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(α, β)-fuzzy Lie algebras over an (α, β)-fuzzy field

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Abstract

The concept of (α, β)-fuzzy Lie algebras over an (α, β)-fuzzy field is introduced. We provide characterizations of an $(\in, \in \vee q)$-fuzzy Lie algebra over an $(\in, \in \vee q)$-fuzzy field.

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1 Introduction

Zadeh [12] formulated the notion of fuzzy sets and after that many scholars developed fuzzy system of different algebraic structures. The idea of quasi-coincidence of a fuzzy point with a fuzzy set, which is mentioned in [10], has played a vital role in generating some different types of fuzzy subgroups. Using the belong-to relation ($\in$) and quasi-coincidence with relation (q) between fuzzy points and fuzzy sets, the concept of (α, β)-fuzzy subgroup was introduced by Bhakat and Das [4]. Akram [1] introduced (α, β)-fuzzy Lie subalgebras and investigated some of its properties. Nanda [9] introduced fuzzy algebra over fuzzy field. It is natural to investigate similar types of generalization of the existing fuzzy subsystem. In [3], we introduced fuzzy Lie algebra over a fuzzy field and some properties were discussed.

In this paper, we introduce the concept of (α, β)-fuzzy Lie algebra over an (α, β)-fuzzy field and investigate some of its properties.

2 Preliminaries

In this section, we present some definitions needed for our study. We denote a complete distributive lattice with the smallest element 0 and the largest element 1 by I. By a fuzzy subset of a nonempty set X, we mean a function from X to I.

Definition 2.1 (see [5])**.** Let X be a field and let F be a fuzzy subset of X. Then F is called a fuzzy field of X if

(i) for all λ, γ in X, $F(\lambda - \gamma) \geq F(\lambda) \wedge F(\gamma)$,
(ii) for all λ, $\gamma \neq 0$ in X, $F(\lambda\gamma^{-1}) \geq F(\lambda) \wedge F(\gamma)$.

Remark 2.2. It is seen that if F is a fuzzy field of X, then

$$F(0) \geq F(1) \geq F(\lambda) = F(-\lambda) = F\big(\lambda^{-1}\big) \quad \text{for all } \lambda \neq 0 \text{ in } X.$$

Definition 2.3. Let A be a fuzzy subset of a Lie algebra L. Then A is called a *fuzzy Lie algebra of L over a fuzzy field F*, if for all $x, y \in L$, $\lambda \in X$,

(i) $A(x-y) \geq A(x) \wedge A(y)$,
(ii) $A(\lambda x) \geq F(\lambda) \wedge A(x)$,
(iii) $A([x,y]) \geq A(x) \wedge A(y)$.

3 The relations *belong to* and *quasi-coincidence with*

Let L be a Lie algebra over a field X, let $A : L \to [0,1]$ be a fuzzy set on L, and let $F : X \to [0,1]$ be a fuzzy set on X. The support of fuzzy set A is the crisp set that contains all elements of L that have nonzero membership grades in A.

Definition 3.1 (see [10])**.** A fuzzy set $A : L \to [0,1]$ of the form

$$A(y) = \begin{cases} t \in (0,1], & \text{if } y = x, \\ 0, & \text{if } y \neq x \end{cases}$$

is said to be a fuzzy point with support x and value t and is denoted by x_t.

For a fuzzy point x_t and a fuzzy set A in a set L, Pu and Liu [10] gave meaning to the symbol $x_t \alpha A$ where $\alpha \in \{\in, q, \in \vee q\}$.

A fuzzy point x_t is said to *belong to* a fuzzy set A, written as $x_t \in A$, if $A(x) \geq t$. A fuzzy point x_t is said to be *quasi-coincident with* a fuzzy set A, denoted by $x_t q A$, if $A(x) + t > 1$.

For a fuzzy set $A : L \to [0,1]$ and $t \in (0,1]$, we denote $A_t = \{x \in L : x_t \in A\}$.

The following notations are used in this paper.

1. $\in \vee q$ means that either *belong to* or *quasi-coincident with*,
2. $\overline{\alpha}$ means that α does not hold.

Let $\min\{t,s\}$ be denoted by $m(t,s)$ and let $\max\{t,s\}$ be denoted by $M(t,s)$. Take $I = [0,1]$ and $\wedge = \min$, $\vee = \max$ with respect to the usual order in Definitions 2.1 and 2.3.

