TOKYO J. MATH.
VOL. 36, NO. 1, 2013

On Convergents of Certain Values of Hyperbolic Functions Formed from Diophantine Equations

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(Communicated by M. Tsuzuki)

Abstract. Let $\xi = \sqrt{v/u}\tanh(uv)^{-1/2}$, where u and v are positive integers, and let $\eta = |h(\xi)|$, where $h(t)$ is a non-constant rational function with algebraic coefficients. We compute upper and lower bounds for the approximation of certain values η of hyperbolic functions by rationals x/y such that x and y satisfy Diophantine equations. We show that there are infinitely many coprime integers x and y such that $|y\eta - x| \ll \log\log y/\log y$ and a Diophantine equation holds simultaneously relating x and y and some integer z. Conversely, all positive integers x and y with $y \geq c_0$ solving the Diophantine equation satisfy $|y\eta - x| \gg \log\log y/\log y$.

1. Introduction and the basic theorem

$\alpha = [a_0; a_1, a_2, \dots]$ denotes the regular (or simple) continued fraction expansion of a real α, where

$$\alpha = a_0 + 1/\alpha_1\,, \quad a_0 = \lfloor \alpha \rfloor\,,$$
$$\alpha_n = a_n + 1/\alpha_{n+1}\,, \quad a_n = \lfloor \alpha_n \rfloor \quad (n \geq 1)\,.$$

Assume that the continued fraction expansion of a real ξ is quasi-periodic of the form

$$[a_0; a_1, \dots, a_n, \overline{g_1(k), \dots, g_s(k)}]_{k=1}^{\infty}$$
$$= [a_0; a_1, \dots, a_n, g_1(1), \dots, g_s(1), g_1(2), \dots, g_s(2), g_1(3), \dots]\,, \quad (1)$$

where a_0 is an integer, $a_1, \dots, a_n$ are positive integers, $g_1, \dots, g_s$ are positive integer-valued functions for $k = 1, 2, \dots$. If every $g_i(k)$ $(i = 1, 2, \dots, s)$ is a polynomial and at least one of them is not constant, (1) is called *Hurwitz continued fraction* ([16, Viertes Kapitel]). If every $g_i(k)$ $(i = 1, 2, \dots, s)$ is exponential and at least one of them is not constant, (1) is called *Tasoev continued fraction* (see e.g. [11, 17]).

We have the following properties.

Received January 27, 2012; revised April 18, 2012
Mathematics Subject Classification: 11D09, 11D25, 11J04, 11J70
Key words and phrases: hyperbolic functions, Hurwitz continued fractions, Diophantine equations

LEMMA 1. *For all positive integers n, let p_n and q_n be the n-th partial numerator and denominator of the continued fraction of ξ. We have*

1. $q_{(n-1)s+\ell} = g_\ell(n) q_{(n-1)s+\ell-1} + q_{(n-1)s+\ell-2}$ $(\ell = 1, 2, \dots, s)$.
2. *For each $\ell \in \{1, 2, \dots, s-1\}$, there exists $\varepsilon > 0$ such that*

$$\frac{1}{(g_{\ell+1}(n) + \varepsilon)\, q_{(n-1)s+\ell}^2} < \left| \xi - \frac{p_{(n-1)s+\ell}}{q_{(n-1)s+\ell}} \right| < \frac{1}{g_{\ell+1}(n) q_{(n-1)s+\ell}^2} .$$

Moreover,

$$\frac{1}{(g_1(n+1) + \varepsilon)\, q_{ns}^2} < \left| \xi - \frac{p_{ns}}{q_{ns}} \right| < \frac{1}{g_1(n+1) q_{ns}^2} .$$

PROOF. Recall that the denominator of the convergents of the continued fraction of ξ are defined recursively by

$$q_m = a_m q_{m-1} + q_{m-2}$$

where $q_{-1} = 0$, $q_0 = 1$ and a_m is the m-th partial quotient of (1). It is easily verified by induction that, for all positive integers n and for all $\ell \in \{1, 2, \dots, s\}$, we have

$$a_{(n-1)s+\ell} = g_\ell(n) ,$$

from which the first part holds.

