the definitions of $\mathcal{P}_1^2$ and C, we have $\inf_C \mathcal{P}_1^2 \geq \sup_{S_x - C} \mathcal{P}_1^2$. Hence

$$\begin{aligned}
\int_{S_x} \mathcal{P}_1^2 \mathrm{d}\nu' &= \int_C \mathcal{P}_1^2 \mathrm{d}\nu' + \int_{S_x - C} \mathcal{P}_1^2 \mathrm{d}\nu' \\
&\leq \int_C \mathcal{P}_1^2 \mathrm{d}\nu' + \left(\sup_{S_x - C} \mathcal{P}_1^2 \right) \nu'(S_x - C) \\
&\leq \int_C \mathcal{P}_1^2 \mathrm{d}\nu' + \left(\inf_C \mathcal{P}_1^2 \right) A(C) \leq \int_C \mathcal{P}_1^2 \mathrm{d}\nu' + \int_C \mathcal{P}_1^2 \mathrm{d}A \\
&\leq \int_C \mathcal{P}_1^2 \mathrm{d}A.
\end{aligned}$$

By the formula for $\mathcal{P}_1^2$, we have $\mathcal{P}_1^2 = f(1, \phi)$. Therefore

$$\begin{aligned}
\int_C \mathcal{P}_1^2 \mathrm{d}A &= \int_C f(1, \phi) \mathrm{d}A = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\phi'} f(1, \phi) \sin \phi \mathrm{d}\phi \mathrm{d}\theta \\
&= \frac{\alpha}{(c(\rho) - s(\rho))(c(\rho) - s(\rho) + 2\alpha(\rho))}.
\end{aligned}$$

Using hypothesis (iii) and last equation above and equations for $c(\rho)$, $s(\rho)$, we can solve for ρ. This gives

$$\rho \geq \frac{1}{2} \log \frac{\beta(1 - \alpha)}{\alpha(1 - \beta)}$$

which is the desired result. q.e.d.

Lemma 3.3. *Given x, y nonnegative numbers with $x \geq y$ and $x + y \leq 1$, we have*

$$\text{(I)} \qquad \frac{1-x}{y} \geq \frac{1-p}{p}$$

$$\text{(II)} \qquad \frac{(1-x)(1-y)}{xy} \geq \left(\frac{1-p}{p}\right)^2$$

where $p := \frac{1}{2}(x+y)$.

Proof. We will only prove (I). The proof of (II) is similar and is given in [1]. Let us write $x = p + z$ and $y = p - z$ for some nonnegative number $z \in \mathbb{R}$. Then we have $(p-z)(1-x) = y(1-p-z)$, which implies $p(1-x) + y(p-1) = z - 2zp$. But we have $p \leq \frac{1}{2}$, which gives us $p(1-x) + y(p-1) \geq 0$. Hence the result follows. q.e.d.

3.2 Theorem 1.1 and 1.2

Proof of Theorems 1.1 *and Theorem* 1.2. Write $\mathcal{S} = \{\gamma_1, ..., \gamma_k\}$. Let $\Psi := \mathcal{S} \coprod \mathcal{S}^{-1}$, and let $x \in \widetilde{M}$ be an arbitrary point. Then by Theorem 2.18, we have a $\Gamma-$invariant conformal density measure $[\mu_x]^D$ on S_∞ and a family of Borel measures $[\nu_\psi]_{\psi \in \Psi}$ on S_∞ which satisfies (1)–(3) of Theorem 2.18.

Denote $\nu_{\gamma_j}(S_\infty)$ and $\nu_{\gamma_j^{-1}}(S_\infty)$ by, κ_j and ω_j respectively for each $1 \leq j \leq k$. Without loss of generality, we can assume $\kappa_j \leq \omega_j$. Now, from Theorem 2.18 (1) and (2) and the assumption $\kappa_j \leq \omega_j$, we get $0 \leq \omega_j \leq 1$ and $0 \leq \kappa_j \leq 1/2$. It follows from Theorem 2.18 (3) that $\int_{S_\infty} \Theta^D(x, \gamma_j^{-1}x, \xi) \mathrm{d}\nu_{\gamma_j} = 1 - \omega_j$. Hence we can invoke Lemma 3.1 with $\alpha = \kappa_j$ and $\beta = 1 - \omega_j$. This gives

$$\rho_j := \operatorname{dist}(x, \gamma_j x) \geq \frac{1}{D} \log \frac{1-\omega_j}{\kappa_j}.$$

By using Lemma 3.3 (I) with $x = \omega_j$, $y = \kappa_j$, we get

$$\exp(D\rho_j) \geq \frac{1-p_j}{p_j},$$

which implies that

$$\sum_1^k \frac{1}{1+\exp(D\rho_j)} \leq \sum_1^k p_j = \frac{1}{2}\sum_1^k (\omega_j + \kappa_j) = \frac{1}{2}.$$