Lemma 3.2. *A fuzzy subset F of a field X is a fuzzy field if and only if it satisfies the following conditions:*

(i) for all λ, γ in X, $\lambda_t, \gamma_s \in F \Rightarrow (\lambda - \gamma)_{m(t,s)} \in F$,
(ii) for all $\lambda, \gamma \neq 0$ in X, $\lambda_t, \gamma_s \in F \Rightarrow (\lambda\gamma^{-1})_{m(t,s)} \in F$,

for all $t, s \in (0,1]$.

Lemma 3.3. *Let L be a Lie algebra over a field X. Then a fuzzy subset A of Lie algebra L is a fuzzy Lie algebra over a fuzzy field F of X if and only if it satisfies the following conditions:*

(i) $x_t, y_s \in A \Rightarrow (x-y)_{m(t,s)} \in A$,
(ii) $x_t \in A$, $\lambda_r \in F \Rightarrow (\lambda x)_{m(r,t)} \in A$,
(iii) $x_t, y_s \in A \Rightarrow ([x,y])_{m(t,s)} \in A$,

for all $x, y \in L$, for all $\lambda \in X$, for all $t, s, r \in (0,1]$.

4 (α, β)-fuzzy Lie algebras over an (α, β)-fuzzy field

Let α and β denote any one of $\in, q, \in \vee q$ unless otherwise specified.

Definition 4.1. Let X be a field and let $F : X \to [0, 1]$ be a fuzzy subset of X. Then F is called an (α, β)-fuzzy field of X, if it satisfies the following conditions:

(i) for all λ, γ in X, $\lambda_t \alpha F, \gamma_s \alpha F \Rightarrow (\lambda - \gamma)_{m(t,s)} \beta F$,
(ii) for all $\lambda, \gamma \neq 0$ in X, $\lambda_t \alpha F, \gamma_s \alpha F \Rightarrow (\lambda \gamma^{-1})_{m(t,s)} \beta F$,

for all $t, s \in (0, 1]$.

Definition 4.2. Let L be a Lie algebra over a field X, and let $F : X \to [0, 1]$ be an (α, β)-fuzzy field of X. Then a fuzzy subset $A : L \to [0, 1]$ is called an (α, β)-fuzzy Lie algebra of L over an (α, β)-fuzzy field F of X, if it satisfies the following conditions:

(i) $x_t \alpha A, y_s \alpha A \Rightarrow (x - y)_{m(t,s)} \beta A$,
(ii) $x_t \alpha A, \lambda_r \alpha F \Rightarrow (\lambda x)_{m(r,t)} \beta A$,
(iii) $x_t \alpha A, y_s \alpha A \Rightarrow ([x, y])_{m(t,s)} \beta A$,

for all $x, y \in L$, for all $\lambda \in X$, for all $t, s, r \in (0, 1]$.

Example 4.3. In the real vector space $\mathbb{R}^3$, define $[x, y] = x \times y$, where '$\times$' is cross product of vectors for all $x, y \in \mathbb{R}^3$. Then $\mathbb{R}^3$ is a Lie algebra over the field $\mathbb{R}$.

Define $A : \mathbb{R}^3 \to [0, 1]$ for all $x = (a, b, c) \in \mathbb{R}^3$ by

$$A(a, b, c) = \begin{cases} 1 & \text{if } a = b = c = 0, \\ 0.5 & \text{if } a \neq 0, b = 0, c = 0, \\ 0 & \text{otherwise,} \end{cases}$$

and define $F : \mathbb{R} \to [0, 1]$ for all $\lambda \in \mathbb{R}$, by

$$F(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \mathbb{Q}, \\ 0.5 & \text{if } \lambda \in \mathbb{Q}(\sqrt{2}) - \mathbb{Q}, \\ 0 & \text{if } \lambda \in \mathbb{R} - \mathbb{Q}(\sqrt{2}). \end{cases}$$

(i) Then by actual computation, it follows that F is an $(\in, \in)$-fuzzy field of $\mathbb{R}$ and A is an $(\in, \in)$-fuzzy Lie algebra of $\mathbb{R}^3$ over the $(\in, \in)$-fuzzy field F of $\mathbb{R}$. Also it can be verified that A is an $(\in, \in \vee q)$-fuzzy Lie algebra of $\mathbb{R}^3$ over an $(\in, \in \vee q)$-fuzzy field F of $\mathbb{R}$.

