TOKYO J. MATH.
VOL. 36, NO. 1, 2013

Isomorphism Classes and Zeta-functions of Some Nilpotent Groups

Fumitake HYODO

Waseda University

(Communicated by J. Murakami)

Abstract. In this article, we study a class of groups which are commensurable with a direct product of the discrete Heisenberg group and a free abelian group, or a free abelian group by using zeta functions of groups defined by Grunewald, Segal, and Smith as generating functions of the number of subgroups of given index n. We will show that zeta functions determine their isomorphism classes for groups belonging to the above class

1. Introduction

In this paper, we study a certain type of groups by using zeta functions of groups that were introduced in [2], by F. J. Grunewald, D. Segal, and G. C. Smith to study the subgroup growth of finitely generated groups.

Given a finitely generated group G, let $a_n(G)$ be the number of subgroups of index n, for each positive integer n. The zeta function of G is defined as the Dirichlet series associated to the sequence $\{a_n(G)\}_n$, and this function is denoted by ζ_G i.e.,

$$\zeta_G(s) := \sum_{n=1}^{\infty} a_n(G)n^{-s} = \sum_{H:[G:H]<\infty} [G : H]^{-s} .$$

For each prime p, $\zeta_{G,p}(s)$ is also defined as

$$\zeta_{G,p}(s) := \sum_{k=0}^{\infty} a_{p^k}(G)p^{-ks} .$$

EXAMPLE 1. If $G = \mathbb{Z}$, the subgroup of each index of G is unique, so that $\zeta_{\mathbb{Z}}(s)$ is equal to the Riemann zeta function denoted by $\zeta(s)$, and $\zeta_{\mathbb{Z},p}(s)$ is its Euler p-factor $\zeta_p(s)$ i.e., $(1 - p^{-s})^{-1}$.

EXAMPLE 2. **([2])** If $G = \mathbb{Z}^m$, we have

$$\zeta_{\mathbb{Z}^m}(s) = \zeta(s)\zeta(s-1)\cdots\zeta(s-m+1) .$$

Received September 26, 2011; revised March 26, 2012; revised April 16, 2012

Next example is the case where G is a non-abelian group.

EXAMPLE 3 ([4], Theorem 2.22). Define a subgroup $U_3(\mathbb{Z})$ of $GL_3(\mathbb{Z})$ as follows:

$$U_3(\mathbb{Z}) := \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; a, b, c \in \mathbb{Z} \right\},$$

and put $G = U_3(\mathbb{Z}) \times \mathbb{Z}^{m-1}$ for a positive integer m. Then we have

$$\zeta_{U_3(\mathbb{Z}) \times \mathbb{Z}^{m-1}}(s) = \frac{\zeta_{\mathbb{Z}^{m+1}}(s)\zeta(2s-m-1)\zeta(2s-m-2)}{\zeta(3s-m-2)}.$$

The above group $U_3(\mathbb{Z})$ is usually called the discrete Heisenberg group.

In general, zeta functions are expected to have good properties such as natural Euler product expansions, mermorphic extensions to the whole complex plane, and functional equations.

It was shown that if G is torsion-free, finitely generated, and nilpotent, the zeta function of G satisfies following good properties (cf. [2], [6]):

1. $\{a_n(G)\}_n$ has a polynomial growth i.e., there exists $c \in \mathbb{Z}_{>0}$ such that $a_n(G) < n^c$. Hence $\zeta_G(s)$ has a non-empty domain of convergence.
2. $\{a_n(G)\}_n$ is multiplicative, hence $\zeta_G(s) = \prod_p \zeta_{G,p}(s)$.
3. For all p, there exists a rational function $f_p(X) \in \mathbb{Q}(X)$ such that $\zeta_{G,p}(s) = f_p(p^{-s})$, hence $\zeta_{G,p}(s)$ can be continued to a meromorphic function on the whole complex plane.
4. For all but finitely many primes p, $\zeta_{G,p}(s)$ satisfies a local functional equation (cf. [6]).

Now a fundamental question that arises here is to ask, what kind of equivalence class of G is determined by the zeta function $\zeta_G(s)$. In particular, does $\zeta_G(s) = \zeta_{G'}(s)$ implies $G \cong G'$? Unfortunately, the latter question has a negative answer (cf. [1], Proposition B, or [3], Example 4), but it is known that normal zeta functions determine finitely generated, torsion free, nilpotent groups in a certain class up to commensurability (Corollary 8.3 in [2]), where normal zeta functions mean generating functions obtained by counting only normal subgroups. Recall that G and H are commensurable if and only if there exists a group which is isomorphic to a subgroup of finite index in each of G and H.

The purpose of this article is to give a positive answer to the latter question in the case G belongs to a class of finitely generated, torsion-free, and nilpotent groups to which the discrete Heisenberg group belongs and which every G in this class is commensurable with $U_3(\mathbb{Z}) \times \mathbb{Z}^{m-1}$ for some m, or a free abelian group. It is achieved by computing the explicit form of the zeta function for G belonging to our class.

The construction of this paper is as follows. In section 2, we introduce the class of groups which we study in this paper. Section 3 is a preparation to section 4, where we state the main theorems, and give a part of the proof of them. Finally, we complete this by calculating the zeta functions in section 5 and 6.

