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Simple Ribbon Moves and Primeness of Knots

Dedicated to Professor Kazuaki Kobayashi on his 70th birthday

Tetsuo SHIBUYA* and Tatsuya TSUKAMOTO†

Osaka Institute of Technology

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Abstract. In [3], local moves, called *simple ribbon moves* for links are defined. In this paper, we study primeness of knots which can be transformed into the trivial knot by a single simple ribbon move.

1. Introduction

All links are assumed to be ordered and oriented, and they are considered up to ambient isotopy in an oriented 3-sphere S^3. A knot which is the connected sum of two non-trivial knots is said to be *composite*. A non-trivial knot which is not composite is said to be *prime*. It is known that any knot with unknotting number one is prime by [5]. The local moves as illustrated in Figure 1 are called the *pass move* ([1]) and the Δ-*move* ([4]). There is a non-prime knot which can be transformed into the trivial knot O by a single pass-move. The square knot is an example (see Figure 2). On the other hand, it is not known whether any knot which can be transformed into the trivial knot by a single Δ-move is prime.

In [3], local moves, called *simple ribbon moves* or *SR-moves* for links are defined. In this paper, we study primeness of knots which can be transformed into the trivial knot by a single simple ribbon move.

Let H be a 3-ball in S^3 and $\mathcal{D} = D_1 \cup \cdots \cup D_m$ (resp. $\mathcal{B} = B_1 \cup \cdots \cup B_m$) a union of mutually disjoint disks in int H (resp. H) satisfying the following:

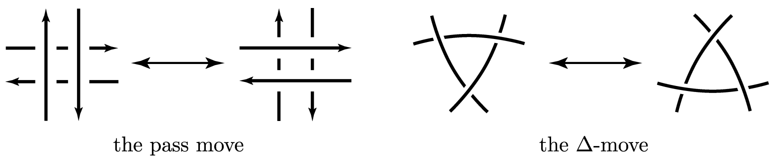

FIGURE 1

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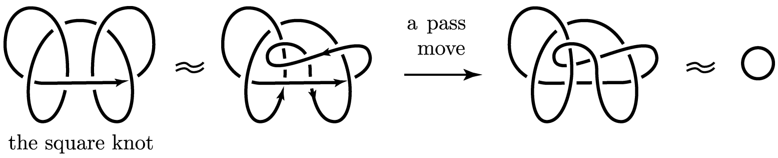


FIGURE 2

(i) $B_i \cap \partial H = \partial B_i \cap \partial H$ is an arc;
(ii) $B_i \cap \partial \mathcal{D} = \partial B_i \cap \partial D_i$ is an arc; and
(iii) $B_i \cap \operatorname{int} \mathcal{D} = B_i \cap \operatorname{int} D_{\pi(i)}$ is a single arc of ribbon type (Figure 3), where π is a certain permutation on $\{1, 2, \ldots, m\}$.

Then we call $\bigcup_i (\partial(B_i \cup D_i) - \operatorname{int}(B_i \cap \partial H))$ an *SR-tangle* and denote it by $\mathcal{T}$, and we call each B_i a *band*.

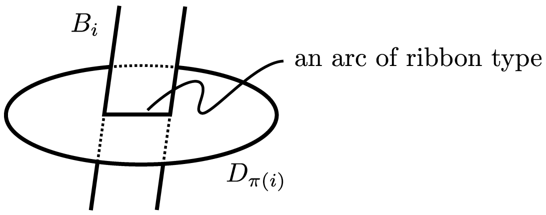


FIGURE 3

Let ℓ be a link in S^3 such that $\ell \cap H = \ell \cap \partial H$ consists of arcs. Take an SR-tangle $\mathcal{T}$ such that $\mathcal{B} \cap \partial H = \ell \cap \partial H$. Then let L be the link obtained from ℓ by substituting $\mathcal{T}$ for $\ell \cap \partial H$. We call the transformation either from ℓ to L or from L to ℓ a *simple ribbon-move* or an *SR-move*, and H (resp. $\mathcal{T}$) the *associated* 3-*ball* (resp. *tangle*) *of* the SR-move. The transformation from ℓ to L (resp. from L to ℓ) is called an *SR^+-move* (resp. *SR^--move*)(see Figure 4 for an example).

