TOKYO J. MATH.
VOL. 39, NO. 3, 2017

The Direct Image Sheaf $f_*(O_X)$

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(Communicated by N. Suwa)

Abstract. We prove $f_*(O_X) = O_S$ for a proper flat surjective morphism $f : X \to S$ of noetherian schemes under a mild condition.

1. Introduction

Let X and S be noetherian schemes, and O_X (resp. O_S) the structure sheaf of X (resp. S). Let $f : X \to S$ be a morphism of schemes. We mean by f a pair $f = (\psi, \theta) : (X, O_X) \to (S, O_S)$ in the sense of [3, I, Def. 2.2.1] where $\psi : X \to S$ is the map of underlying topological spaces, and $\theta : O_S \to f_*(O_X)$ is the homomorphism of structure sheaves. For any morphism $T \to S$, we denote the fiber product $X \times_S T$ by X_T and the natural projection of X_T to T by f_T. Let s be a point of S, $k(s)$ the residue field of s, and $X_s = f^{-1}(s) := X \times_S \operatorname{Spec} k(s)$ the fiber of f over s.

By [3, III_1, Th. 3.2.1], $(f_T)_*(O_{X_T})$ is a coherent sheaf on T if f is a proper morphism of schemes.

The main result of this note is the following.

LEMMA 1.1. *Let $f = (\psi, \theta) : (X, O_X) \to (S, O_S)$ be a proper flat surjective morphism of noetherian schemes such that $H^0(X_s, O_{X_s}) = k(s)$ for any closed point s of S. Then the natural homomorphism $\theta : O_S \to f_*(O_X)$ is an isomorphism. Moreover the isomorphism $\theta : O_S \simeq f_*(O_X)$ commutes with base change $T \to S$, that is, for any morphism $t : T \to S$, we have a commutative diagram of natural isomorphisms*

Received February 17, 2015; revised January 26, 2016
2010 *Mathematics Subject Classification*: 14F05 (Primary), 13D02, 13H99 (Secondary)
Key words and phrases: Flatness, Direct image sheaf
The first author is partially supported by the Grant-in-Aid for Young Scientists (B) (No. 25800018) and the Grant-in-Aid for Scientific Research (S) (No. 24224001), JSPS. The second author was partially supported by the Grants-in-aid for Scientific Research (S) (No. 20340001, 24224001), JSPS.

$$\begin{array}{ccccc} \theta_T & : & O_T & \xrightarrow{\simeq} & (f_T)_*(O_{X_T}) \\ & & \uparrow\simeq & & \uparrow \\ t^*\theta & : & t^*O_S & \xrightarrow{\simeq} & t^*f_*(O_X), \end{array}$$

where f_T is the base change of f by t.

The following is a corollary of [1, Th. 7.3, p. 67] and Lemma 1.1.

COROLLARY 1.2. *Let $f : X \to S$ be a proper flat surjective morphism of noetherian schemes such that $H^0(X_s, O_{X_s}) = k(s)$ for any closed point s of S. Then the Picard functor for f is representable by an algebraic space* $\mathrm{Pic}_{X/S}$ *locally of finite presentation.*

We remark that $H^0(Y, O_Y) = k$ for any proper scheme Y over a field k that is geometrically reduced and geometrically connected over k. Lemma 1.1 is important for applications such as the above corollary. However it seems that there are no adequate literatures for Lemma 1.1, and that this is not well-known even to specialists. Note that Lemma 1.1 does not assume that S is reduced, and that it is proved by using [7, Cor. 2, p. 48] when S is reduced.

2. Faithful flatness

THEOREM 2.1 [5, Th. 2, p. 25]. *Let A be a ring and M an A-module. Then the following conditions are equivalent*:

(1) *M is faithfully A-flat*;
(2) *M is A-flat, and for any A-module $N \neq 0$ we have $N \otimes_A M \neq 0$*;
(3) *M is A-flat, and for any maximal ideal m of A, we have $mM \neq M$.*

This theorem does not assume that M is a finite A-module. Thus we can apply it to any local ring $M = B$ over a local ring A.