(ii) Let $x = (1, 0, 0)$, $y = (2, 0, 0)$, $t = 0.4$, $s = 0.3$. Then $A(x - y) = 0.5$ and $m(t, s) = 0.3$. $A(x - y) + m(t, s) < 1$. So $(x - y)_{m(t,s)} \overline{q} A$. Hence A is not an $(\in, q)$-fuzzy Lie algebra.

(iii) Let $x = (0, 0, 0)$, $y = (2, 0, 0)$ be elements in $\mathbb{R}^3$ and $t = 0.4$, $s = 0.6$. Then $x_t q A$ and $y_s q A$. But $A(x - y) + m(t, s) = 0.5 + 0.4 < 1$. This shows that $(x - y)_{m(t,s)} \overline{q} A$. Hence A is not a (q, q)-fuzzy Lie algebra.

Theorem 4.4. *Let X be a field. Then a fuzzy subset $F : X \to [0, 1]$ is a fuzzy field if and only if F is an $(\in, \in)$-fuzzy field of X.*

Proof. The result follows immediately from Lemma 3.2. □

Theorem 4.5. *Let L be a Lie algebra over a field X. Then a fuzzy subset A of L is a fuzzy Lie algebra over a fuzzy field F of X if and only if A is an $(\in, \in)$-fuzzy Lie algebra of L over an $(\in, \in)$-fuzzy field F of X.*

Proof. The result follows immediately from Lemmas 3.2 and 3.3. □

Theorem 4.6. *Let X be a field and let $F : X \to [0,1]$ be a fuzzy subset of X. Then F is an $(\in, \in\vee q)$-fuzzy field of X if and only if*

(i) for all λ, γ in X, $F(\lambda - \gamma) \geq m(F(\lambda), F(\gamma), 0.5)$,
(ii) for all $\lambda, \gamma \neq 0$ in X, $F(\lambda\gamma^{-1}) \geq m(F(\lambda), F(\gamma), 0.5)$.

Proof. Suppose that F is an $(\in, \in\vee q)$-fuzzy field of X. It is clear that

$$m(F(\lambda), F(\gamma), 0.5) = m(m(F(\lambda), F(\gamma)), 0.5).$$

We consider two possibilities.

Case 1. Let $m(F(\lambda), F(\gamma)) < 0.5$. Then, $m(F(\lambda), F(\gamma), 0.5) = m(F(\lambda), F(\gamma))$. If possible, let $F(\lambda - \gamma) < m(F(\lambda), F(\gamma), 0.5) = m(F(\lambda), F(\gamma))$. Let $r, s \in (0,1]$ be such that $F(\lambda - \gamma) < r < s < m(F(\lambda), F(\gamma))$. Then $F(\lambda) > r$, $F(\gamma) > s$ and so $\lambda_r \in F$ and $\gamma_s \in F$. Also $F(\lambda - \gamma) < m(r,s)$ shows that $(\lambda - \gamma)_{m(r,s)} \overline{\in} F$ and $F(\lambda - \gamma) + m(r,s) < m(r,s) + m(r,s) < 1$ shows that $(\lambda - \gamma)_{m(r,s)} \overline{q} F$. Therefore, $(\lambda - \gamma)_{m(r,s)} \overline{\in\vee q} F$, a contradiction.

Case 2. Let $m(F(\lambda), F(\gamma)) \geq 0.5$. Then, $m(F(\lambda), F(\gamma), 0.5) = 0.5$. If possible, let $F(\lambda - \gamma) < 0.5$. Then $\lambda_{0.5} \in F$, $\gamma_{0.5} \in F$, but $(\lambda - \gamma)_{0.5} \overline{\in\vee q} F$, a contradiction. Therefore, it follows that $F(\lambda - \gamma) \geq m(F(\lambda), F(\gamma), 0.5)$. Similarly, (ii) is proved.