Since every permutation is a product of cyclic permutations, we rename the indices of the bands and disks as

$$\mathcal{B} = \bigcup_{k=1}^{n} \mathcal{B}^k = \bigcup_{k=1}^{n} (\bigcup_{i=1}^{m_k} B_i^k) \quad \text{and} \quad \mathcal{D} = \bigcup_{k=1}^{n} \mathcal{D}^k = \bigcup_{k=1}^{n} (\bigcup_{i=1}^{m_k} D_i^k), \text{ where}$$

(1) $1 \leq m_1 \leq m_2 \leq \cdots \leq m_n$;
(2) $B_i^k \cap \partial \mathcal{D} = \partial B_i^k \cap \partial D_i^k$ is an arc; and
(3) $B_i^k \cap \operatorname{int} \mathcal{D} = B_i^k \cap \operatorname{int} D_{i+1}^k$ is a single arc of ribbon type, where the lower indices are considered modulo m_k.

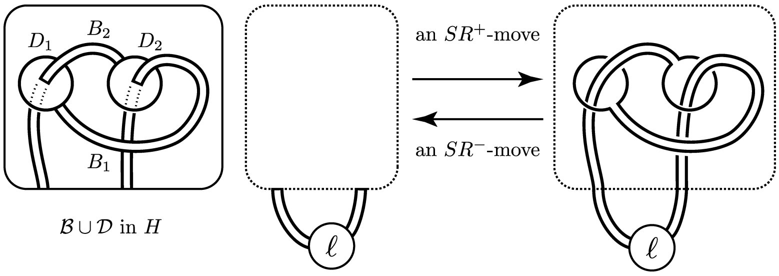


FIGURE 4

For an SR-tangle $\mathcal{T}$, we call $\bigcup_{i=1}^{m_k} (\partial(B_i^k \cup D_i^k) - \mathrm{int}(B_i^k \cap \partial H))$ the (k-th) *component* of the SR-move or of the SR-tangle, denote it by $\mathcal{T}^k$, and call m_k the *index* of the component ($k = 1, 2, \ldots, n$). The *type* of the SR-move or of the SR-tangle is the ordered set $(m_1, m_2, \ldots, m_n)$ of the indices.

Let $T_i^k = \partial(B_i^k \cup D_i^k) - \mathrm{int}\,(B_i^k \cap \partial H)$. We say that the string T_i^k of the SR-tangle is *trivial* if $T_i^k \cup (B_i^k \cap \partial H)$ bounds a non-singular disk in H whose interior is in int H and does not intersect with $\mathcal{T}$. We say that the k-th component $\mathcal{T}^k$ of the SR-tangle is *trivial* if T_i^k is trivial for any i. In fact, $\mathcal{T}^k$ is trivial if T_i^k is trivial for some i, which is easy to see. We say that an SR-tangle is *reducible* if T_i^k is trivial for a pair of i and k. Otherwise we say that the SR-tangle is *irreducible*. We say that an SR-tangle is *trivial* if T_i^k is trivial for any i and k.

Consider an SR-move transforming ℓ into L. We say that a string T_i^k of the SR-move is *trivial* if $T_i^k \cup (B_i^k \cap \partial H)$ bounds a non-singular disk in S^3 whose interior does not intersect with L . We say that the k-th component $\mathcal{T}^k$ of the SR-move is *trivial* if T_i^k is trivial for any i. We say that an SR-move is *reducible* if T_i^k is trivial for a pair of i and k. Otherwise we say that the SR-move is *irreducible*. We say that an SR-move is *trivial* if T_i^k is trivial for any i and k. Clearly any trivial SR-move does not change the link type.