COROLLARY 2.2 [5, Corollary, p. 27]. *Let A and B be local rings*[1], *and $\phi : A \to B$ a homomorphism of local rings. If B is A-flat via ϕ, then B is faithfully A-flat.*

PROOF. Let p (resp. q) be the maximal ideal of A (resp. B). Since $q \supset \phi(p)$, $B \neq qB \supset \phi(p)B$. Hence by Th. 2.1, B is faithfully A-flat. □

The following is due to [3, IV_2, Cor. 2.2.8].

LEMMA 2.3. *Let $f = (\psi, \theta) : (X, O_X) \to (S, O_S)$ be a flat surjective morphism of schemes and $S =$ Spec A. Then $H^0(\theta) := H^0(S, \theta) : A = H^0(S, O_S) \to H^0(X, O_X)$ is injective.*

[1] A is not necessarily noetherian in Cor. 2.2.

PROOF. Let $J = \mathrm{Ker}\,(H^0(\theta))$. We shall prove $J = 0$ (without assuming that A is noetherian). Let p be any prime ideal of A. Since f is a flat surjective morphism, there exists an open affine subset $U = \mathrm{Spec}\ B$ of X and a prime ideal q of B such that $p = \theta^{-1}(q)$ by the natural pullback homomorphism $\theta : A \to B$. Since f is flat, B is A-flat via θ, hence the local ring B_q is A_p-flat (hence A-flat) via the localization (denoted by θ) of θ.

From the exact sequence $0 \to J \to A \to A/J \to 0$ we infer an exact sequence by the A-flatness of B_q:

$$0 \to J \otimes_A B_q \to B_q \to (A/J) \otimes_A B_q \to 0\,.$$

Since $(A/J)\otimes_A B_q \simeq B_q/H^0(\theta)(J)B_q \simeq B_q$, we have $(J\otimes_A A_p)\otimes_{A_p} B_q = J\otimes_A B_q = 0$. Since B is A-flat via $H^0(\theta)$, B_q is faithfully A_p-flat by Cor. 2.2. It follows from Th. 2.1 that $J \otimes_A A_p = 0$ for any prime ideal p of A, hence $J = 0$ by [6, Th. 4.6, p. 27]. See also [2, Prop. 3.8, p. 40]. □

3. Proof of Lemma 1.1

To prove Lemma 1.1, we may assume $S = \mathrm{Spec}\ A$. We apply the argument of Mumford [7, Cor. 3, p. 50] to $f_*(O_X)$.

By [7, Theorem, p. 44], there exists a complex

$$K^\bullet = (K^p; \partial^p : K^p \to K^{p+1})$$

of finite A-modules K^p $(p \in \mathbf{Z})$ such that

(K-i) $K^p = 0$ except for $0 \le p \le g$,
(K-ii) K^0 is A-flat, K^p is A-projective $(1 \le p \le g)$,
(K-iii) there is an isomorphism of functors

$$H^q(X \times_S \mathrm{Spec}\ B, O_X \otimes_A B) \simeq H^q(K^\bullet \otimes_A B)\,, \quad (q \ge 0)$$

on the category of A-algebras B.

Let $H^0(\theta) := H^0(S, \theta) : A = H^0(S, O_S) \to H^0(X, O_X)$. Since $H^0(\theta)$ is injective by Lemma 2.3, we have a sequence of A-modules

$$A \overset{H^0(\theta)}{\hookrightarrow} H^0(X, O_X) \overset{\phi}{\hookrightarrow} K^0 \overset{\partial^0}{\to} K^1\,, \tag{1}$$

where ϕ is the composite of the isomorphism $H^0(X, O_X) \simeq \ker(\partial^0)$ and the natural inclusion $\ker(\partial^0) \hookrightarrow K^0$. Let $f_0 := \phi H^0(\theta)(1_A) \in K^0$ for the unit 1_A of A. By tensoring (1) with B, we obtain a sequence of B-modules

$$B \xrightarrow{H^0(\theta)\otimes_A B} H^0(X, O_X) \otimes_A B \xrightarrow{\phi\otimes_A B} K^0 \otimes_A B \xrightarrow{\partial^0\otimes_A B} K^1 \otimes_A B\,.$$

In what follows, we shall prove that, for any A-algebra B, the homomorphism $\phi H^0(\theta) \otimes_A B$ induces an isomorphism

$$B \simeq \ker(\partial^0 \otimes_A B : K^0 \otimes_A B \to K^1 \otimes_A B)\,. \tag{2}$$

Let s be any closed point of S. By localizing S at s, we may assume that A is a local ring with maximal ideal p (corresponding to s).

