

The Uniqueness Theorem for the Heat Equation on the Heisenberg Group

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Abstract. The aim of this paper is to establish the uniqueness theorem for the Cauchy problem for the heat equation with the Tikhonov condition on the Heisenberg group. To do this, we give Green's formula and show the existence of a Lipschitz cut-off function on the Heisenberg group in accordance with the idea in [7].

1. Introduction

It is known that the Cauchy problem for the heat equation

$$\begin{cases} \frac{\partial u_t(x)}{\partial t} = \Delta u_t(x), & \text{in } \mathbb{R}^d \times (0, T) \\ u_t(x)|_{t=0} = u_0(x) \end{cases}$$

for any continuous function u_0 has at most one bounded solution. Furthermore, if $u_t(x)$ solves the Cauchy problem with an initial value $u_0 = 0$ and satisfies the condition

$$|u_t(x)| \leq C e^{C|x|^2}$$

for some constant C and all $t > 0$, $x \in \mathbb{R}^d$, then $u \equiv 0$. This is called *the Tikhonov theorem* on the Euclidean space (see [17]).

On the other hand, the Tikhonov theorem on a Riemannian manifold was given in [6] and [7] with a complete proof. Moreover this theorem on the Carnot groups having the Heisenberg group as a typical example was given in [9] with a comment that one can obtain exactly the same proof in [7] by existence of Lipschitz cut-off function and integration by part.

Accordingly, we focus on the Heisenberg group and give a complete proof of the Tikhonov theorem for the Cauchy problem of the heat equation on the Heisenberg group,

$$\begin{cases} \frac{\partial U_s(g)}{\partial s} = \Delta_{\mathbb{H}^d} U_s(g), & \text{in } \mathbb{H}^d \times (0, S) \\ U_s(g)|_{s=0} = U_0(g), \end{cases}$$

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following the proof in [7].

The Heisenberg group is also the simplest and typical example of the sub-Riemannian manifold. Moreover it is a step 2 sub-Riemannian manifold. The sub-Riemannian manifold may be interpreted as a generalization of the Riemannian manifold. The difference is that the motion for the sub-Riemannian manifold is restricted to the horizontal direction (for the Riemannian manifold, we can measure the velocity and distance in all directions). Since it is difficult to consider the sub-Riemannian manifolds, we only consider the Heisenberg group case which has been investigated by many mathematicians.

Recently, we have studied the heat kernel method on the Heisenberg group (see [13] and [14]). The heat kernel method, introduced in [11] by T. Matsuzawa, is the method to characterize the generalized functions on the Euclidean space by the initial value of the solutions of the heat equation. The Tikhonov theorem on the Heisenberg group plays an important role to construct the heat kernel method on the Heisenberg group.

The plan of this paper is as follows: In section 2, we will recall the definition and the properties of the Heisenberg group and give the sub-Riemannian metric on the Heisenberg group. In section 3, we consider the Lipschitz functions and the gradient in the Heisenberg group. We also give Green’s formula on the Heisenberg group. Finally, in section 4, we will give the statement and the proof of the our main theorem.

2. The Heisenberg group \mathbb{H}^d

First of all, we fix some notations. We use a multi-index $\alpha \in \mathbb{Z}_+^d$, namely, $\alpha = (\alpha_1, \dots, \alpha_d)$, where $\alpha_i \in \mathbb{Z}$ and $\alpha_i \geq 0$. So, for $x \in \mathbb{R}^d$, $x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$ and $\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d}$, where $\partial_{x_j}^{\alpha_j} = (\partial/\partial x_j)^{\alpha_j}$. Moreover $\Delta = \sum_{j=1}^d \partial^2/\partial x_j^2$.

We recall the definition and the properties of the Heisenberg group. We refer to [1], [2], [4], [8], [15] and [16]. Let $g = (x, y, t)$ and $g' = (x', y', t') \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} = \mathbb{R}^{2d+1}$. Then we define the group law of \mathbb{R}^{2d+1} by

$$(x, y, t)(x', y', t') = (x + x', y + y', t + t' + 2(x' \cdot y - x \cdot y')), \tag{1}$$

where $x \cdot y = \sum_{j=1}^d x_j y_j$. The group \mathbb{R}^{2d+1} with respect to the group law defined by (1) is called the Heisenberg group and denoted by \mathbb{H}^d . Its identity element is $e = (0, 0, 0)$ and the inverse of the element (x, y, t) is $(x, y, t)^{-1} = (-x, -y, -t)$. The Heisenberg group \mathbb{H}^d is a locally compact Hausdorff group and its Haar measure is the Lebesgue measure $dx dy dt$. The left-invariant vector fields in the Heisenberg group \mathbb{H}^d as \mathbb{R}^{2d+1} are represented by