2. The group $(\mathbb{Z}^n, \mathbb{Z}^m; f)$

The group we study in this paper is given for a pair of nonnegative integers $n,\ m \in \mathbb{Z}_{\geq 0}$ and a bilinear map $f : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}^m$, and we shall denoted by $(\mathbb{Z}^n, \mathbb{Z}^m; f)$.

DEFINITION 1. Let n, m and f be as above. Then we define the group $(\mathbb{Z}^n, \mathbb{Z}^m; f)$ as follows:

1. As a set we have $(\mathbb{Z}^n, \mathbb{Z}^m; f) = \mathbb{Z}^n \times \mathbb{Z}^m$.
2. The composition of $(\mathbf{a}, \mathbf{b}), (\mathbf{a}', \mathbf{b}') \in \mathbb{Z}^n \times \mathbb{Z}^m$ is defined by

$$(\mathbf{a}, \mathbf{b})(\mathbf{a}', \mathbf{b}') := (\mathbf{a} + \mathbf{a}', \mathbf{b} + \mathbf{b}' + f(\mathbf{a}, \mathbf{a}')).$$

The associative law for this composition is easily checked using that f is a bilinear form. Note also that the identity element of this group is $(\mathbf{0}, \mathbf{0})$, and the inverse element of $(\mathbf{a}, \mathbf{b})$ is $(-\mathbf{a}, -\mathbf{b} + f(\mathbf{a}, \mathbf{a}))$.

Free abelian groups, the discrete Heisenberg group are isomorphic to $(\mathbb{Z}^n, \mathbb{Z}^m; f)$ for some n, m, f as follows.

EXAMPLE 4. If $n = 0$, then this group is isomorphic to $\mathbb{Z}^m$.

EXAMPLE 5. If $n = 2, m = 1$ and $f : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{Z}$ is given by

$$\left(\begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} a' \\ c' \end{pmatrix} \right) \mapsto ac',$$

then this group is isomorphic to the discrete Heisenberg group.

It is easy to see that the group $G = (\mathbb{Z}^n, \mathbb{Z}^m; f)$ has the following properties.

1. For every $k \in \mathbb{Z}$, $(\mathbf{a}, \mathbf{b})^k = (k\mathbf{a}, k\mathbf{b} + \frac{k(k-1)}{2} f(\mathbf{a}, \mathbf{a}))$, so this group is torsion free.
2. The commutator of $(\mathbf{a}, \mathbf{b}), (\mathbf{a}', \mathbf{b}')$ is
$$[(\mathbf{a}, \mathbf{b}), (\mathbf{a}', \mathbf{b}')] = (\mathbf{0}, f(\mathbf{a}, \mathbf{a}') - f(\mathbf{a}', \mathbf{a})).$$
Especially, we have
$$[(\mathbf{a}, \mathbf{b}), (\mathbf{0}, \mathbf{b}')] = (\mathbf{0}, \mathbf{0}).$$
3. The subgroup $N = \{(\mathbf{0}, \mathbf{b}) \mid \mathbf{b} \in \mathbb{Z}^m\}$ is contained in the center of G, and G/N is isomorphic to $\mathbb{Z}^n$. Hence G is nilpotent.
4. Let $\{\mathbf{e}_i\}_{i=1}^n,\ \{\mathbf{e}'_j\}_{j=1}^m$ be the standard basis of $\mathbb{Z}^n,\ \mathbb{Z}^m$ respectively. Then G is generated by $\{(\mathbf{e}_i, \mathbf{0})\}, \{(\mathbf{0}, \mathbf{e}'_j)\}$. Hence G is finitely generated.

Thus we see that the group $(\mathbb{Z}^n, \mathbb{Z}^m; f)$ is finitely generated, torsion free, and nilpotent of class 2 and Hirsch length $n + m$. From now on, we consider the following two problems.

1. To classify the groups $\{(\mathbb{Z}^n, \mathbb{Z}^m; f)\}$ up to isomorphism.
2. To compute explicitly the zeta function of $(\mathbb{Z}^n, \mathbb{Z}^m; f)$.

These questions are not independent. It is indeed an interesting question to ask if the zeta function of $(\mathbb{Z}^n, \mathbb{Z}^m; f)$ determines its isomorphism class.

3. Expressions by matrices

Let $M_n(\mathbb{Z}^m)$ be the set of $n \times n$ matrices whose entries are elements of the abelian group $\mathbb{Z}^m$. It has a natural additive group structure. We define actions of the rings $M_m(\mathbb{Z})$ and $M_n(\mathbb{Z})$ on this group. For

$$A = \begin{pmatrix} \mathbf{a}_{11} & \dots & \mathbf{a}_{1n} \\ \vdots & \ddots & \vdots \\ \mathbf{a}_{n1} & \dots & \mathbf{a}_{nn} \end{pmatrix} \in M_n(\mathbb{Z}^m), \quad \mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \in \mathbb{Z}^n,$$

we define the product $A\mathbf{b}$ by

$$A\mathbf{b} := \begin{pmatrix} \mathbf{a}_{11}b_1 + \mathbf{a}_{12}b_2 + \cdots + \mathbf{a}_{1n}b_n \\ \vdots \\ \mathbf{a}_{n1}b_1 + \mathbf{a}_{n2}b_2 + \cdots + \mathbf{a}_{nn}b_n \end{pmatrix}$$

the product ${}^t\mathbf{b}A$ is defined similarly, where ${}^t\mathbf{b}$ is the transpose of $\mathbf{b}$. Using these products we put, for a matrix $B = (\mathbf{b}_1, \dots, \mathbf{b}_n) \in M_n(\mathbb{Z})$,

$$AB := (A\mathbf{b}_1, \dots, A\mathbf{b}_n) \in M_n(\mathbb{Z}^m),$$

$${}^tBA := ({}^t\mathbf{b}_1 A, \dots, {}^t\mathbf{b}_n A) \in M_n(\mathbb{Z}^m).$$