We have a complex of $k(s)$-vector spaces:

$$A \otimes_A k(s) \overset{\phi H^0(\theta)\otimes_A k(s)}{\longrightarrow} K^0 \otimes_A k(s) \overset{\partial^0 \otimes_A k(s)}{\to} K^1 \otimes_A k(s)\,.$$

Since $H^0(X_s, O_{X_s})$ is, by the assumption, the $k(s)$-vector space consisting of constant functions with values in $k(s)$, we obtain by (K-iii)

$$0 \neq \operatorname{im}(\phi H^0(\theta) \otimes_A k(s)) \subset \ker(\partial^0 \otimes_A k(s)) \simeq H^0(X_s, O_{X_s}) = k(s)\,. \tag{3}$$

It follows from (3)

$$\ker(\partial^0 \otimes_A k(s)) = \operatorname{im}(\phi H^0(\theta) \otimes_A k(s)) = k(s)(f_0 \otimes_A 1_{k(s)})\,.$$

Since $K^p \otimes_A k(s)$ is a finite-dimensional $k(s)$-vector space, we have a $k(s)$-vector subspace $\overline{W}_2$ of $K^0 \otimes_A k(s)$, a $k(s)$-vector subspace $\overline{U}_1$ of $K^1 \otimes_A k(s)$ such that

$$\begin{aligned} K^0 \otimes_A k(s) &= k(s)(f_0 \otimes 1_{k(s)}) \oplus \overline{W}_2\,, \\ K^1 \otimes_A k(s) &\simeq \overline{U}_1 \oplus \overline{W}_2\,, \end{aligned} \tag{4}$$

and there is a commutative diagram of $k(s)$-homomorphisms

$$\begin{array}{ccc} K^0 \otimes_A k(s) & \xrightarrow{\partial^0 \otimes_A k(s)} & K^1 \otimes_A k(s) \\ \simeq \downarrow & & \downarrow \simeq \\ k(s)(f_0 \otimes 1_{k(s)}) \oplus \overline{W}_2 & \xrightarrow{0 \oplus \mathrm{id}_{\overline{W}_2}} & \overline{U}_1 \oplus \overline{W}_2\,. \end{array} \tag{5}$$

Since K^p is A-flat or A-projective by (K-ii) and A is a local ring, K^p is A-free by [5, (3.G), p. 21]. Then there exist an A-free submodule W_2 of K^0, an A-free submodule U_1 of K^1 and a commutative diagram of A-homomorphisms such that

$$W_2 \otimes_A k(s) = \overline{W}_2\,, \quad U_1 \otimes_A k(s) = \overline{U}_1\,, \tag{6}$$

$$\begin{array}{ccc} K^0 & \xrightarrow{\partial^0} & K^1 \\ \simeq \downarrow & & \downarrow \simeq \\ Af_0 \oplus W_2 & \xrightarrow{0 \oplus \mathrm{id}_{W_2}} & U_1 \oplus W_2\,, \end{array} \tag{7}$$

where $Af_0 \simeq A$ because $\phi H^0(\theta)$ in (1) is injective.

We prove it in what follows. First we choose elements e_i $(i \in I)$ of K^0 such that $e_i \otimes 1_{k(s)}$ $(i \in I)$ is a $k(s)$-basis of $\overline{W}_2$. Next let

$$W_2 := \sum_{i\in I} Ae_i\,,\ F := Af_0 + W_2\,.$$

Then F is an A-submodule of K^0 such that $(K^0/F) \otimes_A k(s) = 0$. Hence by Nakayama's lemma, we have $K^0 = F$. Moreover e_i $(i \in I)$ and f_0 is an A-free basis of K^0. Indeed, this is shown as follows. e_i $(i \in I)$ and f_0 is a minimal basis of K^0 in the sense of [6, Th. 2.3, p. 8]. Since K^0 is A-free, K^0 has an A-free basis, which is a minimal basis of K^0 by [6, Th. 2.3 (i)]. Hence e_i $(i \in I)$ and f_0 is an A-free basis by [6, Th. 2.3 (iii)].