Conversely, suppose that conditions (i) and (ii) are satisfied by a fuzzy set F of X. Let $\lambda_r \in F$, $\gamma_s \in F$, for $\lambda, \gamma \in X$ and $r, s \in (0,1]$. Then $F(\lambda) \geq r$, $F(\gamma) \geq s$ and so $m(F(\lambda), F(\gamma)) \geq m(r,s)$. Since F satisfies condition (i),

$$F(\lambda - \gamma) \geq m\big(F(\lambda), F(\gamma), 0.5\big) \geq m(r, s, 0.5).$$

Now consider the possibilities $m(r,s) \leq 0.5$ or $m(r,s) > 0.5$. If $m(r,s) \leq 0.5$, then $m(r,s,0.5) = m(r,s)$ and $F(\lambda - \gamma) \geq m(r,s)$ and so $(\lambda - \gamma)_{m(r,s)} \in F$. If $m(r,s) > 0.5$, then $m(r,s,0.5) = 0.5$ and $F(\lambda - \gamma) \geq 0.5$. So, $F(\lambda - \gamma) + m(r,s) \geq 0.5 + m(r,s) > 0.5 + 0.5 = 1$ and hence $(\lambda - \gamma)_{m(r,s)} q F$. Therefore, it follows that if $\lambda_r \in F$, $\gamma_s \in F$, then $(\lambda - \gamma)_{m(r,s)} \in\vee q F$. Similarly, if $\lambda_r \in F$, $\gamma_s \in F$ for all $\lambda, \gamma \neq 0$ in X, then $(\lambda\gamma^{-1})_{m(r,s)} \in\vee q F$. Hence F is an $(\in, \in\vee q)$-fuzzy field of X. □

Theorem 4.7. *Let L be a Lie algebra over a field X. Then a fuzzy subset A of L is an $(\in, \in\vee q)$-fuzzy Lie algebra of L over an $(\in, \in\vee q)$-fuzzy field F of X if and only if*

(i) for all $x, y \in L$, $A(x - y) \geq m(A(x), A(y), 0.5)$,
(ii) for all $x \in L$, $\lambda \in X$, $A(\lambda x) \geq m(F(\lambda), A(x), 0.5)$,
(iii) for all $x, y \in L$, $A([x,y]) \geq m(A(x), A(y), 0.5)$.

Proof. Suppose that A is an $(\in, \in\vee q)$-fuzzy Lie algebra over an $(\in, \in\vee q)$-fuzzy field F of X. It is clear that $m(F(\lambda), A(x), 0.5) = m(m(F(\lambda), A(x)), 0.5)$. We consider two possibilities.

Case 1. Let $m(F(\lambda), A(x)) < 0.5$. Then, $m(F(\lambda), A(x), 0.5) = m(F(\lambda), A(x))$. If possible, let $A(\lambda x) < m(F(\lambda), A(x), 0.5) = m(F(\lambda), A(x))$. Let $t \in (0,1]$ be such that $A(\lambda x) < t < m(F(\lambda), A(x))$. Then, $F(\lambda) > t$ and $A(x) > t$. So, $\lambda_t \in F$ and $x_t \in A$. But $A(\lambda x) < t$ and $A(\lambda x) + t < t + t < 2m(F(\lambda), A(x)) < 1$. This shows that $(\lambda x)_t \overline{\in\vee q} A$, a contradiction.

Case 2. Let $m(F(\lambda), A(x)) \geq 0.5$. If possible, let $A(\lambda x) < m(F(\lambda), A(x), 0.5) = 0.5$. Then we have $\lambda_{0.5} \in F$ and $x_{0.5} \in A$, but $(\lambda x)_{0.5} \overline{\in\vee q} A$, a contradiction. Therefore, it follows that $A(\lambda x) \geq m(F(\lambda), A(x), 0.5)$. Thus, (ii) is proved. Similarly, (i) and (iii) are proved.