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad X_{d+j} = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t} \quad \text{and} \quad X_{2d+1} = \frac{\partial}{\partial t}$$

for $j = 1, 2, \dots, d$ and these make a basis for the Lie algebra of \mathbb{H}^d . Since their first brackets

$$[X_j, X_{d+j}] = -4X_{2d+1},$$

the induced geometry is step 2. The sub-Laplacian $\Delta_{\mathbb{H}^d}$ on \mathbb{H}^d is defined by

$$\Delta_{\mathbb{H}^d} = \sum_{j=1}^{2d} X_j^2.$$

We consider the heat operator

$$\frac{\partial}{\partial s} - \Delta_{\mathbb{H}^d}$$

on $\mathbb{H}^d \times (0, \infty)$.

Let $\lambda > 0$. Then we define the dilations δ_λ by

$$\delta_\lambda(x, y, t) = (\lambda x, \lambda y, \lambda^2 t)$$

for $(x, y, t) \in \mathbb{H}^d$. The homogeneous dimension Q of \mathbb{H}^d is given by $Q = 2d + 2$. Moreover, a function u from \mathbb{H}^d to \mathbb{C} is called the Heisenberg-homogeneous of degree $k \in \mathbb{Z}$ if $u \circ \delta_\lambda = \lambda^k u$ for $\lambda > 0$. Especially the Heisenberg-homogeneous of degree of the distance function ρ defined by $\rho(g) = ((x^2 + y^2)^2 + t^2)^{\frac{1}{4}}$ for $g = (x, y, t) \in \mathbb{H}^d$ is one, that is, $\rho(\lambda x, \lambda y, \lambda^2 t) = \lambda \rho(x, y, t)$. The following estimate also holds:

$$\rho(g'^{-1}g) \leq \rho(g) + \rho(g'). \quad (2)$$

The distance between two points g and g' in \mathbb{H}^d is given by $d_K(g, g') := \rho(g'^{-1}g)$. Especially, we denote by $d_K(g)$ the distance from the origin. This distance function ρ is called Korányi norm and the distance d_K is called Korányi distance.

The horizontal distribution is defined by $\mathcal{H}_g = \text{span}_g\{X_1, X_2, \dots, X_{2d}\}$. Then we shall consider the non-degenerate, positive definite bilinear form $\langle \cdot, \cdot \rangle : \mathcal{H}_g \times \mathcal{H}_g \rightarrow \mathbb{R}$ at any point $g \in \mathbb{H}^d$ such that $\langle X_i, X_j \rangle = \delta_{i,j}$ ($i, j = 1, 2, \dots, 2d$), where $\delta_{i,j}$ means Kronecker's delta. The length $l(\gamma)$ of the horizontal curve $\gamma(t)$, $t \in [a, b]$ is defined by

$$l(\gamma) = \int_a^b \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle dt.$$

The Carnot-Carathéodry distance $d_{CC}(g, g')$ between two points $g, g' \in \mathbb{H}^d$ is defined by the infimum of the lengths of all smooth horizontal curves joining g to g' (see [1]). These distances are bi-Lipschitz equivalent. Thus, there exists a constant $C_1 > 1$ such that

$$\frac{1}{C_1} d_{CC} \leq d_K \leq C_1 d_{CC} \quad (3)$$

(for instance, see [10]).

3. The Lipschitz functions and the gradient $\tilde{\nabla}$ in \mathbb{H}^d

Let f be a function on \mathbb{H}^d . Then we say that f is L -Lipschitz on \mathbb{H}^d if there exists a constant L such that

$$|f(g) - f(g')| \leq L d_{CC}(g, g')$$

for any $g, g' \in \mathbb{H}^d$. The constant L is called the Lipschitz constant of f . The set of all Lipschitz functions on \mathbb{H}^d is denoted by $\text{Lip}(\mathbb{H}^d)$. We define the Lipschitz semi norm $\|f\|_{\text{Lip}(\mathbb{H}^d)}$ of f by

$$\|f\|_{\text{Lip}(\mathbb{H}^d)} = \sup_{g, g' \in \mathbb{H}^d, g \neq g'} \frac{|f(g) - f(g')|}{d_{CC}(g, g')}.$$

From the definition, we can see that $\text{Lip}(\mathbb{H}^d) \subset C(\mathbb{H}^d)$ immediately.