In this way we regard $M_n(\mathbb{Z}^m)$ as a right and left $M_n(\mathbb{Z})$-module. Also for $B \in M_m(\mathbb{Z})$, we define:

$$BA := \begin{pmatrix} B\mathbf{a}_{11} & \dots & B\mathbf{a}_{1n} \\ \vdots & \ddots & \vdots \\ B\mathbf{a}_{n1} & \dots & B\mathbf{a}_{nn} \end{pmatrix} \in M_n(\mathbb{Z}^m).$$

Thus $M_n(\mathbb{Z}^m)$ is regarded as a left $M_m(\mathbb{Z})$-module as well.

From a bilinear map $f : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}^m$, we define the matrix A by

$$A := \begin{pmatrix} f(\mathbf{e}_1, \mathbf{e}_1) & \dots & f(\mathbf{e}_1, \mathbf{e}_n) \\ \vdots & \ddots & \vdots \\ f(\mathbf{e}_n, \mathbf{e}_1) & \dots & f(\mathbf{e}_n, \mathbf{e}_n) \end{pmatrix} \in M_n(\mathbb{Z}^m).$$

Then we have $f(\mathbf{a}, \mathbf{a}') = {}^t\mathbf{a}A\mathbf{a}'$ for every $(\mathbf{a}, \mathbf{a}') \in \mathbb{Z}^n \times \mathbb{Z}^n$.

So we shall often denote the group $(\mathbb{Z}^n, \mathbb{Z}^m; f)$ by $(\mathbb{Z}^n, \mathbb{Z}^m; A)$. With this notation Example 5 can be stated as follows:

EXAMPLE 6. If $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, then $(\mathbb{Z}^2, \mathbb{Z}^1; A)$ is isomorphic to $U_3(\mathbb{Z})$

The n-th discrete Heisenberg groups H_n (cf. [5]) is isomorphic to $(\mathbb{Z}^{2n}, \mathbb{Z}; A)$ for some $A \in M_{2n}(\mathbb{Z})$:

EXAMPLE 7. If $A = \begin{pmatrix} O & E_n \\ O & O \end{pmatrix}$, H_n is isomorphic to $(\mathbb{Z}^{2n}, \mathbb{Z}; A)$, where E_n is the identity matrix in $M_n(\mathbb{Z})$.

4. Classification of isomorphism classes of $\{(\mathbb{Z}^n, \mathbb{Z}^m; A)\}$

According to the problem in section 2, we classify the set of groups $\{(\mathbb{Z}^n, \mathbb{Z}^m; A) \mid A \in M_n(\mathbb{Z}^m)\}$ up to isomorphism. First of all the following two isomorphisms are obvious :

PROPOSITION 1.

$$\begin{array}{llcll} (1) & (\mathbb{Z}^n, \mathbb{Z}^m; {}^tXAX) & \cong & (\mathbb{Z}^n, \mathbb{Z}^m; A) & (X \in GL_n(\mathbb{Z})) \\ & (\mathbf{a}, \mathbf{b}) & \mapsto & (X\mathbf{a}, \mathbf{b}) & \end{array}$$

$$\begin{array}{llcll} (2) & (\mathbb{Z}^n, \mathbb{Z}^m; A) & \cong & (\mathbb{Z}^n, \mathbb{Z}^m; YA) & (Y \in GL_m(\mathbb{Z})) \\ & (\mathbf{a}, \mathbf{b}) & \mapsto & (\mathbf{a}, Y\mathbf{b}) & \end{array}$$

In what follows we shall fix n and m, and denote our group by $G(A) := (\mathbb{Z}^n, \mathbb{Z}^m; A)$ for $A \in M_n(\mathbb{Z}^m)$. We consider the relations of a system of generators of $G(A)$. Put $x_i = (\mathbf{e}_i, \mathbf{0})$ $(1 \le i \le n)$, $y_i = (\mathbf{0}, \mathbf{e}'_j)$ $(1 \le j \le m)$, $z_{ij}(A) = [x_i, x_j]$ $(1 \le i < j \le n)$. Then $z_{ij}(A)$'s are generated by $y_1, \ldots, y_m$. Moreover, x_i's and y_j's form a system of generators of $G(A)$, and satisfy the following relations:

1. $[x_i, x_j] = z_{ij}(A) \quad (1 \le i < j \le n)$
2. $[x_i, y_j]$ is the identity element $\quad (1 \le i \le n,\ 1 \le j \le m)$
3. $[y_i, y_j]$ is the identity element $\quad (1 \le i < j \le m)$.

Let $\tilde{G}(A)$ denote the quotient group of the free group on the words $\{x_1, \ldots, x_n, y_1, \ldots, y_m\}$ by the above relations. Then we have:

LEMMA 1. *The canonical homomorphism*

$$\tilde{G}(A) \longrightarrow G(A)$$

is an isomorphism.