Conversely, suppose that the conditions (i), (ii), and (iii) are satisfied by a fuzzy set A of L. Let $x_t \in A$, $y_s \in A$, for $x, y \in L$ and $t, s \in (0, 1]$. Then, $A(x) \geq t$, $A(y) \geq s$ and so $m(A(x), A(y)) \geq m(t, s)$. Since A satisfies condition (iii),

$$A\big([x, y]\big) \geq m\big(A(x), A(y), 0.5\big) \geq m(t, s, 0.5).$$

Now consider the possibilities $m(t, s) \leq 0.5$ or $m(t, s) > 0.5$. If $m(t, s) \leq 0.5$, then, $m(t, s, 0.5) = m(t, s)$ and $A([x, y]) \geq m(t, s)$, and so $([x, y])_{m(t,s)} \in A$. If $m(t, s) > 0.5$, then, $m(t, s, 0.5) = 0.5$ and $A([x, y]) \geq 0.5$. So $A([x, y]) + m(t, s) \geq 0.5 + m(t, s) > 0.5 + 0.5 = 1$ and hence $([x, y])_{m(t,s)} qA$. Therefore, it follows that if $x_t \in A$, $y_s \in A$, then $([x, y])_{m(t,s)} \in \vee qA$. Similarly, if $x_t \in A$, $y_s \in A$, then $(x - y)_{m(t,s)} \in \vee qA$ and if $\lambda_r \in F$, $x_t \in A$, then $(\lambda x)_{m(r,t)} \in \vee qA$. Hence, A is an $(\in, \in \vee q)$-fuzzy Lie algebra of L over an $(\in, \in \vee q)$-fuzzy field F of X. □

Proposition 4.8. *Let L be a Lie algebra over a field X. Then every $(\in, \in)$-fuzzy Lie algebra of L over an $(\in, \in)$-fuzzy field of X is an $(\in, \in \vee q)$-fuzzy Lie algebra of L over an $(\in, \in \vee q)$-fuzzy field of X.*

Proof. Suppose A is an $(\in, \in)$-fuzzy Lie algebra of L over an $(\in, \in)$-fuzzy field F of X. Let $\lambda, \gamma \in X$, $r, s \in (0, 1]$. Since F is an $(\in, \in)$-fuzzy field of X, $\lambda_r \in F$, $\gamma_s \in F \Rightarrow (\lambda - \gamma)_{m(r,s)} \in F$, then $F(\lambda - \gamma) \geq m(r, s)$ shows that $(\lambda - \gamma)_{m(r,s)} \in \vee qF$. Similarly, $(\lambda\gamma^{-1})_{m(r,s)} \in \vee qF$ for all $\lambda, \gamma \neq 0$ in X. So F is an $(\in, \in \vee q)$-fuzzy field of X. Since A is an $(\in, \in)$-fuzzy Lie algebra, for $x, y \in L$, $t, s \in (0, 1]$, $x_t \in A$, $y_s \in A \Rightarrow ([x, y])_{m(t,s)} \in A$. Thus, $A([x, y]) \geq m(t, s)$. Then by definition $([x, y])_{m(t,s)} \in \vee qA$. Similarly, $x_t \in A$, $y_s \in A \Rightarrow (x - y)_{m(t,s)} \in \vee qA$ and $x_t \in A$, $\lambda_s \in F \Rightarrow (\lambda x)_{m(t,s)} \in \vee qA$. Hence A is an $(\in, \in \vee q)$-fuzzy Lie algebra of L over an $(\in, \in \vee q)$-fuzzy field F of X. □

Remark 4.9. The converse of this proposition may not be true as seen in the following example.

Example 4.10. Let $L = \mathbb{R}^3$ and $[x, y] = x \times y$, where '$\times$' is cross product for all $x, y \in L$. Then L is a Lie algebra over the field $\mathbb{R}$. Define $A : \mathbb{R}^3 \to [0, 1]$ for all $x = (a, b, c) \in \mathbb{R}^3$ by

$$A(a, b, c) = \begin{cases} 0.6 & \text{if } a = b = c = 0, \\ 0.8 & \text{if } a \neq 0,\ b = 0,\ c = 0, \\ 0.5 & \text{otherwise}, \end{cases}$$

and define $F : \mathbb{R} \to [0, 1]$ for all $\lambda \in \mathbb{R}$ by

$$F(\lambda) = \begin{cases} 0.6 & \text{if } \lambda \in \mathbb{Q}, \\ 0.8 & \text{if } \lambda \in \mathbb{Q}(\sqrt{2}) - \mathbb{Q}, \\ 0.5 & \text{if } \lambda \in \mathbb{R} - \mathbb{Q}(\sqrt{2}). \end{cases}$$

Then by Theorem 4.7, it follows that A is an $(\in, \in \vee q)$-fuzzy Lie algebra of $\mathbb{R}^3$ over an $(\in, \in \vee q)$-fuzzy field F of $\mathbb{R}$.