The second part follows immediately from the fact that

$$\begin{aligned} \frac{1}{(a_{m+1} + 1/\alpha_{m+2} + q_{m-1}/q_m) q_m^2} = \left| \xi - \frac{p_m}{q_m} \right| &= \frac{1}{(\alpha_{m+1} q_m + q_{m-1}) q_m} \\ &< \frac{1}{(a_{m+1} q_m + q_{m-1}) q_m} < \frac{1}{a_{m+1} q_m^2} . \end{aligned}$$

□

Let $h : \bar{\mathbf{Q}} \to \mathbf{R}$ be a non-constant rational function which is in the class $C^1[0+\delta, 1]$, where δ is an arbitrary small positive number. In particular, we choose a rational function h such that for each $p, q (> 0) \in \mathbf{Z}$ the function h takes the form

$$h\left(\frac{p}{q}\right) = \frac{h_1(p,q)}{h_2(p,q)}$$

where $h_1, h_2 \in \mathbf{Z}[p, q]$. Assume that there is a polynomial P, whose coefficients are in $\mathbf{Z}$, such that

$$P(h_1, h_2, h_3(h_1, h_2)) = 0 . \tag{2}$$

By the mean value theorem, there exists $t \in (p/q, \xi)$ if $p/q < \xi$; $t \in (\xi, p/q)$ if $\xi < p/q$

such that

$$\left|h(\xi) - h\left(\frac{p}{q}\right)\right| = |h'(t)|\left|\xi - \frac{p}{q}\right| .$$

We apply Lemma 1 to the above equation and obtain:

LEMMA 2. *Let $n \in \mathbf{N}$.*

1. *For each $\ell = 1, 2, \ldots, s-1$, there exist $\varepsilon > 0$ and a constant t such that*

$$\frac{|h'(t)|}{(g_{\ell+1}(n) + \varepsilon)\, q^2_{(n-1)s+\ell}} < \left|h(\xi) - h\left(\frac{p_{(n-1)s+\ell}}{q_{(n-1)s+\ell}}\right)\right| < \frac{|h'(t)|}{g_{\ell+1}(n) q^2_{(n-1)s+\ell}}.$$

2. *For $\ell = s$, there exist $\varepsilon := \varepsilon(s, n) > 0$ and a constant t such that*

$$\frac{|h'(t)|}{(g_1(n+1) + \varepsilon)\, q^2_{ns}} < \left|h(\xi) - h\left(\frac{p_{ns}}{q_{ns}}\right)\right| < \frac{|h'(t)|}{g_1(n+1) q^2_{ns}} .$$

Putting the above information together, we have:

THEOREM 1. *Let $\xi \in \mathbf{R}$ whose continued fraction expansion is given by* (1). *Then,*

1. *for each $\ell = 1, 2, \ldots, s-1$, there exist $\varepsilon > 0$ and a constant t such that*

$$\frac{|h'(t)|}{(g_{\ell+1}(n) + \varepsilon)\, q^2_{(n-1)s+\ell}} < \left|h(\xi) - h\left(\frac{p_{(n-1)s+\ell}}{q_{(n-1)s+\ell}}\right)\right| < \frac{|h'(t)|}{g_{\ell+1}(n) q^2_{(n-1)s+\ell}} ;$$

2. *for $\ell = s$, there exist $\varepsilon > 0$ and a constant t such that*

$$\frac{|h'(t)|}{(g_1(n+1) + \varepsilon)\, q^2_{ns}} < \left|h(\xi) - h\left(\frac{p_{ns}}{q_{ns}}\right)\right| < \frac{|h'(t)|}{g_1(n+1) q^2_{ns}}$$

for all the points (p_m, q_m) from the m-th convergents p_m/q_m of (1) *lying on the curve* (2).

2. Hyperbolic tangent functions

Let p_n/q_n denote the n-th convergent of the number

$$\xi := \sqrt{\frac{v}{u}} \tanh\left(\frac{1}{\sqrt{uv}}\right) = [0; \overline{(4k-3)u, (4k-1)v}\,]_{k=1}^{\infty} ,$$

where u and v are positive integers.