PROOF. The surjectivity is trivial. By the above relations, we see that each $x \in G(A)$ is uniquely expressed in the following way

$$x = \prod_{i=1}^{n} x_i^{a_i} \prod_{j=1}^{m} y_j^{b_j}, \quad a_i,\ b_j \in \mathbb{Z}.$$

Since $\tilde{G}(A)$ has the same property, this morphism is injective. □

PROPOSITION 2. *If $A - A'$ is symmetric, then the map $(\mathbb{Z}^n, \mathbb{Z}^m; A) \to (\mathbb{Z}^n, \mathbb{Z}^m; A')$ induced by the identity map of $\mathbb{Z}^n \times \mathbb{Z}^m$ is an isomorphism.*

PROOF. By the above Lemma, it is sufficient to prove $\tilde{G}(A) = \tilde{G}(A')$. For all $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^n \times \mathbb{Z}^m$, the commutator of $\mathbf{x}$ and $\mathbf{y}$ in $(\mathbb{Z}^n, \mathbb{Z}^m; A)$ is equal to the commutator of them in $(\mathbb{Z}^n, \mathbb{Z}^m; A')$. So the relations of the generators of $\tilde{G}(A)$ and $\tilde{G}(A')$ are the same, Hence $\tilde{G}(A) = \tilde{G}(A')$. □

Now, we can state the main theorem of this paper. It gives the classification and the concrete expression of zeta functions of these groups in the case $n = 2$.

For $\alpha \in \mathbb{Z}$ we put

$$G(\alpha) = \left(\mathbb{Z}^2, \mathbb{Z}; \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}\right).$$

THEOREM 1. (1) *For each $A \in M_2(\mathbb{Z}^m)$, there exists a unique $\alpha \in \mathbb{Z}_{\geq 0}$ such that $(\mathbb{Z}^2, \mathbb{Z}^m; A)$ is isomorphic to $G(\alpha) \times \mathbb{Z}^{m-1}$.*
(2) *The Euler p-factor of $\zeta_{G(\alpha)\times\mathbb{Z}^{m-1}}(s)$ is given as*

$$\zeta_{G(\alpha)\times\mathbb{Z}^{m-1},p}(s) = \zeta_{\mathbb{Z}^{m+2},p}(s)\left(1 - p^{-\rho(s,\alpha)}\frac{\zeta_p(2s-m-1)\zeta_p(2s-m-2)}{\zeta_p(s)\zeta_p(s-1)}\right),$$

where $\rho(s, \alpha) = (s - m - 1)(v_p(\alpha) + 1)$, and $v_p(*)$ denotes the p-adic additive valuation. (We make the convention that $p^{-\rho(s,0)} \equiv 0$.)
(3) *If $\alpha = 1$, we recover the result of Example 3, i.e.,*

$$\zeta_{G(1)\times\mathbb{Z}^{m-1}}(s) = \frac{\zeta_{\mathbb{Z}^{m+1}}(s)\zeta(2s-m-1)\zeta(2s-m-2)}{\zeta(3s-m-2)},$$

REMARK 1. This theorem implies $\zeta_{G(\alpha)\times\mathbb{Z}^{m-1}}$ is *finitely uniform* in the sense of [4],1.2.4.

REMARK 2. For each $A \in M_2(\mathbb{Z}^m)$, there exists a non negative integer α such that $(\mathbb{Z}^2, \mathbb{Z}^m; A)$ is isomorphic to $G(\alpha) \times \mathbb{Z}^{m-1}$. If $\alpha \neq 0$, $G(\alpha)$ is isomorphic to the finite index subgroup of $G(1)$. This embedding is defined by $(\mathbf{e}_1, 0) \mapsto (\alpha\mathbf{e}_1, 0)$, $(\mathbf{e}_2, 0) \mapsto (\mathbf{e}_2, 0)$, $(\mathbf{0}, 1) \mapsto (\mathbf{0}, 1)$. If $\alpha = 0$, $G(\alpha)$ is isomorphic to $\mathbb{Z}^3$. Thus $(\mathbb{Z}^2, \mathbb{Z}^m; A)$ is commensurable with $U_3(\mathbb{Z}) \times \mathbb{Z}^{m-1}$ or $\mathbb{Z}^{m+2}$.

Let α, α' be distinct nonnegative integers, and put $G = G(\alpha) \times \mathbb{Z}^{m-1}$, $G' = G(\alpha') \times \mathbb{Z}^{m-1}$. By the above theorem, G is not isomorphic to G', and there exists a prime number p such that $\zeta_{G,p}(s) \neq \zeta_{G',p}(s)$. Hence we have:

THEOREM 2. *A group* $G \in \{(\mathbb{Z}^2, \mathbb{Z}^m; A) \mid A \in M_2(\mathbb{Z}^m),\ m \in \mathbb{Z}_{\geq 1}\}$ *is determined by* $\zeta_G(s)$ *up to isomorphism.*