But this is not an $(\in, \in)$-fuzzy Lie algebra of $\mathbb{R}^3$ over an $(\in, \in)$-fuzzy field of $\mathbb{R}$. Let $x = (1, 0, 0)$. Then $A(1, 0, 0) = 0.8 > 0.65 > 0.62$. So $x_{0.65} \in A$ and $x_{0.62} \in A$. But $(x - x)_{m(0.65, 0.62)} = (0)_{0.62} \overline{\in} A$. It is clear that $A(0) + 0.62 = 0.6 + 0.62 > 1$ and so $(0)_{0.62} \in \vee qA$. Therefore A is not an $(\in, \in)$-fuzzy Lie algebra of $\mathbb{R}^3$ over an $(\in, \in)$-fuzzy field F of $\mathbb{R}$.

Theorem 4.11. *Let A be an $(\in, \in \vee q)$-fuzzy Lie algebra of L over an $(\in, \in \vee q)$-fuzzy field F of X such that $M(A(x), F(\lambda)) < 0.5$ for all $x \in L$ and for all $\lambda \in X$. Then A is an $(\in, \in)$-fuzzy Lie algebra of L over an $(\in, \in)$-fuzzy field F of X.*

Proof. Suppose that A is an $(\in, \in \vee q)$-fuzzy Lie algebra of L over an $(\in, \in \vee q)$-fuzzy field F of X. Let $\lambda, \gamma \in X$ and $t, s \in (0, 1]$ be such that $\lambda_t \in F$, $\gamma_s \in F$. Then, $F(\lambda) \geq t, F(\gamma) \geq s$ and so $m(F(\lambda), F(\gamma)) \geq m(t, s)$. It follows from Theorem 4.6 that $F(\lambda - \gamma) \geq m(F(\lambda), F(\gamma), 0.5)$. Given that $M(A(x), F(\lambda)) < 0.5$ for all $x \in L$, for all $\lambda \in X$,

then, we have $m(F(\lambda), F(\gamma)) < 0.5$.
So $F(\lambda - \gamma) \geq m(F(\lambda), F(\gamma)) \geq m(t, s)$.
Therefore, $(\lambda - \gamma)_{m(t,s)} \in F$.
Similarly, $(\lambda\gamma^{-1})_{m(t,s)} \in F$ for all $\lambda, \gamma \neq 0$ in X.
Therefore, F is an $(\in, \in)$-fuzzy field of X.

Let $x, y \in L$ and $t_1, t_2 \in (0, 1]$ be such that $x_{t_1} \in A$, $y_{t_2} \in A$. Then, $A(x) \geq t_1, A(y) \geq t_2$ and so $m(A(x), A(y)) \geq m(t_1, t_2)$. From Theorem 4.7, $A(x - y) \geq m(A(x), A(y), 0.5)$ and from the given condition we get $m(A(x), A(y)) < 0.5$. Therefore, $A(x - y) \geq m(t_1, t_2)$ and so, $(x - y)_{m(t_1,t_2)} \in A$. Let $x \in L$, $\lambda \in X$, $s, t \in (0, 1]$ be such that $\lambda_s \in F$, $x_t \in A$. Then $F(\lambda) \geq s$, $A(x) \geq t$ and so $m(F(\lambda), A(x)) \geq m(s, t)$. By Theorem 4.7,

$$A(\lambda x) \geq m\big(A(x), F(\lambda), 0.5\big) = m\big(A(x), F(\lambda)\big) \geq m(s, t).$$

So $(\lambda x)_{m(s,t)} \in A$. Similarly, $x_{t_1} \in A$, $y_{t_2} \in A \Rightarrow ([x, y])_{m(t_1,t_2)} \in A$. Therefore, A is an $(\in, \in)$-fuzzy Lie algebra of L over an $(\in, \in)$-fuzzy field F of X. □

Proposition 4.12. *If A is an $(\in, \in \vee q)$-fuzzy Lie algebra of L over an $(\in, \in \vee q)$-fuzzy field F, then*

(1) $A(0) \geq m(A(x), 0.5)$,
(2) $A(-x) \geq m(A(x), 0.5)$,
(3) $A(x + y) \geq m(A(x), A(y), 0.5)$.