Let the function $\eta \mapsto \eta_+$ be defined by

$$\eta_+ = \begin{cases} \eta, & \eta > 0, \\ 0, & \eta \leq 0 \end{cases}$$

for $\eta \in \mathbb{R}$.

PROPOSITION 1. *Let $d_{CC}(g, E) = \inf_{z \in E} d_{CC}(g, z)$ for any non-empty set $E \subset \mathbb{H}^d$ and any point $g \in \mathbb{H}^d$. Then the function $g \mapsto (d_{CC}(g, E) - R)_+$ for some $R > 0$ is 1-Lipschitz function on \mathbb{H}^d .*

PROOF. We prove Proposition 1 dividing it in two cases, (i) case $d_{CC}(g, E) > R$ and $d_{CC}(g', E) \leq R$ ($d_{CC}(g, E) \leq R$ and $d_{CC}(g', E) > R$), (ii) $d_{CC}(g, E) > R$ and $d_{CC}(g', E) > R$. Assume that $d_{CC}(g, E) > R$ and $d_{CC}(g', E) \leq R$. Then we can see that

$$(d_{CC}(g, E) - R)_+ - (d_{CC}(g', E) - R)_+ = d_{CC}(g, E) - R \leq d_{CC}(g, g').$$

Next we shall consider the case (ii) $d_{CC}(g, E) > R$ and $d_{CC}(g', E) > R$. For any $\varepsilon > 0$, there exists $z \in E$ such that

$$d_{CC}(g', E) \geq d_{CC}(g', z) - \varepsilon.$$

Thus, by using the triangle inequality, we obtain

$$\begin{aligned} (d_{CC}(g, E) - R)_+ - (d_{CC}(g', E) - R)_+ &= d_{CC}(g, E) - d_{CC}(g', E) \\ &\leq d_{CC}(g, z) - (d_{CC}(g', z) - \varepsilon) \\ &= d_{CC}(g, z) - d_{CC}(g', z) + \varepsilon \\ &\leq d_{CC}(g, g') + \varepsilon \end{aligned}$$

for any $\varepsilon > 0$. Therefore for any two point $g, g' \in \mathbb{H}^d$, we have

$$(d_{CC}(g, E) - R)_+ - (d_{CC}(g', E) - R)_+ \leq d_{CC}(g, g').$$

This completes the proof of Proposition 1. □

Let $d_{CC}(g) = d_{CC}(g, \mathbf{e})$. Then from Proposition 1, we can see that the function $(d_{CC}(g) - R)_+$ is 1-Lipschitz function. On the other hand, let the gradient $\tilde{\nabla}_{\mathbb{H}^d}$ on \mathbb{H}^d be defined by $\tilde{\nabla}_{\mathbb{H}^d} = (X_1, \dots, X_d, X_{d+1}, \dots, X_{2d})$.

The following Proposition 2 is known (see [3], [5] and [12]):

PROPOSITION 2. *For any L -Lipschitz function f on \mathbb{H}^d , the value $X_j f(g)$ exists for almost all $g \in \mathbb{H}^d$, $j = 1, 2, \dots, 2d$, and*

$$|Xf| = \left(\sum_{j=1}^{2d} (X_j f(g))^2 \right)^{\frac{1}{2}} \leq L \text{ a.e.}$$

This Proposition 2 implies the following Proposition 3:

PROPOSITION 3. *We have the following estimate:*

$$|\tilde{\nabla}_{\mathbb{H}^d}(d_{CC}(g) - R)_+| \leq 1.$$

As a remark, in [12], R. Monti also obtain the following result

$$|\tilde{\nabla}_{\mathbb{H}^d} d_{CC}(g)| = 1$$

for all $g = (x, y, t) \in \mathbb{H}^d$ such that $(x, y) \neq (0, 0)$.