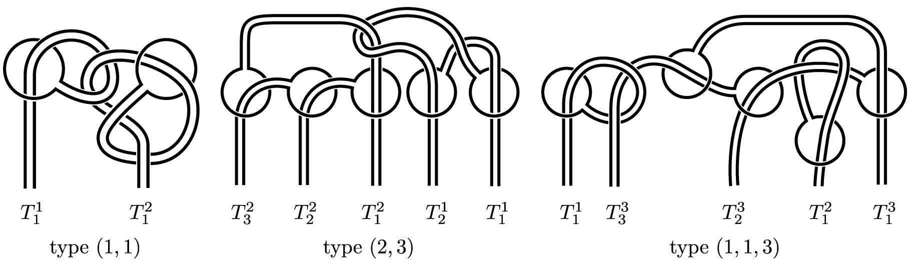
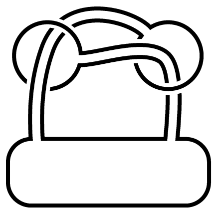


FIGURE 5

From the definitions, an SR-move is reducible (resp. trivial) if its associated tangle is reducible (resp. trivial). The opposite holds for non-split links.

PROPOSITION 1.1 ([3, Theorem 1.11]). *An SR-move on a non-split link is reducible* (*resp. trivial*) *if and only if its associated tangle is reducible* (*resp. trivial*).

It is easy to see that any SR-move of type (1) is trivial. Thus any knot which can be transformed into the trivial knot by a single SR^{-}-move of type (1) is trivial, and hence not prime. Let K be the knot as illustrated in Figure 6, which can be transformed into the trivial knot by a single SR^{-}-move of type (2). It is easily to see that K is the square knot (Figure 2), and thus K is not prime.



K

FIGURE 6

An SR-tangle is said to be *separable* if there exists a non-singular disk F properly embedded in $H - \mathcal{T}$ such that each component of $H - F$ contains a component of $\mathcal{T}$. Then the following is our main theorem.

THEOREM 1.2. *Let K be a knot in S^3 which is not the square knot. If K can be transformed into the trivial knot by a single SR^{-}-move whose associated tangle is neither type* (1) *nor separable*, *then K is prime.*

REMARK 1.3. From Corollaries 1.12, 1.15, 1.21 of [3], K in the statement is non-trivial.

The following is used in the proof of the theorem.

LEMMA 1.4 ([3, Corollary 1.20]). *If an SR-tangle is reducible, then it is separable.*

2. Proof of Theorem 1.2

Let K be a composite knot in S^3 which is not the square knot and can be transformed into the trivial knot by a single SR^{-}-move whose associated tangle is not separable. Let $\mathcal{B} \cup \mathcal{D}$ be the set of bands and disks which gives the SR^{-}-move. Since K can be transformed into the trivial knot by a single SR^{-}-move, there exist a non-singular disk $D_0 \subset (S^3 - H)$ such that ∂D_0 is a certain trivial knot and a set of bands $\mathcal{B}' = \bigcup_{k=1}^{n} (\bigcup_{i=1}^{m_k} B'^{k}_{i}) \subset (S^3 - \mathrm{int} H)$ such that each band B'^{k}_{i} satisfies that $B'^{k}_{i} \cap \partial H = \partial B'^{k}_{i} \cap \partial H$ is an arc, that $B'^{k}_{i} \cap \partial D_0 = \partial B'^{k}_{i} \cap \partial D_0$

is an arc, and that ${B'}_i^k \cap \mathrm{int}D_0$ consists of arcs of ribbon type (may be empty). Then we have a ribbon disk $\mathcal{C} = D_0 \cup (\mathcal{B} \cup \mathcal{B}') \cup \mathcal{D}$ for K. For a convenience, in the following we denote $B_i^k \cup {B'}_i^k$ by B_i^k, and $\mathcal{B} \cup \mathcal{B}'$ by $\mathcal{B}$.

