

Four-Dimensional Nilpotent Diassociative Algebras

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Abstract

The paper is devoted to structural properties of diassociative algebras. We introduce the notions of nilpotency, solvability of the diassociative algebras and study their properties. The list of all possible nilpotent diassociative algebra structures on four-dimensional complex vector spaces is given.

Keywords: Associative algebra; Diassociative algebra; Isomorphism; Nilpotency; Solvability

Introduction

In 1993, Loday introduced the notion of Leibniz algebra, that is a generalization of Lie algebra, where the skew-symmetry of the bracket is dropped and the Jacobi identity is replaced by the Leibniz identity (the identity has been called Leibniz identity by Loday due to its similarity to Leibniz rule, this is the reason for the class to be called by the name of Leibniz). Loday also showed that the link between Lie and associative algebras can be extended to analogous link between Leibniz algebras and a so-called associative dialgebras which are a generalization of associative algebras possessing two composition laws. Loday showed that if $D=(V, \vdash, \dashv)$ is a diassociative algebra structure on a vector space V then the Leibniz algebra structure on V is defined by the bracket $[x, y] := x \dashv y - y \vdash x$. Conversely, the universal enveloping algebra of Leibniz algebra has the structure of diassociative algebra. Loday has given some examples to motivate the study of these classes of algebras. On the structure of algebras from these classes not so much is known. Researchers (Loday, Pirashvili and others) mainly focused on their (co)homological problems.

The study of structural properties of Leibniz algebras has been initiated by Ayupov and Omirov [1]. Casas gave the list of isomorphism classes of three-dimensional complex Leibniz algebras [2] (two-dimensional case has been given by Loday himself). The list of nilpotent Leibniz algebras in dimension four has been obtained in [3]. In higher dimensions there are classification results with stronger conditions.

Diassociative algebra is a vector space with two bilinear binary associative operations \vdash, \dashv , satisfying certain conditions [4]. Associative algebras are particular case of the diassociative algebras when the two operations coincide. The main motivation of Loday to introduce this class of algebras was the search of an "obstruction" to the periodicity in algebraic K-theory. Besides this purely algebraic motivation some their relations with classical and non-commutative geometry, and physics have been recently discovered.

The classification problem of algebras is one of the important problems of modern algebra. The problem has been successfully solved for semisimple parts of many classes of finite-dimensional algebras. However, the complete classification of solvable and nilpotent parts is still unsolved. Particularly, for Lie, Leibniz and associative algebras the solution to the problem in low-dimensional cases has been given with some conditions.

One of the approaches is to use the conditions on structure constants due to axioms of the class of algebras considered. As for diassociative algebras none of the above mentioned parts are studied. In this paper we are interested in description of diassociative algebra

structures on low-dimensional complex vector spaces. We construct a diassociative algebra structure on a vector space as a combination of two associative algebras. Accordingly, we need complete lists of associative algebras (both, unital and non unital) in the dimensions considered. The lists can be found in [5,6].

We remind that the classification problems of low-dimensional complex Lie and Leibniz algebras have been considered in [1,3,7,8]. In this paper we give the list of isomorphism classes of four-dimensional complex nilpotent diassociative algebras. We restrict our discussion to four-dimensional nilpotent case since in lower dimensions the classifications have already been given in [9,10].

The organization of the paper is as follows. It is split into four sections including the introduction. In Section 2 we give necessary definitions needed and present the lists of complex diassociative algebras in dimensions two and three from [9,10]. The main results of this paper are in Sections 3 and 4. In Section 3 we propose the notions of nilpotency and solvability for diassociative algebras and give criteria of nilpotency and solvability of dialgebras. In Section 4 the complete list of isomorphism classes of four-dimensional nilpotent complex diassociative algebras are given.

Loday Algebras

Around 1990, Loday introduced several new classes of algebras called Leibniz algebra, diassociative algebra, dendriform algebra and Zinbiel algebra [11,4]. He also has given a categorical diagram on links between these and classical algebras. In this section we briefly review definitions and remind the classification results of Loday algebras.

Definition 1: A Leibniz algebra is a vector space L equipped with a binary operation, called Leibniz bracket, $[\bullet, \bullet] : L \times L \rightarrow L$ which satisfies the Leibniz identity

$$[[x, y], z] = [[x, z], y] + [x, [y, z]], \text{ for all } x, y, z \in L. \quad (1)$$

Obviously, if the bracket is skew-symmetric, then L is Lie algebra.

It is well known that an associative algebra (A, \bullet) gives rise to a Lie

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algebra by $[a, b] = a \bullet b - b \bullet a$. There is a generalization of associative algebra which plays the same role to Leibniz algebra as associative algebra to Lie algebra in the above link. This generalization is called associative dialgebra. The definition of the associative dialgebra is as follows.

Definition 2: Associative dialgebra (the term diassociative algebra also is used) D is an algebra equipped with two bilinear binary associative operations, \dashv and \vdash , called left and right products, respectively, satisfying the following axioms:

$$(x \dashv y) \dashv z = x \dashv (y \vdash z),$$

$$(x \vdash y) \dashv z = x \vdash (y \dashv z),$$

$$(x \dashv y) \vdash z = x \vdash (y \vdash z),$$

for all $x, y, z \in D$.

Let (D, \dashv, \vdash) be a diassociative algebra structure on a vector space V then it is immediate to check that the bracket $[x, y] := x \dashv y - y \vdash x$ satisfies the Leibniz identity (1) and defines a Leibniz algebra structure on V .

Some examples and applications of diassociative algebras are given in [12-14].

We remind the definitions of a few more classes of algebras also introduced by Loday and (co)homologically closely related to the above mentioned two classes of algebras in order to complete Loday's categorical diagram. One of these classes is called the class of dendriform algebras.

Definition 3: Dendriform algebra E is algebra with two binary operations

$$\succ: E \times E \rightarrow E, \prec: E \times E \rightarrow E$$

satisfying the axioms:

$$(x \prec y) \prec z = x \prec (y \prec z) + x \prec (y \succ z),$$

$$(x \succ y) \prec z = x \succ (y \prec z),$$

$$(x \prec y) \succ z + (x \succ y) \succ z = x \succ (y \succ z),$$

for all $x, y, z \in E$.