By applying [7, Corollary 1], the following identities hold for $n \geq 2$.

$$(4n-5)q_{2n} - \big((4n-1)(4n-3)(4n-5)uv + (8n-6)\big)q_{2n-2} + (4n-1)q_{2n-4} = 0 ,$$

$$(4n-3)q_{2n+1} - \big((4n+1)(4n-1)(4n-3)uv + (8n-2)\big)q_{2n-1} + (4n+1)q_{2n-3} = 0 .$$

The same identities also hold for p_{2n}'s and p_{2n+1}'s instead of q_{2n}'s and q_{2n+1}'s. Then, for $n \geq 1$ there exist $\varepsilon_1, \varepsilon_2 > 0$ such that

$$\frac{1}{\big((4n+1)u+\varepsilon_1\big)q_{2n}^2} < \left|\xi - \frac{p_{2n}}{q_{2n}}\right| < \frac{1}{(4n+1)uq_{2n}^2}\,, \tag{3}$$

$$\frac{1}{\big((4n-1)v+\varepsilon_2\big)q_{2n-1}^2} < \left|\xi - \frac{p_{2n-1}}{q_{2n-1}}\right| < \frac{1}{(4n-1)vq_{2n-1}^2}\,. \tag{4}$$

Let $h(x)$ be a function defined in the previous section. Then we have the following, which shall be proven in the next section. The case where ξ is replaced by $e^{1/s}$ is proven in [8, Theorem 3].

LEMMA 3. *For any of* $(P_n, Q_n) = (p_{2n}, q_{2n})$ *and* $(P_n, Q_n) = (p_{2n-1}, q_{2n-1})$, *the inequalities*

$$C_1 \frac{\log\log Q_n}{Q_n^2 \log Q_n} \leq \left|h(\xi) - h\left(\frac{P_n}{Q_n}\right)\right| \leq C_2 \frac{\log\log Q_n}{Q_n^2 \log Q_n} \quad (n \geq 3)\,,$$

hold, where C_1 *and* C_2 *are effectively computable positive constants depending only on* u, v *and the function* h.

It has long been known, see e.g. [16, p. 124], that the exponential value $e^{1/s}$ has a quasi-periodic continued fraction expansion of the form $[1; s-1, 1, 1, \overline{s(2k-1)-1, 1, 1}]_{k\geq 2}$. Using this particular explicit form and the concept of leaping convergents, very good rational approximations of several numbers related to $e^{1/s}$, such as $\sinh(1/s)$, $\cosh(1/s)$ and $\tanh(1/s)$, have been obtained in 2007 by Elsner, Komatsu and Shiokawa ([8]). The authors have remarkably shown that by choosing appropriate rational functions h, such approximations can be made to involve values of rationals satisfying certain diophantine equations. In 2009, Elsner-Komatsu-Shiokawa ([9]) gave more results dealing with hyperbolic and trigonometric functions which can be approximated by rationals satisfying more diophantine equations.

Various values $h(e^{1/s})$ of hyperbolic and trigonometric functions approximated by rationals x/y such that x and y satisfy Diophantine equations are obtained in [8] and [9], based upon the ideas in [5], [6] and [12]. It is also mentioned without giving any proof in the last section of [9] that similar results would be established even if $e^{1/s}$ is replaced by $h(\tan(1/s))$. We shall show that the similar results are established even if $e^{1/s}$ is replaced by $\sqrt{v/u}\tanh(uv)^{-1/2}$. Namely, we shall show the following results.

THEOREM 2. *Let*

$$\eta := \frac{1}{2\sqrt{uv}}\left((u-v)\coth\frac{2}{\sqrt{uv}} + (u+v)\operatorname{cosech}\frac{2}{\sqrt{uv}}\right).$$

In addition, assume that $u \geq v$. *Then there are infinitely many triples* (x, y, z) *of integers* *satisfying simultaneously*

$$|y\eta - x| < C_3 \frac{\log\log y}{\log y} \quad \textit{and} \quad x^2 + y^2 = z^2 .$$

Conversely, for given integers x, y with $x^2 + y^2 = z^2$, we have the inequality

$$|y\eta - x| > C_4 \frac{\log\log y}{\log y} .$$

THEOREM 3. *Let*

$$\eta_2 := \frac{1}{2\sqrt{uv}} \left((u+v) \coth \frac{2}{\sqrt{uv}} + (u-v) \operatorname{cosech} \frac{2}{\sqrt{uv}} \right) .$$