Let $f_{\mathcal{C}} : D_0^* \cup \big(\cup_{i,k} D_i^{k*}\big) \cup \big(\cup_{i,k} B_i^{k*}\big) \to S^3$ be an immersion of a disk such that $f_{\mathcal{C}}(D_0^*) = D_0$, $f_{\mathcal{C}}(D_i^{k*}) = D_i^k$ and $f_{\mathcal{C}}(B_i^{k*}) = B_i^k$. We denote $\big(\cup_{i,k} D_i^{k*}\big)$ (resp. $\big(\cup_{i,k} B_i^{k*}\big)$) by $\mathcal{D}^*$ (resp. $\mathcal{B}^*$) and $D_0^* \cup \mathcal{D}^* \cup \mathcal{B}^*$ by $\mathcal{C}^*$. In the followings, we omit the upper index k unless we need to emphasize it. Denote the arc of $B_{i-1} \cap \mathrm{int}D_i$ by α_i, and the pre-image of α_i on D_i^* (resp. B_{i-1}^*) by α_i^* (resp. $\dot{\alpha}_i^*$). Denote the arc of $B_i \cap \partial H$ by $\beta_{i,0}$, and the pre-image of $\beta_{i,0}$ on B_i^* by $\beta_{i,0}^*$. Each B_i may intersect with $\mathrm{int}D_0$, and then denote the arc of $B_i \cap \mathrm{int}D_0$ by $\beta_{i,1}, \ldots, \beta_{i,t_i}$, and their pre-images on B_i^* (resp. on D_0^*) by $\beta_{i,1}^*, \ldots, \beta_{i,t_i}^*$ (resp. $\dot{\beta}_{i,1}^*, \ldots, \dot{\beta}_{i,t_i}^*$), where we assign the indices so that $\beta_{i,j}^*$ is closer to $\beta_{i,0}^*$ than $\beta_{i,l}^*$ on B_i^* if $j < l$ (see Figure 7).

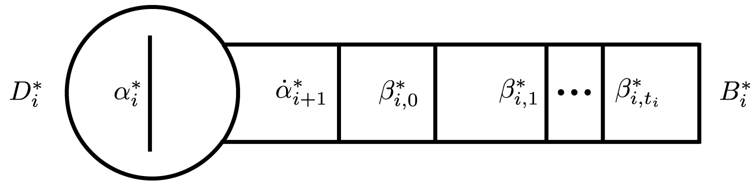


FIGURE 7

Since K is composite, there is a decomposing sphere Σ for K such that $K = k_1 \sharp k_2$. We may assume that Σ intersects with $\mathcal{C}$ and with ∂H transversely. Since Σ intersects with $K = \partial \mathcal{C}$ in two points, the pre-image $\mathcal{S}^*$ of $\Sigma \cap \mathcal{C}$ on $\mathcal{C}^*$ consists of an arc γ^* and loops, which are mutually disjoint. Let $n_{\mathcal{C}}$ be the number of such loops and n_H be the number of loops of $\Sigma \cap \partial H$. Since $\mathcal{D}$ and D_0 are in $\mathrm{int}H$ and in $S^3 - H$, respectively, a triple point of $\Sigma \cup \mathcal{C} \cup \partial H$ is made of Σ, B_i, and one from D_0, D_j, and ∂H. Let n_t be the number of the triple points and let n_d the number of intersections of Σ and $\partial \mathcal{B} \cap \partial D_0$. Define the *complexity* of Σ as the lexicographically ordered set $(n_{\mathcal{C}}, n_H, n_t, n_d)$.

PROOF OF THEOREM 1.2. Suppose that there exists a composite knot K in S^3 which is not the square knot and can be transformed into the trivial knot by a single SR^--move whose associated tangle is not separable. Take a ribbon disk $\mathcal{C}(= D_0 \cup \mathcal{D} \cup \mathcal{B})$ for K so that the number of intersections of $\mathcal{B} \cap D_0$ is minimal among such ribbon disks. Then take a decomposing sphere Σ for K with the minimal complexity.

First take a look at $\mathcal{S}^* \cap (\mathcal{D}^* \cup \mathcal{B}^*)$. Let ρ^* be a connected component of it, and $\rho = f_{\mathcal{C}}(\rho^*)$. Assume that ρ^* is on $D_i^* \cup B_i^*$.

CLAIM 2.1. *ρ^* is not a loop which bounds a disk in $D_i^* \cup B_i^* - (\alpha_i^* \cup \dot{\alpha}_{i+1}^* \cup \beta_{i,0}^* \cup \cdots \cup \beta_{i,t_i}^*)$.*

PROOF. Assume otherwise. We may assume that ρ^* is innermost on $D_i^* \cup B_i^*$, i.e., the disk δ^* which ρ^* bounds on $D_i^* \cup B_i^*$ does not contain any other loops of $\mathcal{S}^* \cap (D_i^* \cup B_i^*)$.