PROOF OF THEOREM 1. We prove first the existence of α in the statement (1). It is easy to see that, for $A \in M_2(\mathbb{Z}^m)$ there exists $\mathbf{a} \in \mathbb{Z}^m$ such that $A - \begin{pmatrix} \mathbf{0} & \mathbf{a} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$ is symmetric. Then Proposition 2 implies that $(\mathbb{Z}^2, \mathbb{Z}^m; A)$ is isomorphic to $(\mathbb{Z}^2, \mathbb{Z}^m; \begin{pmatrix} \mathbf{0} & \mathbf{a} \\ \mathbf{0} & \mathbf{0} \end{pmatrix})$. Considering the actions of $GL_m(\mathbb{Z})$ described above, we see that the $GL_m(\mathbb{Z})$-orbit of $\begin{pmatrix} \mathbf{0} & \mathbf{a} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$ contains a unique element of the form $\begin{pmatrix} \mathbf{0} & \alpha\mathbf{e}'_1 \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$. (Consider the elementary transformation of the $m \times 1$ matrix $\mathbf{a}$.) So by Proposition 1 we have $(\mathbb{Z}^2, \mathbb{Z}^m; \begin{pmatrix} \mathbf{0} & \mathbf{a} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}) \cong (\mathbb{Z}^2, \mathbb{Z}^m; \begin{pmatrix} \mathbf{0} & \alpha\mathbf{e}'_1 \\ \mathbf{0} & \mathbf{0} \end{pmatrix})$. Since it is clear that $(\mathbb{Z}^2, \mathbb{Z}^m; \begin{pmatrix} \mathbf{0} & \alpha\mathbf{e}'_1 \\ \mathbf{0} & \mathbf{0} \end{pmatrix}) \cong G(\alpha) \times \mathbb{Z}^{m-1}$, it follows that $(\mathbb{Z}^2, \mathbb{Z}^m; A) \cong G(\alpha) \times \mathbb{Z}^{m-1}$. $\square$

Next, assuming the statement (2), we prove the uniqueness of α in the first statement (The proof of (2) will be given in section 5 and 6). It is sufficient to show that, if α, α' are distinct nonnegative integers then $G(\alpha) \times \mathbb{Z}^{m-1}$ and $G(\alpha') \times \mathbb{Z}^{m-1}$ are not isomorphic. The assumption $\alpha \neq \alpha'$ implies $v_p(\alpha) \neq v_p(\alpha')$ for some prime p. From the result of (2), it follows that $\zeta_{G(\alpha)\times\mathbb{Z}^{m-1},p} \neq \zeta_{G(\alpha')\times\mathbb{Z}^{m-1},p}$, which clearly implies that $G(\alpha) \times \mathbb{Z}^{m-1}$ is not isomorphic to $G(\alpha') \times \mathbb{Z}^{m-1}$.

REMARK 3. For each $n \geq 3$, we consider the problem whether zeta functions of groups determine the isomorphism classes of $\{(\mathbb{Z}^n, \mathbb{Z}^m; A) \mid A \in M_2(\mathbb{Z}^m),\ m \in \mathbb{Z}_{\geq 1}\}$. If $n = 4$, there exists a counter example (cf. [1], Proposition B, or [3], Example 4). For each $G = (\mathbb{Z}^n, \mathbb{Z}^m; A)$, let

$$B = \begin{pmatrix} A & O \\ O & O \end{pmatrix} \in M_{n+k}(\mathbb{Z}^m)\,,$$

then $G \times \mathbb{Z}^k \cong (\mathbb{Z}^{n+k}, \mathbb{Z}^m; B)$ is an element of $\{(\mathbb{Z}^{n+k}, \mathbb{Z}^m; A) \mid A \in M_2(\mathbb{Z}^m)\}$. Hence in the case $n \geq 4$, the above problem has an negative answer. Thus our remaining problem is the case $n = 3$. Our study on this problem is currently in progress.

5. Explicit form of the zeta function

We shall prove the statement (2) of Theorem 1. We can also prove this by the method in [2], section 2, but here, we will give an elementary proof. For the proof, we shall define some symbols.

Let us consider an exact sequence of groups:

$$1 \to N \to G \to \mathfrak{g} \to 1\,.$$

We choose and fix a subgroup M of N and a subgroup $\mathfrak{h}$ of $\mathfrak{g}$. Let $\mathcal{A}(M, \mathfrak{h})$ denote the set of subgroups H of G such that $H \cap N = M$, and that the image of H by the homomorphism $G \to \mathfrak{g}$ is equal to $\mathfrak{h}$, i.e., the following diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & \mathfrak{g} & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & M & \longrightarrow & H & \longrightarrow & \mathfrak{h} & \longrightarrow & 1 \end{array}$$

is commutative, and the second horizontal sequence is exact. It follows that $[G : H] = [\mathfrak{g} : \mathfrak{h}][N : M]$ for each $H \in \mathcal{A}(M, \mathfrak{h})$. Let $\mathfrak{a}(M, \mathfrak{h})$ denote the cardinality of $\mathcal{A}(M, \mathfrak{h})$. Then the zeta function of G is expressed as

$$\zeta_G(s) = \sum_{M,\mathfrak{h}} \mathfrak{a}(M, \mathfrak{h})[N : M]^{-s}[\mathfrak{g} : \mathfrak{h}]^{-s}\,,$$

where the sum is taken over all $M, \mathfrak{h}$ satisfying $[N : M] < \infty$, $[\mathfrak{g} : \mathfrak{h}] < \infty$. Similarly the local zeta function at a prime p is expressed as

$$\zeta_{G,p}(s) = \sum_{M,\mathfrak{h}} \mathfrak{a}(M, \mathfrak{h})[N : M]^{-s}[\mathfrak{g} : \mathfrak{h}]^{-s}\,,$$

where the sum is taken over all $M, \mathfrak{h}$ for which $[N : M]$, $[\mathfrak{g} : \mathfrak{h}]$ are p-th power.