The papers [4,15-17] are devoted to the dendriform algebras and their relations.

Another class of algebras introduced by Loday is a class called Zinbiel algebras. The definition of the Zinbiel algebra is as follows.

Definition 4: Zinbiel algebra R is an algebra with a binary operation $\bullet: R \times R \rightarrow R$, satisfying the condition:

$$(x \bullet y) \bullet z = x \bullet (y \bullet z) + x \bullet (z \bullet y), \text{ for all } x, y, z \in R.$$

The essential progress on Zinbiel algebras has been made by A. Dzhamalidaev et al. in [18], where the authors proved that any finite-dimensional Zinbiel algebra over an algebraic closed field is nilpotent, nil and solvable. A few years before Omirov [19] have given the classification of two-dimensional complex Zinbiel algebras. Most studied class among the classes of algebras introduced by Loday is the class of Leibniz algebras. Complete classifications up to dimension three are given [2], nilpotent case in dimension four can be found in [3], the solvable Leibniz algebras have been studied in [20,21]. In higher dimensions there are classifications of subclasses of nilpotent part (filiform Leibniz algebras) [22-24].

The results intertwining Loday algebras are best expressed in the framework of algebraic operads. The notion of diassociative algebra defines an algebraic operad $Dias$, which is binary and quadratic. According to the theory of Ginzburg and Kapranov there is a well-defined "dual operad" $Dias^!$ Loday has showed that this is exactly the operad $Dend$ of the dendriform algebras, in other words dual diassociative algebra is nothing but a dendriform algebra. The similar duality can be established between the algebraic operads $Leib$ defined by the notion of Leibniz algebra and the algebraic operads $Zinb$ defined by the notion of Zinbiel algebra. The categories of algebras over these operads assemble into a commutative diagram of functors which reflects the Koszul duality [4].

The classification of these new classes of algebras is of great interest. In this paper we deal with the classification problem of diassociative algebras.

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Now and what it follows we use the following notations: As_n^q and $Dias_n^q$ stand for q th associative and q th diassociative algebra structures in n -dimensional vector space, respectively. All algebras and vector spaces are assumed to be finite-dimensional and over the field of complex numbers \mathbb{C} , unless is stated otherwise.

There is the following naive approach for classifying of algebra structures on a vector space which often is being used up to now. It runs as follows. Once one fixes a basis of the vector space then according to the identities which the algebra satisfies we get a system of equations with respect to the structure constants of the algebra on this basis. Solving this system of equations we get a redundant, in general, list of algebras via the tables of multiplications. The second step is to make the obtained list irredundant. The irredundancy can be achieved by identifying those algebras which are obtained from others by a base change. This approach has been applied to get classifications of two and three-dimensional diassociative algebras over \mathbb{C} . The results are given by the next two theorems followed by a discussion on the structure constants of diassociative algebras.

Let V be an n -dimensional vector space and (D, \dashv, \vdash) be a diassociative algebra structure on V . Choose a basis e_1, e_2, \dots, e_n of V . A diassociative algebra structure on V is determined by the set of structure constants α_{ij}^k and β_{lp}^q .

$$e_i \dashv e_j = \sum_{k=1}^n \alpha_{ij}^k e_k, \quad e_i \vdash e_p = \sum_{l=1}^n \beta_{ip}^l e_l, \quad \text{for } 1 \leq i, j, l, p \leq n.$$

The diassociative algebra axioms give the following constrains for α_{ij}^k and β_{lp}^q :

$$\alpha_{ij}^k \alpha_{rs}^l = \alpha_{rs}^l \alpha_{ij}^k, \quad \alpha_{ij}^k \alpha_{rs}^l = \alpha_{rs}^l \alpha_{ij}^k, \quad \alpha_{ij}^k \beta_{rs}^l = \alpha_{rs}^l \beta_{ij}^k, \quad \alpha_{ij}^k \beta_{rs}^l = \beta_{ij}^k \alpha_{rs}^l, \quad \beta_{ij}^k \beta_{rs}^l = \beta_{rs}^l \beta_{ij}^k.$$

The classification result below for two-dimensional complex diassociative algebras is obtained by solving two-dimensional version of the system of equations above [9].

Theorem 2.1: Any diassociative algebra structure on two-dimensional complex vector space is isomorphic to one of the following classes of algebras:

$$Dias_2^1: e_1 \dashv e_1 = e_1, e_1 \dashv e_2 = e_1, e_2 \dashv e_1 = e_2;$$

$$Dias_2^2: e_1 \dashv e_1 = e_1, e_1 \dashv e_2 = e_2, e_2 \dashv e_1 = e_1;$$

$$Dias_3^3(\alpha): e_1 \dashv e_1 = e_2, e_1 \dashv e_1 = \alpha e_2, \quad \alpha \in \mathbb{C} \setminus \{0\};$$

$$Dias_2^4: e_1 \dashv e_1 = e_1, e_1 \dashv e_2 = e_2, e_1 \dashv e_1 = e_1, e_2 \dashv e_1 = e_2.$$

Here is the classification result in three-dimensional case obtained in [10].

Theorem 2.2: Any three-dimensional complex diassociative algebra can be included in one of the following isomorphism classes of diassociative algebras:

$$Dias_3^1: e_1 \dashv e_1 = e_2, e_2 \dashv e_2 = e_2, e_3 \dashv e_3 = e_3, e_2 \dashv e_2 = e_2, e_3 \dashv e_3 = e_3;$$

$$Dias_3^2: e_1 \dashv e_2 = e_1, e_2 \dashv e_2 = e_2, e_3 \dashv e_3 = e_3, e_2 \dashv e_1 = e_1, e_2 \dashv e_2 = e_2, e_3 \dashv e_3 = e_3;$$

$$Dias_3^3: e_1 \dashv e_2 = e_1, e_2 \dashv e_2 = e_2, e_3 \dashv e_3 = e_3, e_2 \dashv e_2 = e_2, e_3 \dashv e_3 = e_3;$$