In addition, assume that $u \geq v$. Then there are infinitely many triples (x, y, z) of integers satisfying simultaneously

$$|y\eta_2 - x| < C_5 \frac{\log\log y}{\log y} \quad \textit{and} \quad x^2 - y^2 = z^2 .$$

Conversely, for given integers x, y with $x^2 - y^2 = z^2$, we have the inequality

$$|y\eta_2 - x| > C_6 \frac{\log\log y}{\log y} .$$

THEOREM 4. *Let*

$$\eta_3 := \frac{(u-v) \cosh \frac{2}{\sqrt{uv}} + (u+v)}{(u+v) \cosh \frac{2}{\sqrt{uv}} + (u-v)} .$$

In addition, assume that $u \geq v$. Then there are infinitely many triples (x, y, z) of integers satisfying simultaneously

$$|y\eta_3 - x| < C_7 \frac{\log\log y}{\log y} \quad \textit{and} \quad y^2 - x^2 = z^2 .$$

Conversely, for given integers x, y with $y^2 - x^2 = z^2$, we have the inequality

$$|y\eta_3 - x| > C_8 \frac{\log\log y}{\log y} .$$

3. Proof of Lemma 3

We need the following lemma in order to prove Lemma 3.

LEMMA 4. *For* $n \geq 1$

$$n^{2n-1}u^n v^{n-1} < q_{2n-1} < (2n-1)^{2n-1}u^n v^{n-1}\,, \tag{5}$$

$$n^{2n}(uv)^n < q_{2n} < (2n)^{2n}(uv)^n\,. \tag{6}$$

PROOF. By [14, Corollary1], for $n = 1, 2, \dots$ we have

$$q_{2n-1} = \sum_{k=0}^{n-1} \binom{2n+2k}{4k+2}\binom{4k+2}{2k+1}\frac{(2k+1)!}{2^{2k+1}}u^{k+1}v^k,$$

$$q_{2n} = \sum_{k=0}^{n} \binom{2n+2k}{4k}\binom{4k}{2k}\frac{(2k)!}{2^{2k}}(uv)^k\,.$$

By using the recurrence relation $q_n = a_n q_{n-1} + q_{n-2}$ $(n \geq 2)$ together with $a_{2n-1} = (4n-3)u$ and $a_{2n} = (4n-1)v$ $(n \geq 1)$, we get both identities by induction. □

PROOF OF LEMMA 3. First, let $(P_n, Q_n) = (p_{2n}, q_{2n})$. By (6) for a positive constant D_1 depending only on u and v

$$\begin{aligned} 2n\log n < \log q_{2n} &< 2n\log 2n + n\log uv \\ &< D_1 n \log 2n \quad (n \geq 1)\,. \end{aligned}$$

So, for a positive constant D_2

$$\log\log q_{2n} < \log(D_1 n \log 2n) < D_2 \log n \quad (n \geq 2)\,.$$

Thus, for $n \geq 2$

$$n < \frac{\log q_{2n}}{2\log n} < \frac{D_2 \log q_{2n}}{2\log\log q_{2n}}$$

or

$$\frac{1}{n} > \frac{D_3 \log\log q_{2n}}{\log q_{2n}} \quad (D_3 := 2/D_2)\,. \tag{7}$$

Conversely,

$$\log 2n < \log(2n\log n) < \log\log q_{2n} \quad (n \geq 3)\,.$$

Hence,

$$n > \frac{\log q_{2n}}{D_1 \log 2n} > \frac{\log q_{2n}}{D_1 \log\log q_{2n}}$$

or

$$\frac{1}{n} < \frac{D_1 \log\log q_{2n}}{\log q_{2n}} \quad (n \geq 3)\,. \tag{8}$$