By an easy calculation, we have the following product rule and the chain rule of the gradient $\tilde{\nabla}_{\mathbb{H}^d}$ for the C^∞ functions on \mathbb{H}^d :

$$\tilde{\nabla}_{\mathbb{H}^d}(fg) = f\tilde{\nabla}_{\mathbb{H}^d}g + g\tilde{\nabla}_{\mathbb{H}^d}f, \quad \tilde{\nabla}_{\mathbb{H}^d}\psi(f) = \psi'(f)\tilde{\nabla}_{\mathbb{H}^d}f$$

for $\psi \in C^1(\mathbb{R})$. Moreover we can express that

$$\tilde{\nabla}_{\mathbb{H}^d}f = \sum_{i=1}^{2d} X_i(f)X_i$$

for $f \in C_0^\infty(\mathbb{H}^d)$. By an integration by parts, we can obtain

$$\int_{\mathbb{H}^d} X_i(f)X_i(h)dg = - \int_{\mathbb{H}^d} fX_i^2(h)dg \tag{4}$$

for $f, h \in C_0^\infty(\mathbb{H}^d)$.

By (4), we have for $f, h \in C_0^\infty(\mathbb{H}^d)$,

$$\int_{\mathbb{H}^d} \langle \tilde{\nabla}_{\mathbb{H}^d}f, \tilde{\nabla}_{\mathbb{H}^d}h \rangle dg = \int_{\mathbb{H}^d} \left\langle \sum_{i=1}^{2d} X_i(f)X_i, \sum_{j=1}^{2d} X_j(h)X_j \right\rangle dg$$

$$\begin{aligned} &= \sum_{i=1}^{2d} \int_{\mathbb{H}^d} X_i(f)X_i(h)dg \\ &= - \sum_{i=1}^{2d} \int_{\mathbb{H}^d} f X_i^2(h)dg \\ &= - \int_{\mathbb{H}^d} f \Delta_{\mathbb{H}^d} h dg . \end{aligned}$$

PROPOSITION 4. *Let $f, h \in C_0^\infty(\mathbb{H}^d)$. Then we have the following equality:*

$$\int_{\mathbb{H}^d} f \Delta_{\mathbb{H}^d} h dg = - \int_{\mathbb{H}^d} \langle \tilde{\nabla}_{\mathbb{H}^d} f, \tilde{\nabla}_{\mathbb{H}^d} h \rangle dg .$$

4. The uniqueness theorem for the heat equation on \mathbb{H}^d

We will show the following uniqueness of the solution to the heat equation on \mathbb{H}^d .

THEOREM 1. *Let $0 < S \leq \infty$ be fixed and $U_s(g)$ be a solution to the Cauchy problem*

$$\begin{cases} \frac{\partial}{\partial s} U_s(g) = \Delta_{\mathbb{H}^d} U_s(g) , \\ U_0(g) = 0 \end{cases} \tag{5}$$

in $\mathbb{H}^d \times (0, S)$ and be a continuous function in $\mathbb{H}^d \times [0, S]$. Assume that for some $g_0 \in \mathbb{H}^d$ and for any $R > 0$, we have the following estimate:

$$\int_0^S \int_{B(g_0, R)} U_s^2(g) dg ds \leq e^{f(R)} ,$$

where $f(r)$ is a positive increasing function on $(0, +\infty)$ such that

$$\int^\infty \frac{r}{f(r)} dr = \infty .$$

Then $U \equiv 0$ in $\mathbb{H}^d \times (0, S)$.

By using the distance function $ad_{CC}(g)^2$ for some constant $a > 0$ as f , Theorem 1 immediately implies the following Corollary 1:

COROLLARY 1. *Let $U_s(g)$ be a solution to the Cauchy problem (5) in $\mathbb{H}^d \times (0, S)$ and be a continuous function in $\mathbb{H}^d \times [0, S]$ satisfying the condition: There exists a constant $C > 0$ such that*

$$|U_s(g)| \leq C e^{ad_{CC}(g)^2} , \quad (g, s) \in \mathbb{H}^d \times (0, S) \tag{6}$$

for some constant $a > 0$. Then $U \equiv 0$.

By (3), we can rewrite the condition (6) in Corollary 1 by using the Korányi distance d_K as follows: There exists a constant $C > 0$ such that

$$|U_s(g)| \leq C e^{ad_K(g)^2}$$

for some constant $a > 0$.

Let $B_R = B(g_0, R) = \{g \in \mathbb{H}^d \mid d_{CC}(g, g_0) \leq R\}$. Then to prove Theorem 1, we give the following Lemma 1.