$$Dias_3^4: e_1 \dashv e_3 = e_2, e_2 \dashv e_3 = e_2, e_3 \dashv e_3 = e_3, e_3 \dashv e_3 = e_3;$$

$$Dias_3^5: e_1 \dashv e_3 = e_2, e_2 \dashv e_3 = e_2, e_3 \dashv e_3 = e_3, e_3 \dashv e_1 = e_1 - e_2, e_3 \dashv e_3 = e_3;$$

$$Dias_3^6: e_1 \dashv e_3 = e_2, e_2 \dashv e_3 = e_2, e_3 \dashv e_3 = e_3, e_3 \dashv e_1 = e_1, e_3 \dashv e_2 = e_2, e_3 \dashv e_3 = e_3;$$

$$Dias_3^7: e_1 \dashv e_3 = e_2, e_2 \dashv e_3 = e_2, e_3 \dashv e_3 = e_3, e_3 \dashv e_1 = e_2, e_3 \dashv e_2 = e_2, e_3 \dashv e_3 = e_3;$$

$$Dias_3^8: e_1 \dashv e_3 = e_2, e_2 \dashv e_3 = e_2, e_3 \dashv e_3 = e_3, e_3 \dashv e_1 = e_2, e_3 \dashv e_2 = e_2, e_3 \dashv e_3 = e_3;$$

$$Dias_3^9: e_1 \dashv e_3 = e_2, e_2 \dashv e_3 = e_2, e_3 \dashv e_3 = e_3, e_3 \dashv e_1 = e_1, e_3 \dashv e_2 = e_2, e_3 \dashv e_3 = e_3;$$

$$Dias_3^{10}: e_1 \dashv e_1 = e_2, e_2 \dashv e_3 = e_2, e_3 \dashv e_3 = e_3, e_3 \dashv e_1 = e_1, e_3 \dashv e_2 = e_2;$$

$$Dias_3^{11}: e_1 \dashv e_1 = e_1, e_2 \dashv e_3 = e_2, e_3 \dashv e_3 = e_3, e_3 \dashv e_1 = e_1, e_3 \dashv e_2 = e_2, e_3 \dashv e_3 = e_3;$$

$$Dias_3^{12}: e_1 \dashv e_3 = e_1, e_2 \dashv e_3 = e_2, e_3 \dashv e_3 = e_3, e_3 \dashv e_1 = e_1, e_3 \dashv e_2 = e_2, e_3 \dashv e_3 = e_3;$$

$$Dias_3^{13}: e_1 \dashv e_3 = e_2, e_2 \dashv e_3 = e_2, e_3 \dashv e_3 = e_3, e_3 \dashv e_1 = e_1, e_3 \dashv e_2 = e_2, e_3 \dashv e_3 = e_3;$$

$$Dias_3^{14}: e_1 \dashv e_3 = e_2, e_2 \dashv e_3 = e_2, e_3 \dashv e_3 = e_3, e_3 \dashv e_1 = e_1, e_3 \dashv e_2 = e_2, e_3 \dashv e_3 = e_3;$$

$$Dias_3^{15}: e_1 \dashv e_3 = e_2, e_2 \dashv e_3 = \alpha e_2, e_3 \dashv e_3 = e_3, \quad \forall \alpha \in \mathbb{C};$$

$$Dias_3^{16}: e_1 \dashv e_3 = e_2, e_2 \dashv e_3 = ke_2, e_3 \dashv e_3 = me_2, e_3 \dashv e_1 = pe_2, e_3 \dashv e_2 = qe_2, \text{ where } k, m, n, p, q \in \mathbb{C};$$

$$Dias_3^{17}: e_1 \dashv e_2 = e_2, e_1 \dashv e_2 = e_3, e_2 \dashv e_1 = e_3, e_1 \dashv e_2 = e_2 + e_3, e_1 \dashv e_2 = e_3, e_2 \dashv e_1 = e_3;$$

$$Dias_3^{18}: e_1 \dashv e_3 = e_3, e_3 \dashv e_1 = e_2, e_3 \dashv e_2 = e_2, e_3 \dashv e_3 = e_3;$$

$$Dias_3^{19}: e_1 \dashv e_3 = e_1, e_2 \dashv e_3 = e_2, e_3 \dashv e_3 = e_3, e_3 \dashv e_1 = e_1, e_3 \dashv e_2 = e_2, e_3 \dashv e_3 = e_3.$$

According to the theorem, among the nineteen isomorphism classes of three-dimensional diassociative algebras there are seventeen single representatives and two of the classes are represented by parametric family of algebras.

On Nilpotency and Solvability of Diassociative Algebras

In this section we give natural extensions of the concepts of nilpotency and solvability of algebras to diassociative algebras case. Two important criteria (Theorem 3.1 and Theorem 3.2) verifications of nilpotency and solvability of the diassociative algebras are given.

Let M and N be subsets of a diassociative algebra D . We define the following binary operation, denoted \diamond , over the subsets M and N of D :

$$M \diamond N := M \dashv N + M \dashv N,$$

Where

$$M \dashv N = \text{Span}_{\mathbb{C}}\{d_i \dashv d_j \mid d_i \in M, d_j \in N\} \text{ and } M \dashv N = \text{Span}_{\mathbb{C}}\{d_k \dashv d_s \mid d_k \in M, d_s \in N\}.$$

Obviously, if M and N are ideals so is $M \diamond N$. Let us consider the following series (central series) of ideals:

$$I. D^{<1>} = D, D^{<k+1>} = D^{<k>} \diamond D,$$

$$II. D^{(1)} = D, D^{(k+1)} = D \diamond D^{(k)},$$

$$III. D^1 = DD^{k+1} = D^1 \diamond D^k + D^2 \diamond D^{k-1} + \dots + D^k \diamond D^1.$$

It is an easy consequence of the axioms that the members of the series are two-sided ideals and the series have the following properties:

$$D^{<1>} \supseteq D^{<2>} \supseteq D^{<3>} \supseteq \dots \supseteq D^{<k>} \supseteq \dots$$

$$D^{(1)} \supseteq D^{(2)} \supseteq D^{(3)} \supseteq \dots \supseteq D^{(k)} \supseteq \dots$$

$$D^1 \supseteq D^2 \supseteq D^3 \supseteq \dots \supseteq D^k \supseteq \dots$$

The lemma below is on grading properties of the binary operation \diamond .