Now, for every positive integer n there exists a real number t satisfying simultaneously

$$\left| h(\xi) - h\left(\frac{p_{2n}}{q_{2n}}\right) \right| = |h'(t)| \left| \xi - \frac{p_{2n}}{q_{2n}} \right|$$

and

$$t_1 := \frac{p_2}{q_2} \le \frac{p_{2n}}{q_{2n}} < t < t_2 := \xi \, .$$

By the hypotheses on the function h, the choice of δ and the transcendence of ξ , the positive numbers

$$D_6 := \min_{t_1 \le t \le t_2} |h'(t)| \quad \text{and} \quad D_7 := \max_{t_1 \le t \le t_2} |h'(t)|$$

exist. By Theorem 1 with (3) we have

$$\frac{D_6}{((4n+1)u + \varepsilon_1)\, q_{2n}^2} < \left| h(\xi) - h\left(\frac{p_{2n}}{q_{2n}}\right) \right| < \frac{D_7}{(4n+1)u q_{2n}^2} \, . \tag{9}$$

Therefore, together with (7) and (8) we get

$$\frac{D_4 D_6 \log\log q_{2n}}{q_{2n}^2 \log q_{2n}} < \left| h(\xi) - h\left(\frac{p_{2n}}{q_{2n}}\right) \right| < \frac{D_5 D_7 \log\log q_{2n}}{q_{2n}^2 \log q_{2n}} \quad (n \ge 3)\, .$$

Thus, if we put $C_1 = D_4 D_6$ and $C_2 = D_5 D_7$, the proof of the lemma is completed.

Next, let $(P_n, Q_n) = (p_{2n-1}, q_{2n-1})$. By (5) together with (4) for positive constants D_4' and D_5'

$$\frac{D_4' \log\log q_{2n-1}}{q_{2n-1}^2 \log q_{2n-1}} < \left| \xi - \frac{p_{2n-1}}{q_{2n-1}} \right| < \frac{D_5' \log\log q_{2n-1}}{q_{2n-1}^2 \log q_{2n-1}} \, .$$

The rest of the parts are also similar and omitted. □

4. Proof of Theorem 2

We need an auxiliary Lemma in order to Prove Theorem 2. The case where ξ is replaced by $e^{1/s}$ is proven in [9, Lemma 2.1]. The method of proving the following Lemma is similar to the one in [9, Lemma 2.1], so the proof is omitted.

LEMMA 5. *Let $h(t) \in \bar{\mathbf{Q}}(t) \backslash \mathbf{Q}$. Then there exists a closed interval $I = [\xi - \delta, \xi + \delta]$ such that for any coprime integers p and $q (\ge 3)$ the following holds.*

$$\frac{p}{q} \in I \quad \textit{implies} \quad \left| h(\xi) - h\left(\frac{p}{q}\right) \right| > C \frac{\log\log q}{q^2 \log q} \, ,$$

where δ and C are positive constants depending only on u, v and the function h.

Let

$$h(t) := \frac{1}{2}\left(t - \frac{1}{t}\right) \quad (0 < t \le \xi < 1)\,.$$

Notice that $h(t)$ is monotonically increasing for $0 < t < 1$ and $h \in C^{(1)}(0, \infty)$. Then

$$h'(t) = \frac{1}{2} + \frac{1}{2t^2}\,, \quad 1 \le h'(t) \le \frac{1}{2}\left(1 + \frac{1}{t_1^2}\right) \quad (t_1 \le t \le t_2 < 1)\,.$$

Put $x_n := Q_n^2 - P_n^2$, $y_n := 2P_nQ_n$. Notice that $P_n < Q_n$, so that x_n and y_n are always positive. Thus, $x_n^2 + y_n^2 = (P_n^2 + Q_n^2)^2 = z_n^2$. Furthermore,

$$h(\xi) = -\eta\,, \quad h\left(\frac{P_n}{Q_n}\right) = \frac{P_n^2 - Q_n^2}{2P_nQ_n} = -\frac{x_n}{y_n}\,.$$

By $t_1Q_n^2 < P_nQ_n = y_n/2 \le t_2Q_n^2$, we get $t_1Q_n \le P_n \le t_2Q_n$. So, $Q_n \le 2P_nQ_n = y_n$, implying $\log\log Q_n \le \log\log y_n$. On the other hand, by $2t_2 \le Q_n$, we have

$$\log y_n \le \log 2t_2 + 2\log Q_n < 3\log Q_n\,,$$

yielding $\log Q_n > (1/3)\log y_n$.