LEMMA 1. *Let $U_s(g)$ be the solution of the heat equation in $\mathbb{H}^d \times (a, b)$. Assume that $U_s(g)$ extends to a continuous function in $\mathbb{H}^d \times [a, b]$. Assume also that, for any $R > 0$,*

$$\int_a^b \int_{B_R} U_s^2(g) dg ds \leq e^{f(R)},$$

where f is a positive increasing function on $(0, +\infty)$ such that

$$\int^\infty \frac{r}{f(r)} dr = \infty.$$

Then for any $R > 0$ satisfying the condition

$$b - a \leq \frac{R^2}{8f(4R)},$$

we have the following inequality:

$$\int_{B_R} U_b^2(g) dg \leq \int_{B_{4R}} U_a^2(g) dg + \frac{4}{R^2}.$$

By using this Lemma, Theorem 1 can be proved by the same method as the proof of Theorem 11. 9 in [7]. Therefore we will only show the proof of Lemma 1.

The proof of Lemma 1. Let $\sigma(g)$ be a Lipschitz function on \mathbb{H}^d with the Lipschitz constant 1. Then, fix a real $l \notin [a, b]$ and put

$$\xi(g, s) = -\frac{\sigma^2(g)}{4(l-s)} \quad \text{on } \mathbb{H}^d \times [a, b]. \tag{7}$$

Then we have

$$|\tilde{\nabla}_{\mathbb{H}^d} \xi(g, s)| \leq \frac{\sigma(g)}{2(l-s)}.$$

Since

$$\frac{\partial \xi}{\partial s} = -\frac{\sigma^2(g)}{4(l-s)^2},$$

we obtain

$$\frac{\partial \xi}{\partial s} + |\tilde{\nabla}_{\mathbb{H}^d} \xi|^2 \leq 0. \tag{8}$$

Now we define a function $\chi(g)$ by

$$\chi(g) = \min \left\{ \left(3 - \frac{d_{CC}(g, g_0)}{R} \right)_+, 1 \right\}$$

for a given $R > 0$ and $g_0 \in \mathbb{H}^d$. Then we can see that the function χ satisfies $0 \leq \chi \leq 1$ on \mathbb{H}^d , $\chi \equiv 1$ in B_{2R} and $\chi \equiv 0$ outside B_{3R} . Since the function $d_{CC}(g, g_0)$ is the 1-Lipschitz function, we can see that χ is the $1/R$ -Lipschitz function. By Proposition 2, we obtain

$$|\tilde{\nabla}_{\mathbb{H}^d} \chi| \leq 1/R.$$

Since all the balls in \mathbb{H}^d are relatively compact sets, the function $\chi(g)$ has a compact support. Hence for any fixed $s \in [a, b]$, the function $U \chi^2 e^\xi$ of g also has a compact support.

Now multiplying the heat equation

$$\frac{\partial U}{\partial s} = \Delta_{\mathbb{H}^d} U$$

by the function $U \chi^2 e^\xi$ and integrating it over $\mathbb{H}^d \times [a, b]$, we have

$$\int_a^b \int_{\mathbb{H}^d} \frac{\partial U}{\partial s} U \chi^2 e^\xi dg ds = \int_a^b \int_{\mathbb{H}^d} (\Delta_{\mathbb{H}^d} U) U \chi^2 e^\xi dg ds. \tag{9}$$

The integral with respect to the variable s of the left hand side in (9) becomes

$$\int_a^b \frac{\partial U}{\partial s} U \chi^2 e^\xi ds = \frac{1}{2} \int_a^b \frac{\partial (U^2)}{\partial s} \chi^2 e^\xi ds = \frac{1}{2} [U^2 \chi^2 e^\xi]_a^b - \frac{1}{2} \int_a^b \frac{\partial \xi}{\partial s} U^2 \chi^2 e^\xi ds. \tag{10}$$