Lemma 1: For any $g, h \in \mathbb{N}$ the following embeddings are valid:

$$A. D^{<g>} \diamond D^{<h>} \subseteq D^{<g+h>},$$

$$B. D^{(g)} \diamond D^{(h)} \subseteq D^{(g+h)}.$$

Proof: The proof we carry out by the induction on h for arbitrary g . The verification of the validity of A) for $h=1$ is trivial. Suppose that the embedding is true for h and we prove the statement for $h+1$. It can be easily obtained by using the identities of the diassociative algebra and the following chain of inclusions:

$$D^{<g>} \diamond D^{<h+1>} = D^{<g>} \diamond (D^{<h>} \diamond D) \subseteq (D^{<g>} \diamond D^{<h>}) \diamond D \subseteq D^{<g+h>} \diamond D \subseteq D^{<g+h+1>}.$$

The part B) is showed similarly.

Definition 5: Diassociative algebra D is said to be the right (left) nilpotent if there exists a natural number $k \in \mathbb{N} (p \in \mathbb{N})$ such that $D^{<k>} = 0 (D^{(p)} = 0)$.

Definition 6: Diassociative algebra D is said to be nilpotent if there exists a natural number $s \in \mathbb{N}$ such that $D^s = 0$.

Lemma 2: For any $g \in \mathbb{N}$ the following equalities hold:

$$D^{<g>} = D^{(g)} = D^g.$$

Proof: We show that $D^{<g>} = D^g$, the other cases can be showed similarly. It is obvious that $D^{<g>} \subseteq D^g$ for any $g \in \mathbb{N}$. Due to Lemma 1 we have $D^{<g>} \diamond D^{<h>} \subseteq D^{<g+h>}$. It is not difficult to see that $D^{<1>} = D^1, D^{<2>} = D^2$. We make use the mathematical induction method again to show that $D^{g+1} \subseteq D^{<g+1>}$ assuming that $D^{<g>} = D^g$. Indeed,

$$\begin{aligned} D^{g+1} &= D^1 \diamond D^g + D^2 \diamond D^{g-1} + \dots + D^g \diamond D^1 \\ &\subseteq D^{<1>} \diamond D^{<g>} + D^{<2>} \diamond D^{<g-1>} + \dots + D^{<g>} \diamond D^{<1>} \\ &\subseteq D^{<g+1>}. \end{aligned}$$

From this theorem we get the following important corollary.

Corollary 1: The notions of the right nilpotency, left nilpotency and nilpotency of diassociative algebras are equivalent.

Let us now prove the following subsidiary lemma on centralization of monomials in diassociative algebras.

Lemma 3: Let (D, \dashv, \vdash) be a diassociative algebra and $M = x_1 * x_2 * \dots * x_n$ be a monomial in D , with any parenthesizing, where each $*$ is one of the operations \dashv, \vdash .

Then M can be written in the form

$$y_{-m} \dashv y_{-m+1} \dashv y_{-m+2} \dashv \dots \dashv y_{-1} \dashv y_0 \dashv y_1 \dashv y_2 \dashv \dots \dashv y_k,$$

where each y_r is one of the x_s , $r = -m, -m+1, \dots, k$ and $s = 1, 2, \dots, n$.

The element y_0 of the monomial $y_{-m} \dashv y_{-m+1} \dashv y_{-m+2} \dashv \dots \dashv y_{-1} \dashv y_0 \dashv y_1 \dashv y_2 \dashv \dots \dashv y_k$ is said to be its center. Note that no multipliers from the left and right sides of the center also are possible, i.e., the term is monomial with respect to only one binary operation.

Proof: The proof is carried out by the induction with respect to the number of multipliers of monomial. We start with three multipliers case. The following parenthesizes may occur

$$(x_1 * x_2) * x_3, \quad x_1 * (x_2 * x_3),$$

Where $*$ = \dashv, \vdash . It is easy to see that diassociative algebra axioms provide the required forms for these monomials. This gives the base of the induction.

Let us now assume that the assertion is true for $n = s$ and prove it

for $n = s + 1$. Consider $M = x_1 * x_2 * \dots * x_{s+1}$. Then for M we have the following four possibilities:

$$x_1 \dashv h, \quad x_1 \vdash h, \quad h \dashv x_{s+1}, \quad h \vdash x_{s+1},$$

where h has the required form. In the second and third cases M has the required form. As for the other two external cases we repeatedly make use the associativity property of the products and diassociative algebra axioms to bring them to the required form.

As is observed from the definition of diassociative algebra that it is a combination of two associative structures

(D, \dashv) and (D, \vdash) . The following theorem gives an important criterion on verification of nilpotency property of the diassociative algebras. According to that for the verification of the nilpotency of diassociative algebra it suffices to find out one of the associative parts to be nilpotent.

Theorem 3.1: Let (D, \dashv, \vdash) be a finite dimensional diassociative algebra, $A_1 = (D, \dashv)$ and $A_2 = (D, \vdash)$ be corresponding associative algebras. Then the following are equivalent:

- i) The diassociative algebra (D, \dashv, \vdash) is nilpotent,
- ii) The associative algebra A_1 is nilpotent,
- iii) The associative algebra A_2 is nilpotent.

Proof: It is obvious that i) follows ii) and iii).

Let us prove that iii) implies ii). Indeed, if the associative algebra $A = (D, \vdash)$ is nilpotent then there exists $k \in \mathbb{N}$ such that $A^k = 0$ (with respect to the right product \vdash). We show that $A_1^{k+1} = 0$. Let $x \in A_1^{k+1}$, i.e., $x \in \text{Span}_{\mathbb{C}}\{x_1 \dashv x_2 \dashv \dots \dashv x_{k+1} \mid x_i \in A_1\}$.