Then replacing a neighborhood of ρ in Σ with two parallel copies of δ, we can obtain two 2-spheres Σ_1 and Σ_2 one of which, say Σ_1, intersects with K twice. Then Σ_1 is another decomposing sphere with less complexity than that of Σ, which contradicts that Σ has the minimal complexity. □

CLAIM 2.2. *ρ^* does not have a subarc which bounds a disk on $D_i^* \cup B_i^*$ with a subarc of α_i^*, $\dot{\alpha}_{i+1}^*$, or $\beta_{i,j}^*$ whose interior does not intersect with α_i^*, $\dot{\alpha}_{i+1}^*$, or $\beta_{i,j}^*$.*

PROOF. Assume otherwise. Then there may exist several such subarcs, each of which is of ρ^* or of another connected component of $\mathcal{S}^* \cap (D_i^* \cup B_i^*)$. Take a subarc which is innermost among such subarcs, that is, it bounds a disk δ^* on $D_i^* \cup B_i^*$ with a subarc of α_i^* (resp. $\dot{\alpha}_i^*$, $\beta_{i,j}^*$) whose interior does not intersect with any other such subarcs. Here we may assume that the subarc is of ρ^*, and R_1 and R_2 are the ends of the subarc. Since δ^* does not contain any loops from Claim 2.1, we can deform Σ along δ by isotopy so to eliminate R_1 and R_2 (see Figure 8), which contradicts that Σ has the minimal complexity. □

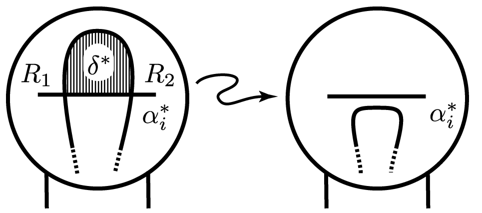


FIGURE 8

CLAIM 2.3. *ρ^* is not a loop.*

PROOF. Suppose that ρ^* is a loop. Then, there are two cases by Claims 2.1 and 2.2: ρ^* bounds a disk in D_i^* which contains α_i^* or only one end of α_i^*. Here we may assume that ρ^* is innermost on $D_i^* \cup B_i^*$, i.e., the disk δ^* which ρ^* bounds on $D_i^* \cup B_i^*$ does not contain any other loops of $\mathcal{S}^* \cap (D_i^* \cup B_i^*)$.

Consider the former case. Since δ intersects with K in two points, one of the two components of $\Sigma - \rho$ does not intersect with K. Let Σ_ρ be the closure of the component and T_i the string $\partial(D_i \cup B_i) \cap H$. Then $T_i \cup (B_i \cap \partial H)$ bounds a non-singular disk $(D_i - \delta) \cup \Sigma_\rho \cup (B_i \cap H)$ in S^3 whose interior does not intersect with K. Thus the SR-move is reducible. Therefore the associated tangle of the SR-move is separable from Proposition 1 and Lemma 1, which is a contradiction.

In the latter case, replacing a neighborhood of ρ in Σ with two parallel copies of δ, we can obtain two 2-spheres Σ_1 and Σ_2 each of which intersects with K twice. Since Σ is a decomposing sphere, either Σ_1 or Σ_2 is also a decomposing sphere, which induces a contradiction that Σ has the minimal complexity. □

From Claim 2.3, ρ^* is an arc. Now let $\xi^*_{i,1}$ be the subarc of $\partial(D^*_i \cup B^*_i) - \partial D^*_0$ such that $\partial \xi^*_{i,1} = \partial \dot{\alpha}^*_{i+1}$ and $\xi^*_{i,2}$ the arc $\partial B^*_i \cap \partial D^*_0$. Let $\xi^*_{i,3}$ be one of the two arcs of $\partial(D^*_i \cup B^*_i) - \mathrm{int}(\xi^*_{i,1} \cup \xi^*_{i,2})$ and $\xi^*_{i,4}$ the other arc (Figure 9). Here we may assume that ρ^* does not have an end on any of $\partial\dot{\alpha}^*_{i+1}$, $\partial\beta^*_{i,0}, \ldots, \partial\beta^*_{i,t_i}$, and $\partial\xi^*_{i,2}$.