Since

$$pq \leq \frac{1}{4} p^2 + q^2$$

for $p > 0$ and $q > 0$, by the product rule, the chain rule and the Schwarz inequality, we have

$$\begin{aligned} -\langle \tilde{\nabla}_{\mathbb{H}^d} U, \tilde{\nabla}_{\mathbb{H}^d} (U \chi^2 e^\xi) \rangle &= -\langle \tilde{\nabla}_{\mathbb{H}^d} U, (\tilde{\nabla}_{\mathbb{H}^d} U) \chi^2 e^\xi + U (\tilde{\nabla}_{\mathbb{H}^d} \chi^2 e^\xi) \rangle \\ &\leq -\chi^2 e^\xi |\tilde{\nabla}_{\mathbb{H}^d} U|^2 + |U| |\tilde{\nabla}_{\mathbb{H}^d} U| |\tilde{\nabla}_{\mathbb{H}^d} \xi| \chi^2 e^\xi \\ &\quad + 2 |\tilde{\nabla}_{\mathbb{H}^d} U| |\chi| |\tilde{\nabla}_{\mathbb{H}^d} \chi| |U| e^\xi \\ &\leq -\chi^2 e^\xi |\tilde{\nabla}_{\mathbb{H}^d} U|^2 + |U| |\tilde{\nabla}_{\mathbb{H}^d} U| |\tilde{\nabla}_{\mathbb{H}^d} \xi| \chi^2 e^\xi \\ &\quad + \frac{1}{2} |\tilde{\nabla}_{\mathbb{H}^d} U|^2 \chi^2 e^\xi + 2 |\tilde{\nabla}_{\mathbb{H}^d} \chi|^2 |U|^2 e^\xi \\ &= \left(-\frac{1}{2} |\tilde{\nabla}_{\mathbb{H}^d} U|^2 + |\tilde{\nabla}_{\mathbb{H}^d} U| |\tilde{\nabla}_{\mathbb{H}^d} \xi| |U| \right) \chi^2 e^\xi + 2 |\tilde{\nabla}_{\mathbb{H}^d} \chi|^2 U^2 e^\xi. \end{aligned}$$

Thus we obtain

$$\begin{aligned}
 -\langle \tilde{\nabla}_{\mathbb{H}^d} U, \tilde{\nabla}_{\mathbb{H}^d} (U \chi^2 e^\xi) \rangle &\leq \left(-\frac{1}{2} |\tilde{\nabla}_{\mathbb{H}^d} U|^2 + |\tilde{\nabla}_{\mathbb{H}^d} U| |\tilde{\nabla}_{\mathbb{H}^d} \xi| |U| \right) \chi^2 e^\xi \\
 &\quad + 2 |\tilde{\nabla}_{\mathbb{H}^d} \chi|^2 U^2 e^\xi.
 \end{aligned} \tag{11}$$

By (8), (9), (10), (11) and Proposition 4, we have

$$\begin{aligned}
 \left[\int_{\mathbb{H}^d} U^2 \chi^2 e^\xi dg \right]_a^b &= \int_a^b \int_{\mathbb{H}^d} \frac{\partial \xi}{\partial s} U^2 \chi^2 e^\xi dg ds + 2 \int_a^b \int_{\mathbb{H}^d} \frac{\partial U}{\partial s} U \chi^2 e^\xi dg ds \\
 &= \int_a^b \int_{\mathbb{H}^d} \frac{\partial \xi}{\partial s} U^2 \chi^2 e^\xi dg ds + 2 \int_a^b \int_{\mathbb{H}^d} (\Delta_{\mathbb{H}^d} U) U \chi^2 e^\xi dg ds \\
 &= \int_a^b \int_{\mathbb{H}^d} \frac{\partial \xi}{\partial s} U^2 \chi^2 e^\xi dg ds - 2 \int_a^b \int_{\mathbb{H}^d} \langle \tilde{\nabla}_{\mathbb{H}^d} U, \tilde{\nabla}_{\mathbb{H}^d} (U \chi^2 e^\xi) \rangle dg ds \\
 &\leq \int_a^b \int_{\mathbb{H}^d} \frac{\partial \xi}{\partial s} U^2 \chi^2 e^\xi dg ds \\
 &\quad + 2 \int_a^b \int_{\mathbb{H}^d} \left\{ \left(-\frac{1}{2} |\tilde{\nabla}_{\mathbb{H}^d} U|^2 + |\tilde{\nabla}_{\mathbb{H}^d} U| |\tilde{\nabla}_{\mathbb{H}^d} \xi| |U| \right) \chi^2 e^\xi + 2 |\tilde{\nabla}_{\mathbb{H}^d} \chi|^2 U^2 e^\xi \right\} dg ds \\
 &\leq \int_a^b \int_{\mathbb{H}^d} (-|\tilde{\nabla}_{\mathbb{H}^d} \xi| U^2 - |\tilde{\nabla}_{\mathbb{H}^d} U|^2 + 2 |\tilde{\nabla}_{\mathbb{H}^d} U| |\tilde{\nabla}_{\mathbb{H}^d} \xi| |U|) \chi^2 e^\xi dg ds \\
 &\quad + 4 \int_a^b \int_{\mathbb{H}^d} |\tilde{\nabla}_{\mathbb{H}^d} \chi|^2 U^2 e^\xi dg ds \\
 &= - \int_a^b \int_{\mathbb{H}^d} (|\tilde{\nabla}_{\mathbb{H}^d} \xi| |U| - |\tilde{\nabla}_{\mathbb{H}^d} U|)^2 \chi^2 e^\xi dg ds + 4 \int_a^b \int_{\mathbb{H}^d} |\tilde{\nabla}_{\mathbb{H}^d} \chi|^2 U^2 e^\xi dg ds \\
 &\leq 4 \int_a^b \int_{\mathbb{H}^d} |\tilde{\nabla}_{\mathbb{H}^d} \chi|^2 U^2 e^\xi dg ds.
 \end{aligned}$$