Then making repeated use of the associativity of \dashv and axiom 1 in the definition of diassociative algebra we get

$$x = (\dots((x_1 \dashv x_2) \dashv x_3) \dashv \dots \dashv x_{k+1}) = x_1 \dashv \underbrace{(x_2 \vdash x_3 \vdash \dots \vdash x_{k+1})}_{=0} = x_1 \dashv 0 = 0.$$

The implication ii) \Rightarrow iii) is proved similarly.

Let show the implication ii) \Rightarrow i). According to the discussion above the statement iii) also holds.

Therefore, there exists $k \in \mathbb{N}$ such that $A_1^k = 0$ and $A_2^{k+2} = 0$. We show that $D^{2k+2} = \{0\}$.

Indeed, since $x \in \text{Span}_{\mathbb{C}}\{x_1 * x_2 * \dots * x_{2k+2} \mid x_i \in D \text{ and } * = \dashv, \vdash\}$ then due to Lemma 3 each of $x_1 * x_2 * \dots * x_{2k+2}$ is written in the following form:

$$y_{-m} \vdash y_{-m+1} \vdash y_{-m+2} \vdash \dots \vdash y_{-1} \vdash y_0 \dashv y_1 \dashv y_2 \dashv \dots \dashv y_n,$$

with $y_j \in D$ and $m + n = 2k + 2$. In each of these terms the number of multipliers from the left and right sides of the center y_0 succeeds $k + 1$, therefore that part equals zero. This completes the proof.

Example 1:

According to the criterion above among the diassociative algebras in Theorem 2.1 there is only one isomorphism class consisting of entirely nilpotent diassociative algebras: $\text{Dias}_3^3(\alpha)$.

There is only one isomorphism class among three-dimensional diassociative algebras which is nilpotent: Dias_3^{10} . (see Theorem 2:2).

If a diassociative algebra D is nilpotent then its opposite (D, \dashv, \vdash) defined by $x \dashv' y = y \vdash x$ and $x \vdash' y = y \dashv x$ also is nilpotent.

Definition 7: An ideal of diassociative algebra is said to be nilpotent

if it is nilpotent as diassociative algebra.

The following fact can be easily proved.

Lemma 4: The sum of two nilpotent ideals of diassociative algebra is nilpotent.

A diassociative algebra has its maximal nilpotent ideal called nilradical.

Similarly to Lie algebras case the concept of solvability for diassociative algebras is defined by using the termination of following recursively defined (derived) series of ideals:

$$IV. \quad D^{[1]} = D, D^{[k+1]} = D^{[k]} \diamond D^{[k]}$$

However, the following theorem shows that there is no non-nilpotent solvable diassociative algebra.

Theorem 3.2: Let (D, \dashv, \vdash) be finite dimensional diassociative algebra and $A_1 = (D, \dashv)$, $A_2 = (D, \vdash)$ be the corresponding associative algebras. Then the following are equivalent:

- i) The diassociative algebra (D, \dashv, \vdash) is solvable,
- ii) A_1 is solvable,
- iii) A_2 is solvable.

Proof: The proof of the parts ii) \Leftrightarrow iii), i) \Rightarrow ii) and i) \Rightarrow iii) are obvious. It remains to show the implication ii) \Rightarrow i). Let A_1 to be solvable, i.e. there exist $k \in \mathbb{N}$ such that $A_1^{[k]} = 0$. Equivalently, A_2 also is solvable, i.e., there exists $l \in \mathbb{N}$ such that $A_2^{[l]} = 0$. these mean that any element of

$$\text{Span}_{\mathbb{C}}\{a_1 \dashv a_2 \dashv \dots \dashv a_p \mid a_i \in A_1, i = 1, 2, \dots, p\} \text{ with } p > 2^{k-1}$$

and

$$\text{Span}_{\mathbb{C}}\{a_1 \vdash a_2 \vdash \dots \vdash a_p \mid a_i \in A_1, i = 1, 2, \dots, p\} \text{ with } p > 2^{l-1}$$

is zero.

We show that there is a natural number m such that $D^{[m]} = \{0\}$. Indeed, let us take $m > \log_2(2^k + 2^l + 1) + 1$. Then

$$D^{[m]} = \text{Span}_{\mathbb{C}}\{d_1 * d_2 * \dots * d_{2^{m-1}} \mid d_i \in D, i = 1, 2, \dots, 2^{m-1} \text{ and } * = \dashv, \vdash\}.$$

According to Lemma 3 the element $d_1 * d_2 * \dots * d_{2^{m-1}}$ can be represented as follows

$$y_{-s} \vdash y_{-s+1} \vdash y_{-s+2} \vdash \dots \vdash y_{-1} \vdash y_0 \dashv y_1 \dashv y_2 \dashv \dots \dashv y_t,$$

Where $s + t = 2^{m-1} + 1$. It is clear that in each such term under the condition either the left or right sides of the center is zero. Therefore $D^{[m]} = \{0\}$.

Corollary 2: There is no non-nilpotent solvable diassociative algebra.

The proof is an immediate consequence of the theorem and the fact that there is no non-nilpotent solvable associative algebra.

Classification of Four Dimensional Nilpotent Diassociative Algebras

In this section we give lists of complex four-dimensional nilpotent diassociative algebras. The idea is as follows. The first part of the diassociative algebra we choose from the list given in [25]. Combining algebra from this list (taking into account the diassociative algebra axioms) with another associative algebra we get a system of equations with respect to the structure constants of the later associative algebra.

Solving the system of equations we find all such possible combinations. Then we distinguish non-isomorphic algebras. The next theorem is the classification result of four-dimensional nilpotent associative algebras which we make use in the paper.