Applying Lemma 3, we have

$$\begin{aligned}\left|\eta - \frac{x_n}{y_n}\right| &= \left|h(\xi) - h\left(\frac{P_n}{Q_n}\right)\right| \\ &\le C_2\frac{\log\log Q_n}{Q_n^2\log Q_n} \\ &< C_2\frac{\log\log y_n}{(y_n/2t_2)(1/3)\log y_n}\,.\end{aligned}$$

Setting $C_3 := 6t_2C_2$ and $(x, y) = (x_n, y_n)$, we get the upper bound in Theorem 2.

Conversely, by Lemma 5, there exists a closed interval $I = [\xi - \delta, \xi + \delta] \subset [0, 1]$ such that for any positive integers $p, q(\ge 3)$, $p/q \in I$ the inequality

$$\left|h(\xi) - h\left(\frac{p}{q}\right)\right| > C\frac{\log\log q}{q^2\log q}$$

holds. Let positive integers $x, y(\ge 3), z$ be given such that $x^2 + y^2 = z^2$. Since $h((0, 1)) = \mathbf{R}_{<0}$, x/y takes every positive rational number. We have

$$x = q^2 - p^2(> 0)\,, \quad y = 2pq\,, \quad z = p^2 + q^2$$

and $h(p/q) = -x/y$. If $p/q = h^{-1}(-x/y) \in I$, then

$$\left|\eta - \frac{x}{y}\right| > C\frac{\log\log q}{q^2\log q}\,.$$

If $p/q \in I = [\xi - \delta, \xi + \delta]$, then $(\xi - \delta)q < p < (\xi + \delta)q$. Since $y = 2pq > 2(\xi - \delta)q^2$ and $y < 2(\xi + \delta)q^2$, there exists a positive constant D_4 such that

$$\begin{aligned}\frac{\log\log q}{q^2 \log q} &> \frac{\log((1/2)\log(y/2(\xi+\delta)))}{(y/4(\xi-\delta))\log(y/2(\xi-\delta))} \\ &> D_4 \frac{\log\log y}{y \log y} \quad (y \geq 3)\,.\end{aligned}$$

Setting $C_4 = D_4 C$, we get the lower bound in Theorem 2.

5. Sketch of the proofs of Theorems 3 and 4

In order to prove Theorem 3, let

$$h(t) = \frac{1}{2}\left(t + \frac{1}{t}\right) \quad (0 < t < 1)\,.$$

Notice that $h(t)$ is monotonically decreasing for $0 < t < 1$ and $h \in C^{(1)}(0, \infty)$. Then

$$h(\xi) = \eta_2 \quad \text{and} \quad h\left(\frac{p}{q}\right) = \frac{p^2 + q^2}{2pq}\,.$$

Put $x = p^2 + q^2$, $y = 2pq$ and $z = q^2 - p^2$ with $0 < p < q$. Then $x > y > 0$ and $z > 0$. Since

$$0 < t_1 \leq \frac{p}{q} \leq t_2 < 1\,,$$

we get

$$2t_1 q^2 \leq y = 2pq \leq 2t_2 q^2\,,$$

yielding the desired evaluations.

In order to prove Theorem 4, let

$$h(t) = \frac{t^2 - 1}{t^2 + 1} \quad (0 < t < 1)\,.$$

Notice that $h(t)$ is monotonically increasing for $0 < t < 1$ and $h \in C^{(1)}(0, \infty)$. Then

$$h(\xi) = -\eta_3 \quad \text{and} \quad h\left(\frac{p}{q}\right) = \frac{p^2 - q^2}{p^2 + q^2}\,.$$

Put $x = q^2 - p^2$, $y = p^2 + q^2$ and $z = 2pq$ with $0 < p < q$. Then $0 < x < y$ and $z > 0$. Since

$$0 < t_1 \leq \frac{p}{q} \leq t_2 < 1\,,$$

we get

$$(t_1^2+1)q^2 \le y = 2pq \le (t_2^2+1)q^2\,,$$

yielding the desired evaluations.

6. More applications

As seen in [9, Section 5], if we apply the results to various functions connected with some suitable Diophantine equations, then we can obtain more results.