Hence we obtain

$$\left[\int_{\mathbb{H}^d} U^2 \chi^2 e^\xi dg \right]_a^b \leq 4 \int_a^b \int_{\mathbb{H}^d} |\tilde{\nabla}_{\mathbb{H}^d} \chi|^2 U^2 e^\xi dg ds. \tag{12}$$

Since $|\tilde{\nabla}_{\mathbb{H}^d} \chi| \leq \frac{1}{R}$, $\text{supp } |\tilde{\nabla}_{\mathbb{H}^d} \chi| \subset \text{supp } \chi = B_{3R}$ and $\chi \equiv 1$ in B_{2R} , by (12) we obtain

$$\int_{B_R} U_b^2 e^{\xi(g,a)} dg \leq \int_{B_{4R}} U_a^2 e^{\xi(g,b)} dg + \frac{4}{R^2} \int_a^b \int_{B_{4R} \setminus B_{2R}} U^2 e^\xi dg ds. \tag{13}$$

By Proposition 3, we can set $\sigma(g) = (d_{CC}(g) - R)_+$. Put $l = 2b - a \notin [a, b]$ in (7). Since

$$b - a \leq l - s \leq 2(b - a)$$

for any $s \in [a, b]$, we have

$$\xi(g, s) = -\frac{\sigma^2(g)}{4(l-s)} \leq -\frac{\sigma^2(g)}{8(b-a)} \leq 0.$$

Hence we can see that

$$\xi(g, b) = -\frac{\sigma^2(g)}{4(l-b)} = 0, \quad g \in B_R,$$

$$\xi(g, a) = -\frac{\sigma^2(g)}{4(l-a)} \leq 0, \quad g \in B_{4R}$$

and

$$\xi(g, s) \leq -\frac{\sigma^2(g)}{8(b-a)} \leq -\frac{R^2}{8(b-a)}, \quad g \in B_{4R} \setminus B_{2R}.$$

Therefore from (13), we obtain the following estimate

$$\int_{B_R} U_b^2 dg \leq \int_{B_{4R}} U_a^2 dg + \frac{4}{R^2} \int_a^b \int_{B_{4R} \setminus B_{2R}} U^2 \exp\left(-\frac{R^2}{8(b-a)}\right) dg ds. \quad (14)$$

By the assumption of Lemma 1, we have

$$\int_a^b \int_{B_{4R}} U^2 dg ds \leq e^{f(4R)} \quad \text{and} \quad b-a \leq \frac{R^2}{8f(4R)}.$$

Hence from (14) we obtain

$$\begin{aligned} \int_{B_R} U_b^2 dg &\leq \int_{B_{4R}} U_a^2 dg + \frac{4}{R^2} \exp\left(-\frac{R^2}{8(b-a)} + f(4R)\right) \\ &\leq \int_{B_{4R}} U_a^2 dg + \frac{4}{R^2} \exp(-f(4R) + f(4R)) \\ &= \int_{B_{4R}} U_a^2 dg + \frac{4}{R^2}. \end{aligned}$$

This completes the proof of Lemma 1. □

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