By (4) and (5) we can find elements u_j $(j \in J)$ of K^1 that $u_j \otimes 1_{k(s)}$ $(j \in J)$ is a $k(s)$-basis of $\overline{U}_1$. Let

$$W_2^* := \sum_{i\in I} Ae_i^*\,, \quad G := W_2^* + \sum_{j\in J} Au_j$$

where $e_i^* = \partial^0 e_i$. It is clear that $W_2 \simeq W_2^*$ as A-modules. Since G is a finite A-submodule of K^1 such that $G \otimes_A k(s) = K^1 \otimes_A k(s)$, by Nakayama's lemma, we have $G = K^1$. Since K^1 is also A-free, e_i^* $(i \in I)$ and u_j $(j \in J)$ is, by [6, Th. 2.3 (i), (iii)], a minimal basis of K^1 and hence an A-free basis of K^1. This proves (6) and (7).

Now we prove (2). For any A-algebra B, we have a commutative diagram of B-homomorphisms for any A-algebra B

$$\begin{tikzcd}
K^0 \otimes_A B \arrow[r, "\partial^0 \otimes_A B"] \arrow[d, "\simeq"'] & K^1 \otimes_A B \arrow[d, "\simeq"'] \\
B(f_0 \otimes 1_B) \oplus (W_2 \otimes_A B) \arrow[r, "0 \oplus \mathrm{id}_{W_2 \otimes_A B}"] & (U_1 \otimes_A B) \oplus (W_2 \otimes_A B)
\end{tikzcd}$$

we infer the isomorphism (2)

$$\begin{aligned} H^0(X \times_S \operatorname{Spec} B, O_X \otimes_A B) &\simeq H^0(K^\bullet \otimes_A B) = \ker(\partial^0 \otimes_A B) \\ &= B(f_0 \otimes 1_B) \simeq B\,, \end{aligned}$$

because $Af_0 \otimes_A B \simeq B$ by $Af_0 \simeq A$. It also follows $H^0(X, O_X) \simeq A$.

Now we shall complete the proof of Lemma 1.1. Since $H^0(X, O_X) \simeq A$ for $S = \operatorname{Spec} A$, we have $O_S \simeq f_*(O_X)$. Let $T \to S$ be any morphism. By (2) and (K-iii), we have a natural isomorphism

$$\theta_T : O_T \simeq (f_T)_*(O_{X_T})\,.$$

It follows that the isomorphism $\theta : O_S \simeq f_*(O_X)$ commutes with base change. This completes the proof of Lemma 1.1.

References

[1] M. ARTIN, Algebraization of formal moduli: I; in "Global Analysis, papers in honor of K. Kodaira," D. C. Spencer and S. Iyanaga, eds., University of Tokyo Press and Princeton University Press, 1969, pp. 21–71.

[2] M. F. ATIYAH and I. G. MACDONALD, *Introduction to Commutative algebra*, Addison-Wesley Series in Mathematics, Westview Press, 1969.

[3] A. GROTHENDIECK, *Éléments de géométrie algébrique, I, II, III, IV*, Publ. Math. IHES **4** (1960), **8** (1961) , **11** (1961), **17** (1963), **20** (1964), **24** (1965), **28** (1966), **32** (1967).

[4] R. HARTSHORNE, *Algebraic Geometry*, Graduate Texts in Mathematics **52**, Springer Verlag, Berlin Heidelberg New York, 1977.

[5] H. MATSUMURA, *Commutative Algebra*, W. A. Benjamin Inc., New York, 1970.

[6] H. MATSUMURA, *Commutative ring theory*, Cambridge studies in advanced mathematics, **8**, Cambridge University Press, 1989.

[7] D. MUMFORD, *Abelian varieties*, Tata Institute of Fundamental Research, Hindustan Book Agency, 2012.

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