We next assume that M is a normal subgroup of G, so that we have the canonical homomorphism $G/M \to \mathfrak{g}$. Let $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{h}, G/M)$ denote the set of all elements in $\mathrm{Hom}(\mathfrak{h}, G/M)$ through which the inclusion map $\mathfrak{h} \to \mathfrak{g}$ factors. Then we have:

LEMMA 2. *There is an bijective map between* $\mathcal{A}(M, \mathfrak{h})$ *and* $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{h}, G/M)$.

PROOF. For each $H \in \mathcal{A}(M, \mathfrak{h})$, the homomorphism composed by the canonical isomorphism $\mathfrak{h} \to H/M$ and the inclusion map $H/M \to G/M$ is an element of $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{h}, G/M)$. On the other hand, for each element $t \in \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{h}, G/M)$, the inverse image of $t(\mathfrak{h})$ by the canonical homomorphism $G \to G/M$ is an element of $\mathcal{A}(M, \mathfrak{h})$. Thus we can construct the two maps between $\mathcal{A}(M, \mathfrak{h})$ and $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{h}, G/M)$. It is easy to see that the two maps are inverse of each other. □

In the following, we consider the case that $\mathfrak{g}$ is isomorphic to $\mathbb{Z}^k$ for some positive integer k, and N is contained in the center of G, $\mathfrak{h}$ and M are of finite index. We remark that M is a normal subgroup of G, $[G, G] \subset N$, and $\mathfrak{h}$ is a free abelian group of rank k. Let $\{h_1, \ldots, h_k\}$ denote a basis of $\mathfrak{h}$ and $\{\tilde{h}_1, \ldots, \tilde{h}_k\}$ denote lifts of $h_1, \ldots, h_k$ to G respectively.

LEMMA 3. *A map* $t : \mathfrak{h} \to G/M$ *is an element of* $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{h}, G/M)$ *if and only if it satisfies the following properties.*

1. *The image of* $t(h_i)$ *under the canonical homomorphism* $G/M \to \mathfrak{g}$ *is equal to* h_i.
2. $t(h_1^{l_1} \cdots h_k^{l_k}) = t(h_1)^{l_1} \cdots t(h_k)^{l_k}$ *for any* $l_1, \dots, l_k \in \mathbb{Z}$.
3. $t(h_i)$ *and* $t(h_j)$ *are commutative for each* i, j.

PROOF. If $t \in \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{h}, G/M)$, it is easy to see that t satisfies the above properties. Suppose, conversely, that t satisfies the above properties. Then by properties 2 and 3, we see that $t \in \mathrm{Hom}(\mathfrak{h}, G/M)$. By the property 1, we also see that $t \in \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{h}, G/M)$. □

COROLLARY 1. *Suppose that* t *is as above, and satisfies property* 1 *and* 2. *Then* $t \in \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{h}, G/M)$ *if and only if* $[\tilde{h}_i, \tilde{h}_j] \in M$ *for all* i, j.

PROOF. Let $\hat{h}_i$, $\hat{h}_j$ denote the images of $\tilde{h}_i$, $\tilde{h}_j$ by the canonical map $G \to G/M$. Then $[\tilde{h}_i, \tilde{h}_j] \in M$ if and only if $[\hat{h}_i, \hat{h}_j] = 1$. By the above lemma, it is sufficient to prove $[\hat{h}_i, \hat{h}_j] = [t(h_i), t(h_j)]$. Since t satisfies the property 1, the images of $\hat{h}_i$ and $t(h_i)$ by $G/M \to \mathfrak{g}$ are the same. Since N/M is contained in the center of G/M, we have $[\hat{h}_i, \hat{h}_j] = [t(h_i), t(h_j)]$. □

COROLLARY 2. *The number of maps* $t : \mathfrak{h} \to G/M$ *which satisfy* 1 *and* 2 *is equal to* $[N : M]^k$.

PROOF. Let φ denote the map $G/M \to \mathfrak{g}$ induced from the map $G \to \mathfrak{g}$. Since $\{h_1, ...h_k\}$ is a basis of $\mathfrak{h}$, we have

$$\begin{aligned}
&\#\{t : \mathfrak{h} \to G/M \mid t \text{ satisfies 1 and 2 }\} \\
&= \#\prod_{i=1}^{k} \varphi^{-1}(h_i) \\
&= \prod_{i=1}^{k} \#\varphi^{-1}(h_i) \\
&= [N : M]^k .
\end{aligned}$$
□

We notice that the condition $[\tilde{h}_i, \tilde{h}_j] \in M$ is independent of t. Therefore we have:

PROPOSITION 3.

$$\begin{cases} \mathfrak{a}(M, \mathfrak{h}) = [N : M]^k , & \textit{if } [\tilde{h}_i, \tilde{h}_j] \in M \textit{ for all } i, j \\ \mathfrak{a}(M, \mathfrak{h}) = 0 , & \textit{otherwise.} \end{cases}$$

Now we apply these results to the computation of $\zeta_G(s)$.