Theorem 4.1: Any four-dimensional complex nilpotent associative algebra can be included in one of the following isomorphism classes of algebras:

- $As_4^1 : e_1e_1 = e_3, e_2e_2 = e_4;$
- $As_4^2 : e_1e_2 = e_3, e_2e_1 = e_4;$
- $As_4^3 : e_1e_2 = e_4, e_3e_1 = e_4;$
- $As_4^4 : e_1e_2 = e_3, e_2e_1 = e_3, e_2e_2 = e_4;$
- $As_4^5 : e_1e_2 = e_3, e_2e_2 = e_4, e_2e_1 = e_3;$
- $As_4^6 : e_1e_2 = e_4, e_3e_3 = e_4, e_2e_1 = e_4;$
- $As_4^7(\alpha) : e_1e_2 = e_4, e_2e_2 = e_3, e_2e_1 = \frac{1+\alpha}{1-\alpha}e_4, \alpha \in \mathbb{C} \setminus \{1\};$
- $As_4^8 : e_1e_1 = e_3, e_1e_3 = e_4, e_2e_2 = e_4, e_3e_1 = e_4;$
- $As_4^9 : e_1e_1 = e_4, e_2e_1 = e_3, e_1e_4 = e_3, e_4e_1 = e_3;$
- $As_4^{10} : e_1e_1 = e_4, e_1e_2 = e_4, e_2e_1 = e_4, e_3e_3 = e_4;$
- $As_4^{11} : e_1e_1 = e_4, e_1e_2 = e_3, e_2e_1 = e_3, e_2e_2 = -2e_3 + e_4;$
- $As_4^{12}(\alpha) : e_1e_1 = e_4, e_1e_2 = e_3, e_2e_1 = -\alpha e_4, e_2e_2 = -e_3, \alpha \in \mathbb{C};$
- $As_4^{13} : e_1e_1 = e_4, e_1e_4 = -e_3, e_2e_1 = e_3, e_2e_2 = e_3, e_4e_1 = e_3;$
- $As_4^{14} : e_1e_2 = e_4, e_1e_3 = e_4, e_2e_1 = e_4, e_2e_2 = e_4, e_3e_1 = e_4;$
- $As_4^{15}(\alpha) : e_1e_1 = e_4, e_1e_2 = \alpha e_4, e_2e_1 = -\alpha e_4, e_2e_2 = e_4, e_3e_3 = e_4, \alpha \in \mathbb{C};$
- $As_4^{16} : e_1e_1 = e_2, e_1e_2 = e_3, e_1e_3 = e_4, e_2e_1 = e_3, e_2e_2 = e_4, e_3e_1 = e_4;$

Now we prove the following theorem which gives all isomorphism classes of four-dimensional complex nilpotent diassociative algebras.

Theorem 4.2: Any four-dimensional complex nilpotent diassociative algebra is isomorphic to one of the following algebras:

- $Dias_4^1 : e_1 \dashv e_1 = e_1, e_1 \dashv e_2 = e_1, e_1 \dashv e_3 = e_1, e_1 \dashv e_4 = e_1, e_2 \dashv e_1 = ae_1 + be_2, e_2 \dashv e_2 = ce_1 + de_2, e_2 \dashv e_3 = fe_1 + ge_2, e_2 \dashv e_4 = he_1 + ke_2;$
- $Dias_4^2 : e_1 \dashv e_2 = e_1, e_1 \dashv e_3 = e_1, e_1 \dashv e_4 = e_1, e_2 \dashv e_1 = ae_1 + be_2, e_2 \dashv e_2 = ce_1 + de_2, e_2 \dashv e_3 = fe_1 + ge_2, e_2 \dashv e_4 = he_1 + ke_2;$
- $Dias_4^3 : e_1 \dashv e_2 = e_1, e_1 \dashv e_3 = e_1, e_1 \dashv e_4 = e_1, e_2 \dashv e_1 = ae_1 + be_2, e_2 \dashv e_2 = ce_1 + de_2, e_2 \dashv e_3 = fe_1 + ge_2, e_2 \dashv e_4 = he_1 + ke_2, e_3 \dashv e_1 = le_1;$
- $Dias_4^4 : e_1 \dashv e_2 = e_1, e_1 \dashv e_3 = e_1, e_1 \dashv e_4 = e_1, e_2 \dashv e_1 = ae_1 + be_2, e_2 \dashv e_2 = ce_1 + de_2, e_2 \dashv e_3 = fe_1 + ge_2, e_2 \dashv e_4 = he_1 + ke_2;$
- $Dias_4^5 : e_1 \dashv e_2 = e_1, e_1 \dashv e_3 = e_1, e_1 \dashv e_4 = e_1, e_2 \dashv e_1 = ae_1 + be_2, e_2 \dashv e_2 = ce_1 + de_2, e_2 \dashv e_3 = fe_1 + ge_2, e_2 \dashv e_4 = he_1 + ke_2;$
- $Dias_4^6 : e_1 \dashv e_2 = e_1, e_1 \dashv e_3 = e_1, e_1 \dashv e_4 = e_1, e_2 \dashv e_1 = ae_1 + be_2, e_2 \dashv e_2 = ce_1 + de_2, e_2 \dashv e_3 = fe_1 + ge_2, e_2 \dashv e_4 = he_1 + ke_2;$
- $Dias_4^7 : e_1 \dashv e_2 = e_1, e_1 \dashv e_3 = e_1, e_1 \dashv e_4 = e_1, e_2 \dashv e_1 = ae_1 + be_2, e_2 \dashv e_2 = ce_1 + de_2, e_2 \dashv e_3 = fe_1 + ge_2, e_2 \dashv e_4 = he_1 + ke_2, a \in \mathbb{C} \setminus \{1\};$
- $Dias_4^8 : e_1 \dashv e_2 = e_1, e_1 \dashv e_3 = e_1, e_1 \dashv e_4 = e_1, e_2 \dashv e_1 = ae_1 + be_2, e_2 \dashv e_2 = ce_1 + de_2, e_2 \dashv e_3 = fe_1 + ge_2, e_2 \dashv e_4 = he_1 + ke_2;$
- $Dias_4^9 : e_1 \dashv e_2 = e_1, e_1 \dashv e_3 = e_1, e_1 \dashv e_4 = e_1, e_2 \dashv e_1 = ae_1 + be_2, e_2 \dashv e_2 = ce_1 + de_2, e_2 \dashv e_3 = fe_1 + ge_2, e_2 \dashv e_4 = he_1 + ke_2;$
- $Dias_4^{10} : e_1 \dashv e_2 = e_1, e_1 \dashv e_3 = e_1, e_1 \dashv e_4 = e_1, e_2 \dashv e_1 = ae_1 + be_2, e_2 \dashv e_2 = ce_1 + de_2, e_2 \dashv e_3 = fe_1 + ge_2, e_2 \dashv e_4 = he_1 + ke_2, he_1, e_1 \dashv e_1 = ke_1, e_1 \dashv e_2 = le_1;$
- $Dias_4^{11} : e_1 \dashv e_2 = e_1, e_1 \dashv e_3 = e_1, e_1 \dashv e_4 = e_1, e_2 \dashv e_1 = ae_1 + be_2, e_2 \dashv e_2 = ce_1 + de_2, e_2 \dashv e_3 = fe_1 + ge_2, e_2 \dashv e_4 = he_1 + ke_2;$