Now let us also assume the hypothesis of Theorem 1.2. Then by Proposition 2.3 and Proposition 2.10 and Corollary 2.13 we have $[\mu_x] = c\mathfrak{M}_x^D$ for some constant c. Since both are normalized probability measures, we have $c = 1$. By $D = 2$ and Theorem 2.18 (2) we get $\nu_{\gamma_j} \leq \mathfrak{M}_x^2$. Hence we can invoke Lemma 3.2 with $\alpha = \kappa_j$ and $\beta = 1 - \omega_j$ to obtain

$$\rho_j \geq \frac{1}{2} \log \frac{(1-\omega_j)(1-\kappa_j)}{\kappa_j \omega_j}.$$

Then, by using Lemma 3.3 (II), we get

$$\rho_j \geq \log \frac{1-p_j}{p_j}$$

and hence

$$\frac{1}{1+\exp \rho_j} \leq \frac{1}{2}(\omega_j + \kappa_j).$$

Therefore

$$\sum_{j=1}^{k} \frac{1}{1+\exp \rho_j} \leq \frac{1}{2} \sum_{j=1}^{k} (\omega_j + \kappa_j) = \frac{1}{2}.$$

q.e.d.

Proof of Corollary 1.3. Suppose M is rank-1 locally symmetric manifold, then it follows from results of Sullivan, Bishop-Jones [3] and Fernández-Melián [6] that the Hausdorff dimension of the conical limit set $\Lambda_c(\Gamma)$ is equal to D. Hence we have $D \leq \mathfrak{D}$, which gives the desired result. q.e.d.

Remark 3.4. In the proof of Theorem 1.2, the condition that Γ is divergent is used only to conclude that Γ is ergodic with respect to μ_x. Hence we can replace the condition that Γ is divergent by the condition that Γ is ergodic with respect to μ_x.

Theorem 3.5. *Suppose that $M = \widetilde{M}/\Gamma$ is a 3-manifold, that Γ is free and that $\mathfrak{M}_x^2(\Lambda(\Gamma)) > 0$. Suppose that $D = 2$ and Γ is ergodic with respect to μ_x. Then*

$$\sum_{\gamma \in \mathcal{S}} \frac{1}{1+\exp \operatorname{dist}(x, \gamma x)} \leq \frac{1}{2}.$$

Theorem 1.1 yields the following corollaries.

Corollary 3.6. *Let Γ be free on a generating set $\mathcal{S} = \{\alpha_1, \cdots, \alpha_k\}$. If Γ is convex-cocompact, then*

$$\sum_1^k \frac{1}{1+\exp(\mathfrak{D}\,\mathrm{dist}(x, \alpha_j x))} \leq \frac{1}{2}$$

It is resonable to believe that the equality of D and the Hausdorff dimension of conical limit set of Γ should remain true for the case of variable (pinched negative) curvature. If this is the case, the hypothesis that Γ is convex-cocompact can be removed from Corollary 3.6.

Corollary 3.7. *Let Γ be free on a generating set $\mathcal{S} = \{\alpha_1, \cdots, \alpha_k\}$. There exists at least $1 \leq i \leq k-1$ distinct generators such that*

$$\mathrm{dist}(x, \alpha_{j_i} x) \geq \frac{1}{D} \log(2i+1)$$

for any $x \in \widetilde{M}$.

Corollary 3.8. *Let Γ be free on a generating set $\mathcal{S} = \{\alpha_1, \cdots, \alpha_k\}$. Let $x \in \widetilde{M}$, and let us assume elements of $\mathcal{S}$ is arranged so that the displacement function of $x \in \widetilde{M}$ under α_1 has least value. Let m be a nonnegative interger. Then each element of the generating set $\{\alpha_1^m \alpha_2, \alpha_1^{m+1}\alpha_2, \alpha_3, \cdots, \alpha_k\}$ have displacement value $\geq \frac{\log 3}{D}$.*

Proof. It is sufficient to prove this for Γ of rank 2. The general case follows from induction on k. Let $\{\alpha_1, \alpha_2\}$ be any free basis of Γ, and denote the critical exponent of Γ by D_Γ. By Corollary 3.7 we have at most one generator say α_1 such that $\mathrm{dist}(x, \alpha_1 x) < \frac{\log 3}{D_\Gamma}$. Then for any integer $m \geq 0$, $\{\alpha_1, \alpha_1^m \alpha_2\}$ is also a free basis for Γ by Nielsen's transformation, hence $\mathrm{dist}(x, \alpha_1^m \alpha_2 x) \geq \frac{\log 3}{D_\Gamma}$ by Corollary 3.7. q.e.d.

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