THEOREM 5. *Let u and v be integers with $u \ge v > 0$. Let*

$$\eta_4 := \frac{(u-v)\cosh\frac{2}{\sqrt{uv}} + 2\sqrt{uv}\sinh\frac{2}{\sqrt{uv}} + (u+v)}{(u-v)\cosh\frac{2}{\sqrt{uv}} - 2\sqrt{uv}\sinh\frac{2}{\sqrt{uv}} + (u+v)} \quad \text{and} \quad h(t) = \frac{t^2-2t-1}{t^2+2t-1}\,.$$

Then there are infinitely many triples (x, y, z) of integers satisfying simultaneously

$$|y\eta_4 - x| < C_9\frac{\log\log y}{\log y} \quad \text{and} \quad x^2+y^2 = 2z^2\,.$$

Conversely, for given positive integers x, $y(\ge 3)$ with $y > x$, $h^{-1}(x/y) > \sqrt{2}-1$ and $x^2+y^2 = 2z^2$, we have the inequality

$$|y\eta_4 - x| > C_{10}\frac{\log\log y}{\log y}\,.$$

REMARK. If $h(t) = (t^2+2t-1)/(t^2-2t-1)$, then

$$\eta_4 := \frac{(u-v)\cosh\frac{2}{\sqrt{uv}} - 2\sqrt{uv}\sinh\frac{2}{\sqrt{uv}} + (u+v)}{(u-v)\cosh\frac{2}{\sqrt{uv}} + 2\sqrt{uv}\sinh\frac{2}{\sqrt{uv}} + (u+v)}\,.$$

THEOREM 6. *Let u and v be integers with $u \ge v > 0$. Let*

$$\eta_5 := \frac{\bigl((u-v)\cosh\frac{2}{\sqrt{uv}} + (u+v)\bigr)^2 - 4uv\sinh^2\frac{2}{\sqrt{uv}}}{4\sqrt{uv}\sinh\frac{2}{\sqrt{uv}}\bigl((u-v)\cosh\frac{2}{\sqrt{uv}} + (u+v)\bigr)} \quad \text{and} \quad h(t) = \frac{(t^2-1)^2-4t^2}{4t(t^2-1)}\,.$$

Then there are infinitely many triples (x, y, z) of integers satisfying simultaneously

$$|y\eta_5 - x| < C_{11}\frac{\sqrt{y}\log\log y}{\log y} \quad \text{and} \quad x^2+y^2 = z^4\,.$$

Conversely, for given positive integers x, $y(\ge 3)$ with $x^2+y^2 = z^4$, we have the inequality

$$|y\eta_5 - x| > C_{12}\frac{\sqrt{y}\log\log y}{\log y}\,.$$

THEOREM 7. *Let u and v be integers with $u \ge v > 0$. Let*

$$\eta_6 := \frac{(u-v)\cosh\frac{2}{\sqrt{uv}} + (u+v)}{2\sqrt{uv}\sinh\frac{2}{\sqrt{uv}} + u\big(\cosh\frac{2}{\sqrt{uv}} + 1\big)} \quad \text{and} \quad h(t) = \frac{t^2-1}{2t+1}\,.$$

Then there are infinitely many triples (x, y, z) of integers satisfying simultaneously

$$|y\eta_6 - x| < C_{13}\frac{\log\log y}{\log y} \quad \text{and} \quad x^2 + xy + y^2 = z^2\,.$$

Conversely, for given positive integers x, $y(\ge 3)$ with $x^2 + xy + y^2 = z^2$, we have the inequality

$$|y\eta_6 - x| > C_{14}\frac{\log\log y}{\log y}\,.$$

THEOREM 8. *Let u and v be integers with $u \ge v > 0$. Let*

$$\eta_7 := \frac{\big((u+v)\cosh\frac{2}{\sqrt{uv}} + (u-v)\big)\big((u-v)\cosh\frac{2}{\sqrt{uv}} + (u+v)\big)}{4uv\sinh^2\frac{2}{\sqrt{uv}}}$$

and

$$h(t) = \frac{1}{4}\left(t^2 - \frac{1}{t^2}\right)\,.$$

Then there are infinitely many triples (x, y, z, w) of integers satisfying simultaneously

$$|y\eta_7 - x| < C_{15}\frac{\sqrt{y}\log\log y}{\log y} \quad \text{and} \quad x^2 + y^2 = z^4 - w^2\,.$$