- $Dias_4^{12} : e_1 \dashv e_2 = e_1, e_1 \dashv e_3 = e_1, e_1 \dashv e_4 = e_1, e_2 \dashv e_1 = ae_1 + be_2, e_2 \dashv e_2 = ce_1 + de_2, e_2 \dashv e_3 = fe_1 + ge_2, e_2 \dashv e_4 = he_1 + ke_2;$
- $Dias_4^{13} : e_1 \dashv e_2 = e_1, e_1 \dashv e_3 = e_1, e_1 \dashv e_4 = e_1, e_2 \dashv e_1 = ae_1 + be_2, e_2 \dashv e_2 = ce_1 + de_2, e_2 \dashv e_3 = fe_1 + ge_2, e_2 \dashv e_4 = he_1 + ke_2;$
- $Dias_4^{14} : e_1 \dashv e_2 = e_1, e_1 \dashv e_3 = e_1, e_1 \dashv e_4 = e_1, e_2 \dashv e_1 = ae_1 + be_2, e_2 \dashv e_2 = ce_1 + de_2, e_2 \dashv e_3 = fe_1 + ge_2, e_2 \dashv e_4 = he_1 + ke_2;$
- $Dias_4^{15} : e_1 \dashv e_2 = e_1, e_1 \dashv e_3 = e_1, e_1 \dashv e_4 = e_1, e_2 \dashv e_1 = ae_1 + be_2, e_2 \dashv e_2 = ce_1 + de_2, e_2 \dashv e_3 = fe_1 + ge_2, e_2 \dashv e_4 = he_1 + ke_2;$
- $Dias_4^{16} : e_1 \dashv e_2 = e_1, e_1 \dashv e_3 = e_1, e_1 \dashv e_4 = e_1, e_2 \dashv e_1 = ae_1 + be_2, e_2 \dashv e_2 = ce_1 + de_2, e_2 \dashv e_3 = fe_1 + ge_2, e_2 \dashv e_4 = he_1 + ke_2, a, b, c, d, f, g, h, k, l \in \mathbb{C}.$

Proof: We provide the proof only for one case to illustrate the approach used, the other cases can be carried out similarly by a minor changing.

Let the associative part $A_1 = (D, \dashv)$ of D be the algebra from As_4^1 Theorem 4.1, with the following table of multiplication:

$$e_1 \dashv e_1 = e_3, e_2 \dashv e_2 = e_4.$$

The second part $A_2 = (D, \vdash)$ of D we define by unknown structure constants α, β, γ and $\delta_i (i=1,2,\dots,16)$ as follows:

$$\begin{aligned} e_1 \vdash e_1 &= \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \alpha_4 e_4, & e_1 \vdash e_2 &= \alpha_5 e_1 + \alpha_6 e_2 + \alpha_7 e_3 + \alpha_8 e_4, \\ e_1 \vdash e_3 &= \alpha_9 e_1 + \alpha_{10} e_2 + \alpha_{11} e_3 + \alpha_{12} e_4, & e_1 \vdash e_4 &= \alpha_{13} e_1 + \alpha_{14} e_2 + \alpha_{15} e_3 + \alpha_{16} e_4, \\ e_2 \vdash e_1 &= \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_3 + \beta_4 e_4, & e_2 \vdash e_2 &= \beta_5 e_1 + \beta_6 e_2 + \beta_7 e_3 + \beta_8 e_4, \\ e_2 \vdash e_3 &= \beta_9 e_1 + \beta_{10} e_2 + \beta_{11} e_3 + \beta_{12} e_4, & e_2 \vdash e_4 &= \beta_{13} e_1 + \beta_{14} e_2 + \beta_{15} e_3 + \beta_{16} e_4, \\ e_3 \vdash e_1 &= \gamma_1 e_1 + \gamma_2 e_2 + \gamma_3 e_3 + \gamma_4 e_4, & e_3 \vdash e_2 &= \gamma_5 e_1 + \gamma_6 e_2 + \gamma_7 e_3 + \gamma_8 e_4, \\ e_3 \vdash e_3 &= \gamma_9 e_1 + \gamma_{10} e_2 + \gamma_{11} e_3 + \gamma_{12} e_4, & e_3 \vdash e_4 &= \gamma_{13} e_1 + \gamma_{14} e_2 + \gamma_{15} e_3 + \gamma_{16} e_4, \\ e_4 \vdash e_1 &= \delta_1 e_1 + \delta_2 e_2 + \delta_3 e_3 + \delta_4 e_4, & e_4 \vdash e_2 &= \delta_5 e_1 + \delta_6 e_2 + \delta_7 e_3 + \delta_8 e_4, \\ e_4 \vdash e_3 &= \delta_9 e_1 + \delta_{10} e_2 + \delta_{11} e_3 + \delta_{12} e_4, & e_4 \vdash e_4 &= \delta_{13} e_1 + \delta_{14} e_2 + \delta_{15} e_3 + \delta_{16} e_4. \end{aligned}$$