COROLLARY 3.

$$\zeta_G(s) = \sum_{\mathfrak{a}(M,\mathfrak{h}) \neq 0} [N : M]^{k-s} [\mathfrak{g} : \mathfrak{h}]^{-s} ,$$

where the sum is extended on M, $\mathfrak{h}$ such that $[N : M]$, $[\mathfrak{g} : \mathfrak{h}]$ are finite. Similarly we have

$$\zeta_{G,p}(s) = \sum_{\mathfrak{a}(M,\mathfrak{h})\neq 0} [N : M]^{k-s}[\mathfrak{g} : \mathfrak{h}]^{-s}\,,$$

where the sum is taken on M, $\mathfrak{h}$ such that $[N : M]$, $[\mathfrak{g} : \mathfrak{h}]$ are p-th power. Especially if G is abelian, then

$$\zeta_G(s) = \zeta_{\mathfrak{g}}(s)\zeta_N(s-k)\,,$$

$$\zeta_{G,p}(s) = \zeta_{\mathfrak{g},p}(s)\zeta_{N,p}(s-k)\,,$$

and

$$a_n(G) = \sum_{lm=n} a_m(\mathfrak{g})a_l(N)l^k\,.$$

We remark that the validity of the statements in corollary 1 and proposition 3 are independent of the choice of $\{\tilde{h}_1, \dots, \tilde{h}_k\}$.

COROLLARY 4. *Suppose that $k = 2$ and $x_1, x_2 \in G$ are such that the images of x_1, x_2 to $\mathfrak{g}$ form a basis of $\mathfrak{g}$. Then in order that $\mathfrak{a}(M, \mathfrak{h}) \neq 0$, it is necessary and sufficient that $[x_1, x_2]^{[\mathfrak{g}:\mathfrak{h}]} \in M$.*

PROOF. Let $\bar{x}_1, \bar{x}_2$ be the images of x_1, x_2 respectively. Then there exist $a, b, c, d \in \mathbb{Z}$ satisfying $ad - bc > 0$, such that

$$h_1 := \bar{x}_1^a\bar{x}_2^b\,, \quad h_2 := \bar{x}_1^c\bar{x}_2^d$$

form a basis of $\mathfrak{h}$. Put $\tilde{h}_1 = x_1^a x_2^b$, $\tilde{h}_2 = x_1^c x_2^d$. Since N is contained in the center, we have $[\tilde{h}_1, \tilde{h}_2] = [x_1, x_2]^{ad-bc}$. We also note that $[\mathfrak{g} : \mathfrak{h}] = ad - bc$. Thus we have

$$[\tilde{h}_1, \tilde{h}_2] = [x_1, x_2]^{[\mathfrak{g}:\mathfrak{h}]}. \qquad \square$$

COROLLARY 5. *Let the assumptions be as above, and let $N_n = N/\langle [x_1, x_2]^n\rangle$. Then we have*

$$\zeta_G(s) = \sum_{n\geq 0} a_n(\mathfrak{g})n^{-s}\zeta_{N_n}(s-2)\,,$$

$$\zeta_{G,p}(s) = \sum_{l\geq 0} \frac{1-p^{l+1}}{1-p}p^{-ls}\zeta_{N_{p^l},p}(s-2)\,.$$

PROOF. By corollary 3, we have

$$\begin{aligned}\zeta_G(s) &= \sum_{\mathfrak{h},M:\ \mathfrak{a}(M,\mathfrak{h})\neq 0, [\mathfrak{g}:\mathfrak{h}]<\infty,\ [N:M]<\infty} [\mathfrak{g} : \mathfrak{h}]^{-s}[N : M]^{2-s} \\ &= \sum_{\mathfrak{h}:[\mathfrak{g}:\mathfrak{h}]<\infty} [\mathfrak{g} : \mathfrak{h}]^{-s} \sum_{M:\ \mathfrak{a}(M,\mathfrak{h})\neq 0, [N:M]<\infty} [N : M]^{2-s}\,.\end{aligned}$$

By Corollary 4, we also have

$$\begin{aligned}\zeta_G(s) &= \sum_{\mathfrak{h}} [\mathfrak{g} : \mathfrak{h}]^{-s} \sum_{[x_1,x_2]^{[\mathfrak{g}:\mathfrak{h}]} \in M} [N : M]^{2-s} \\ &= \sum_{n \geq 1} \sum_{[\mathfrak{g}:\mathfrak{h}]=n} n^{-s} \sum_{[x_1,x_2]^n \in M} [N : M]^{2-s} .\end{aligned}$$

Hence we obtain

$$\zeta_G(s) = \sum_{n \geq 0} a_n(\mathfrak{g}) n^{-s} \zeta_{N_n}(s-2)$$

and similarly

$$\zeta_{G,p}(s) = \sum_{l \geq 0} a_{p^l}(\mathfrak{g}) p^{-ls} \zeta_{N_{p^l},p}(s-2) .$$

By Corollary 3, we have

$$\begin{aligned}a_{p^l}(\mathfrak{g}) = a_{p^l}(\mathbb{Z}^2) &= \sum_{0 \leq i \leq l} a_{p^i}(\mathbb{Z}) p^i a_{p^{l-i}}(\mathbb{Z}) \\ &= \sum_{0 \leq i \leq l} p^i . \qquad \square\end{aligned}$$

6. Calculation of the zeta function of $G(\alpha) \times \mathbb{Z}^{m-1}$

Now, we will complete the proof of the theorem 1 by proving (2). We consider the case:

$$\begin{aligned}G &= \langle x_1, x_2, y_1, \ldots, y_m | [x_1, x_2] = y_1^{\alpha}, [x_i, y_j] = 1, [y_i, y_j] = 1 \rangle \\ &\cong G(\alpha) \times \mathbb{Z}^{m-1} , \\ N &= \langle y_1, \ldots, y_m \rangle , \quad \mathfrak{g} = G/N \cong \mathbb{Z}^2\end{aligned}$$

By Corollary 5, we only have to compute the local factor $\zeta_{N_{p^l},p}(s)$.