Conversely, for given positive integers x, $y(\ge 3)$, z and w with $x^2 + y^2 = z^4 - w^2$, if we assume that $x = q^4 - p^4$, $y = 4p^2q^2$, $z = p^2 + q^2$ and $w = 2pq(q^2 - p^2)$ (p and q are positive integer with $p < q$), then we have the inequality

$$|y\eta_7 - x| > C_{16}\frac{\sqrt{y}\log\log y}{\log y}\,.$$

SKETCH OF THE PROOF OF THEOREM 5. The proof is similar to [9, Section 5]. The diophantine equation is due to [2, p.353, Corollary 6.3.14], [9, Lemma 3.2], [15, p.13]. Notice that

$$h(\xi) = \eta_4 \quad \text{and} \quad h\left(\frac{p}{q}\right) = \frac{x}{y} = \frac{p^2 - q^2 - 2pq}{p^2 - q^2 + 2pq}$$

up to an exchange of x and y, and $z = p^2 + q^2$. $h(t)$ is monotonically increasing for $0 < t < \sqrt{2}-1$ and $t > \sqrt{2}-1$, and $h \in C^{(1)}(0, \sqrt{2}-1)$, $h \in C^{(1)}(\sqrt{2}-1, \infty)$. $h(t) > 0$ for $0 < t < \sqrt{2}-1$, $h(t) < 0$ for $\sqrt{2}-1 < t < \sqrt{2}+1$. □

SKETCH OF THE PROOF OF THEOREM 6. The diophantine equation is due to [3, p.466], [4, p.256], [9, Lemma 3.1]. Notice that

$$h(\xi) = -\eta_5 \quad \text{and} \quad h\left(\frac{p}{q}\right) = -\frac{x}{y} = -\frac{p^4 - 6p^2q^2 + q^4}{4pq(q^2 - p^2)}$$

up to an exchange of x and y, and $z = p^2+q^2$. $h(t)$ is monotonically increasing for $0 < t < 1$ and $h \in C^{(1)}(0, 1)$. $h(t) < 0$ for $0 < t < \sqrt{2}-1$, $h(t) > 0$ for $\sqrt{2}-1 < t < 1$. □

SKETCH OF THE PROOF OF THEOREM 7. The diophantine equation is due to [4, p.406], [9, Lemma 3.4]. Notice that

$$h(\xi) = -\eta_6 \quad \text{and} \quad h\left(\frac{p}{q}\right) = -\frac{x}{y} = -\frac{q^2 - p^2}{2pq + q^2}$$

up to an exchange of x and y, and $z = p^2 + pq + q^2$. $h(t)$ is monotonically increasing for $t > -1/2$ and $h \in C^{(1)}(-1/2, \infty)$. $h(t) < 0$ for $0 < t < 1$. □

SKETCH OF THE PROOF OF THEOREM 8. The diophantine equation is due to [4, p.260], [9, Lemma 3.5]. Notice that

$$h(\xi) = -\eta_7 \quad \text{and} \quad h\left(\frac{p}{q}\right) = -\frac{x}{y} = -\frac{q^4 - p^4}{4p^2q^2}$$

up to an exchange of x and y, and $z = p^2+q^2$ and $w = 2pq(q^2-p^2)$. $h(t)$ is monotonically increasing for $t > 0$ and $h \in C^{(1)}(0, \infty)$. $h(t) < 0$ for $0 < t < 1$. □

7. Irrationality of numbers

Hurwitz's criterion on irrationality states that a real number α is irrational if and only if there are infinitely many rational numbers p/q, written in lowest terms, such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\sqrt{5}q^2}\,.$$

Under the view of this criterion, one cannot decide if the numbers η, η_i $(i = 2, 3, \dots, 7)$ in this paper are irrational or not. The speed of the convergence is not so rapid since

$$\frac{\log y}{\log\log y} \ll \sqrt{y} < y\,.$$

ACKNOWLEDGMENT. The second author was supported in part by the Grant-in-Aid for Scientific research (C) (No. 22540005), the Japan Society for the Promotion of Science.

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