Proof. Let $x \in L, y \in L$. Then, from Theorem 4.7, the following hold.

(1) $A(0) = A([x, x]) \geq m(A(x), 0.5)$. So, $A(0) \geq m(A(x), 0.5)$.
(2) $A(-x) = A(0 - x) \geq m(A(0), A(x), 0.5) = m(m(A(x), 0.5), A(0)) = m(A(x), 0.5)$. Therefore, $A(-x) \geq m(A(x), 0.5)$.
(3) $A(x + y) = A(x - (-y)) \geq m(A(x), A(-y), 0.5) \geq m(A(x), m(A(y), 0.5), 0.5) = m(A(x), A(y), 0.5)$. Therefore, $A(x + y) \geq m(A(x), A(y), 0.5)$. □

Theorem 4.13. *Let A be an $(\in, \in \vee q)$-fuzzy Lie algebra of L over an $(\in, \in \vee q)$-fuzzy field F of X. Then, for $t \in (0, 0.5]$, A_t is a Lie subalgebra over F_t when F_t contains at least two elements.*

Proof. For $t \in (0, 0.5]$, suppose F_t contains at least two elements.

Let $\lambda, \gamma \in F_t$. Then $\lambda_t \in F$, $\gamma_t \in F$ and so $F(\lambda) \geq t$, $F(\gamma) \geq t$. This shows that $m(F(\lambda), F(\gamma)) \geq t$ and so $m(F(\lambda), F(\gamma), 0.5) \geq m(t, 0.5)$. Therefore,

$$F(\lambda - \gamma) \geq m\big(F(\lambda), F(\gamma), 0.5\big) \geq m(t, 0.5) = t$$

and hence $(\lambda-\gamma)_t \in F$. Thus, $\lambda-\gamma \in F_t$. Similarly, $\lambda\gamma^{-1} \in F_t$ for all $\lambda,\gamma \neq 0$ in F_t. Therefore, F_t is a subfield of X.

Suppose $x,y \in A_t$. Then $A(x) \geq t$, $A(y) \geq t$ and $m(A(x),A(y),0.5) \geq m(t,0.5) = t$. So $A(x+y) \geq m(A(x),A(y),0.5) \geq t$ and hence $(x+y) \in A_t$. Let $\lambda \in F_t$, $x \in A_t$. Then $F(\lambda) \geq t$, $A(x) \geq t$ and $m(F(\lambda),A(x)) \geq t$. Thus, $m(F(\lambda),A(x),0.5) \geq t$ and so $A(\lambda x) \geq m(F(\lambda),A(x),0.5) \geq t$. Hence, $\lambda x \in A_t$.

Similarly, for $x,y \in A_t$, $[x,y] \in A_t$. Therefore, A_t is a Lie subalgebra over the field F_t. □

Let $f : L \to L'$ be a function. If A and B are fuzzy subsets of L and L', respectively, then $f(A)$ and $f^{-1}(B)$ are defined using Zadeh's extension principle [6]. If α is one of $\{\in, q, \in\vee q\}$, it follows that $x_t \alpha f^{-1}(B)$ if and only if $(f(x))_t \alpha B$ for all $x \in L$ and for all $t \in (0,1]$.

Theorem 4.14. *Let L and L' be Lie algebras over a field X and let $f : L \to L'$ be a homomorphism. If B is an $(\in,\in\vee q)$-fuzzy Lie algebra of L' over an $(\in,\in\vee q)$-fuzzy field F of X, then $f^{-1}(B)$ is an $(\in,\in\vee q)$-fuzzy Lie algebra of L over the $(\in,\in\vee q)$-fuzzy field F of X.*

Proof. Let $x,y \in L$ and $t,s \in (0,1]$ be such that $x_t \in f^{-1}(B)$ and $y_s \in f^{-1}(B)$. Then $(f(x))_t \in B$, $(f(y))_s \in B$. Since B is an $(\in,\in\vee q)$-fuzzy Lie algebra of L' over an $(\in,\in\vee q)$-fuzzy field F of X,

$$\big(f(x-y)\big)_{m(t,s)} = \big(f(x)-f(y)\big)_{m(t,s)} \in \vee qB.$$

So we have $(x-y)_{m(t,s)} \in \vee qf^{-1}(B)$. Similarly, $([x,y])_{m(t,s)} \in \vee qf^{-1}(B)$.