Since $[x_1, x_2] = y_1^{\alpha}$ and $\{y_1, \ldots, y_m\}$ is a basis of $N \cong \mathbb{Z}^m$, we see that $N_{p^l} = N/\langle [x_1, x_2]^{p^l} \rangle$ is isomorphic to $\mathbb{Z}^{m-1} \times \mathbb{Z}/\alpha p^l \mathbb{Z}$.

By Corollary 3, we have then

$$\zeta_{\mathbb{Z}^{m-1} \times \mathbb{Z}/\alpha p^l \mathbb{Z},p}(s) = \zeta_{\mathbb{Z}^{m-1},p}(s) \zeta_{\mathbb{Z}/\alpha p^l \mathbb{Z},p}(s-m+1) .$$

It is easy to see

$$\begin{aligned}\zeta_{\mathbb{Z}/\alpha p^l \mathbb{Z},p}(s-m+1) &= \zeta_{\mathbb{Z}/p^{l+v_p(\alpha)}\mathbb{Z},p}(s-m+1) \\ &= \frac{1 - p^{-(s-m+1)(1+l+v_p(\alpha))}}{1 - p^{-(s-m+1)}} ,\end{aligned}$$

where $v_p(*)$ is the p-adic additive valuation.

Next, we calculate $\zeta_{G,p}(s)$, i.e.,

$$\zeta_{\mathbb{Z}^{m-1},p}(s-2)\sum_{l\geq 0}\frac{1-p^{l+1}}{1-p}p^{-ls}\frac{1-p^{-(s-m-1)(1+l+v_p(\alpha))}}{1-p^{-(s-m-1)}}\,.$$

Put $t = s-m-1$, $\beta = 1+v_p(\alpha)$.

$$\begin{aligned}
&\sum_{l\geq 0}\frac{1-p^{l+1}}{1-p}p^{-ls}\frac{1-p^{-t(l+\beta)}}{1-p^{-t}} \\
&= \frac{1}{(1-p)(1-p^{-t})}\sum_{l\geq 0}p^{-ls}(1-p^{l+1})(1-p^{-t(l+\beta)}) \\
&= \frac{1}{(1-p)(1-p^{-t})}\sum_{l\geq 0}p^{-ls}(1-p^{l+1}-p^{-t(l+\beta)}+p^{-t(l+\beta)+l+1}) \\
&= \frac{1}{(1-p)(1-p^{-t})}\sum_{l\geq 0}(p^{-ls}-p^{1-l(s-1)}-p^{-t\beta-l(s+t)}+p^{1-t\beta-l(s+t-1)}) \\
&= \frac{1}{(1-p)(1-p^{-t})}\left\{\frac{1}{1-p^{-s}}-\frac{p}{1-p^{-(s-1)}}-\frac{p^{-t\beta}}{1-p^{-(s+t)}}+\frac{p^{-t\beta+1}}{1-p^{-(s+t-1)}}\right\} \\
&= \frac{1}{(1-p)(1-p^{-t})}\left\{\frac{1-p}{(1-p^{-s})(1-p^{-(s-1)})}-p^{-t\beta}\frac{1-p}{(1-p^{-(s+t)})(1-p^{-(s+t-1)})}\right\} \\
&= \zeta_p(t)\zeta_p(s)\zeta_p(s-1)\left(1-p^{-t\beta}\frac{\zeta_p(s+t)\zeta_p(s+t-1)}{\zeta_p(s)\zeta_p(s-1)}\right).
\end{aligned}$$

By Corollary 3, $\zeta_{\mathbb{Z}^{m-1},p}(s-2)\zeta_p(s-1)\zeta_p(s)\zeta_p(t) = \zeta_{\mathbb{Z}^{m+2},p}(s)$. Hence we have

$$\zeta_{G,p}(s) = \zeta_{\mathbb{Z}^{m+2},p}(s)\left(1-p^{-t\beta}\frac{\zeta_p(s+t)\zeta_p(s+t-1)}{\zeta_p(s)\zeta_p(s-1)}\right).$$

This completes the proof of theorem 1.

References

[1] F. J. GRUNEWALD and R. SCHARLAU, A note on finitely generated torsion-free groups of class 2. J. Algebra **58** (1979), 162–175.

[2] F. J. GRUNEWALD, D. SEGAL and G. C. SMITH, Subgroups of finite index in nilpotent groups, Invent. Math. **93** (1988), 185–223.

[3] M. P. F. DU SAUTOY and F. J. GRUNEWALD, In Proceedings of the ICM Madrid 2006 volume 2 (2006), 131–149.

[4] M. P. F. DU SAUTOY and L. WOODWARD, *Zeta functions of groups and rings*, Lecture Notes in Math, vol. 1925, Springer-Verlag, Berlin, 2008.

[5] M. STOLL, Rational and transcendental growth series for the higher Heisenberg groups, Invent. Math. **126** (1996), 1, 85–109.

[6] C. VOLL, Functional equations for zeta functions of groups and rings, Ann. of Math. (2) **172** (2010), no. 2, 1181–1218.

Present Address:
WASEDA UNIVERSITY,
OKUBO 3–4–1, SINJUKU-KU, TOKYO, JAPAN.
e-mail: hyft-269@ruri.waseda.jp