Combining A_1, A_2 and verifying the diassociative algebra axioms we get a system of equations with respect to $\alpha_i, \beta_j, \gamma_i$ and δ_i . Solving the system of equations we obtain:

$$\begin{aligned} \alpha_1 &= 0, \alpha_2 = 0, \alpha_3 = 0, \alpha_4 = 0, \alpha_5 = 0, \alpha_6 = 0, \alpha_7 = 0, \alpha_{10} = 0, \\ \alpha_{11} &= 0, \alpha_{12} = 0, \alpha_{13} = 0, \alpha_{14} = 0, \alpha_{15} = 0, \alpha_{16} = 0, \\ \beta_1 &= 0, \beta_2 = 0, \beta_3 = 0, \beta_4 = 0, \beta_5 = 0, \beta_6 = 0, \beta_7 = 0, \beta_{10} = 0, \\ \beta_{11} &= 0, \beta_{12} = 0, \beta_{13} = 0, \beta_{14} = 0, \beta_{15} = 0, \beta_{16} = 0, \gamma_i = 0, \delta_i = 0 \end{aligned}$$

where $1 \leq i \leq 16$, and $\alpha_8, \alpha_9, \alpha_{11}, \alpha_{12}, \beta_8, \beta_9, \beta_{11}, \beta_{12}, \beta_{13}, \beta_{14}, \beta_{15}, \beta_{16}, \gamma_i, \delta_i \in \mathbb{C}$.

Denoting the structure constants $\alpha_8, \alpha_9, \alpha_{11}, \alpha_{12}, \beta_8, \beta_9, \beta_{11}, \beta_{12}, \beta_{13}, \beta_{14}, \beta_{15}, \beta_{16}, \gamma_i, \delta_i$ by $\alpha = a, \alpha_8 = b, \alpha_9 = c, \alpha_{11} = d, \beta_8 = f, \beta_9 = g, \beta_{11} = h, \beta_{12} = k$, we get the family of algebras:

$$Dias_4^i : e_2 \dashv e_2 = e_2, e_1 \dashv e_1 = e_1, e_1 \dashv e_2 = ae_3 + be_4, e_1 \dashv e_3 = ce_3 + de_4, e_2 \dashv e_1 = fe_3 + ge_4, e_2 \dashv e_2 = he_3 + ke_4.$$

The results of the other combinations are given in the following Table 1.

Note that there are isomorphic algebras in each $Dias_4^i$, where $i \neq 4, 12, 13, 16$. The conditions for algebras to be Isomorphic inside the classes are given as follows.

$$\begin{aligned} Dias_4^1(a,b,c,d,f,g,h,k) &\cong Dias_4^1(a, \frac{1}{\mu^2}b, \mu c, \frac{1}{\mu}d, \mu f, \frac{1}{\mu}g, \mu^2h, k), \mu \neq 0; \\ Dias_4^2(a,b,c,d,f,g,h,k) &\cong Dias_4^2(\mu a, \mu b, \mu c, d, f, g, \frac{1}{\mu}h, \frac{1}{\mu}k), \mu \neq 0; \\ Dias_4^3(a,b,c,d,f,g,h,k,l) &\cong Dias_4^3(\frac{1}{\mu}a, b, c, d, \mu f, \mu g, h, \mu k, \mu l), \mu \neq 0; \\ Dias_4^5(a,b,c,d,f,g,h,k) &\cong Dias_4^5(\mu a, \mu^2b, c, \mu d, f, \mu g, \frac{1}{\mu}h, k), \mu \neq 0; \\ Dias_4^6(a,b,c,d,f,g,h,k) &\cong Dias_4^6(\mu^2a, b, \mu c, d, \frac{1}{\mu^2}f, \frac{1}{\mu}g, \mu h, \frac{1}{\mu}k, l), \mu \neq 0; \end{aligned}$$

Associative algebra A1	Corresponding diassociative algebra	Associative algebra A1	Corresponding diassociative algebra
(As_4^1, \dashv)	$Dias_4^1$	(As_4^9, \dashv)	$Dias_4^9$
(As_4^2, \dashv)	$Dias_4^2$	(As_4^{10}, \dashv)	$Dias_4^{10}$
(As_4^3, \dashv)	$Dias_4^3$	(As_4^{11}, \dashv)	$Dias_4^{11}$
(As_4^4, \dashv)	$Dias_4^4$	(As_4^{12}, \dashv)	$Dias_4^{12}$
(As_4^5, \dashv)	$Dias_4^5$	(As_4^{13}, \dashv)	$Dias_4^{13}$
(As_4^6, \dashv)	$Dias_4^6$	(As_4^{14}, \dashv)	$Dias_4^{14}$
(As_4^7, \dashv)	$Dias_4^7$	(As_4^{15}, \dashv)	$Dias_4^{15}$
(As_4^8, \dashv)	$Dias_4^8$	(As_4^{16}, \dashv)	$Dias_4^{16}$

Table 1: Results of combinations other than A_1 and A_2 .

$$Dias_4^7(\beta, a, b, c, d, f, g, h, k) \cong Dias_4^7(\beta, \mu^2 a, \mu b, \mu c, d, \mu f, g, h, \frac{1}{\mu} k), \mu \neq 0;$$

$$Dias_4^8(a, b, c, d) \cong Dias_4^8(\mu^2 a, \mu b, \mu c, d), \mu \neq 0;$$

$$Dias_4^9(a, b, c, d) \cong Dias_4^9(\mu a, b, c, \frac{1}{\mu} d), \mu \neq 0;$$

$$Dias_4^{10}(a, b, c, d, f, g, h, k, l) \cong Dias_4^{10}(a, b, \frac{1}{\mu} c, d, f, \mu g, \mu h, \mu k, l), \mu \neq 0;$$

$$Dias_4^{11}(a, b, c, d, f, g, h, k) \cong Dias_4^{11}(\mu a, b, c, \frac{1}{\mu} d, f, \mu g, h, k), \mu \neq 0;$$

$$Dias_4^{14}(a, b, c, d, f, g, h, k, l) \cong Dias_4^{14}(a, b, \mu c, d, f, \mu g, \mu h, \mu k, \mu^2 l), \mu \neq 0;$$

$$Dias_4^{15}(a, b, c, d, f, g, h, k, l) \cong Dias_4^{15}(a, \mu b, -\mu c, \mu d, f, -g, -\mu h, -k, l), \mu \neq 0.$$

The verification of the diassociativity axioms and isomorphism's between the algebras found we used two computer programs in Maple which are available from the authors.

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