Let $\lambda \in X$, $x \in L$ and $r,t \in (0,1]$ be such that $\lambda_r \in F$ and $x_t \in f^{-1}(B)$. Then $(f(x))_t \in B$ and so

$$\big(f(\lambda x)\big)_{m(r,t)} = \big(\lambda f(x)\big)_{m(r,t)} \in \vee qB$$

and hence $(\lambda x)_{m(r,t)} \in \vee qf^{-1}(B)$.

Therefore, $f^{-1}(B)$ is an $(\in,\in\vee q)$-fuzzy Lie algebra of L over the $(\in,\in\vee q)$-fuzzy field F of X. □

Definition 4.15. A fuzzy set μ of a set Y is said to possess *sup property* if for every nonempty subset S of Y, there exists $x_0 \in S$ such that

$$\mu(x_0) = \operatorname{Sup}\{\mu(x) \mid x \in S\}.$$

Theorem 4.16. *Let L and L' be Lie algebras over a field X and let $f : L \to L'$ be an onto homomorphism. Let A be an $(\in,\in\vee q)$-fuzzy Lie algebra of L over an $(\in,\in\vee q)$-fuzzy field F of X, which satisfies the sup property. Then $f(A)$ is an $(\in,\in\vee q)$-fuzzy Lie algebra of L' over the $(\in,\in\vee q)$-fuzzy field F of X.*

Proof. Let $a,b \in L'$ and $t,s \in (0,1]$ be such that $a_t \in f(A)$ and $b_s \in f(A)$. Then $f(A)(a) \geq t$ and $f(A)(b) \geq s$ and so

$$\operatorname{Sup}\{A(z) \mid z \in f^{-1}(a)\} \geq t \quad \text{and} \quad \operatorname{Sup}\{A(w) \mid w \in f^{-1}(b)\} \geq s.$$

Since f is onto, $f^{-1}(a)$ and $f^{-1}(b)$ are nonempty subsets of L and by the *sup property* of A, there exists $x \in f^{-1}(a)$ and $y \in f^{-1}(b)$ such that

$$A(x) = \operatorname{Sup}\{A(z) \mid z \in f^{-1}(a)\} \quad \text{and} \quad A(y) = \operatorname{Sup}\{A(w) \mid w \in f^{-1}(b)\},$$

then $x_t \in A$ and $y_s \in A$. Since A is an $(\in, \in \vee q)$-fuzzy Lie algebra of L over an $(\in, \in \vee q)$-fuzzy field F of X, we have $([x,y])_{m(t,s)} \in \vee qA$ and so $A([x,y]) \geq m(t,s)$ or $A([x,y]) + m(t,s) > 1$. Now $f(x) = a$, $f(y) = b$ and so $[x,y] \in f^{-1}([a,b])$. Therefore,

$$f(A)\big([a,b]\big) = \operatorname{Sup}\big\{A(z) \mid z \in f^{-1}\big([a,b]\big)\big\} \geq A\big([x,y]\big)$$

and so $f(A)([a,b]) \geq m(t,s)$ or $f(A)([a,b]) + m(t,s) > 1$. Thus, $([a,b])_{m(t,s)} \in \vee qf(A)$. Also $(x-y)_{m(t,s)} \in \vee qA$ shows that $(a-b)_{m(t,s)} \in \vee qf(A)$.

Let $\lambda \in X$, $b \in L'$ and $r,s \in (0,1]$ be such that $\lambda_r \in F$ and $b_s \in f(A)$. Then it follows that $\lambda_r \in F$ and $y_s \in A$. So $(\lambda y)_{m(r,s)} \in \vee qA$. Thus, $A(\lambda y) \geq m(r,s)$ or $A(\lambda y) + m(r,s) > 1$. But $f(A)(\lambda b) = \operatorname{Sup}\{A(w) \mid w \in f^{-1}(\lambda b)\} \geq A(\lambda y)$. This shows that $(\lambda b)_{m(r,s)} \in \vee qf(A)$.

Therefore, $f(A)$ is an $(\in, \in \vee q)$-fuzzy Lie algebra of L' over the $(\in, \in \vee q)$-fuzzy field F of X. □

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