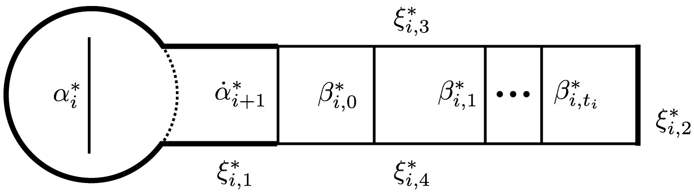


FIGURE 9

CLAIM 2.4. *The ends of ρ^* are on $\xi^*_{i,1} \cup \xi^*_{i,2}$.*

PROOF. Assume otherwise. Then ρ^* has an end p on $\xi^*_{i,3}$ or $\xi^*_{i,4}$. It is sufficient to consider the former case from the symmetry. Let Δ^* be the closure of the component of $B^*_i - (\dot{\alpha}^*_{i+1} \cup \beta^*_{i,0} \cup \cdots \cup \beta^*_{i,t_i} \cup \xi^*_{i,2})$ which contains p. Then we have that ρ^* is in Δ^* or not. If ρ^* is in Δ^*, then we have two cases that the other end of ρ^* than p is on $\xi^*_{i,3}$ or on $\xi^*_{i,4}$.

In the former case, ρ^* bounds a disk δ^* in Δ^* with a subarc of $\xi^*_{i,3}$. Here note that δ^* does not contain any other components of $\mathcal{S}^* \cap (D^*_i \cup B^*_i)$ from Claim 2.1 and that Σ intersects with K in two points. Then $\partial\delta - \rho$ is one of the two components of $K - \Sigma$ and trivial, since δ is an embedded disk in the closure of a component of $S^3 - \Sigma$. Thus it contradicts that Σ is a decomposing sphere for K. In the latter case, let δ^* be the closure of the component of $(D^{k*}_i \cup B^{k*}_i) - \rho^*$ which contains D^{k*}_i. From Claim 2.3 and that Σ intersects with K in two points, we have that $\mathrm{int}\delta^* \cap \mathcal{S}^* = \emptyset$, and thus $\partial\delta - \mathrm{int}\rho$ is the arc of $K \cap \Omega$, where Ω is the closure of the component of $S^3 - \Sigma$ containing δ. Then $\partial\alpha^k_i$ is on $\partial\delta - \mathrm{int}\rho$, since $\partial\alpha^k_i = \mathrm{int}\delta \cap K$. Therefore we have that $\alpha^k_i = \alpha^k_{i+1}$, which tells us that $m_k = 1$. Then we may consider $\partial\delta - \mathrm{int}\rho$ as an SR-tangle of type (1) in Ω, and thus it is trivial. However this contradicts that Σ is a decomposing sphere.

If ρ^* is not in Δ^*, then let q be the point of $\rho^* \cap (\partial\Delta^* - (\xi^*_{i,3} \cup \xi^*_{i,4}))$ such that the interior of the subarc ρ^*_{pq} of ρ^* bounded by p and q does not intersect with $\partial\Delta^* - (\xi^*_{i,3} \cup \xi^*_{i,4})$. Let ζ be the one of $\dot{\alpha}^*_{i+1}$, $\beta^*_{i,0}, \ldots, \beta^*_{i,t_i}$, and $\xi^*_{i,2}$ which contains q. Let s be the point $\zeta \cap \xi^*_{i,3}$, and let ξ^*_{ps} (resp. ζ_{qs}) the subarc of $\xi^*_{i,3}$ (resp. ζ) bounded by p (resp. q) and s. Then ρ^*_{pq}, ξ^*_{ps}, and ζ_{qs} bound a disk δ^*. If $\mathrm{int}\delta^* \cap \mathcal{S}^* = \emptyset$, then we can deform Σ along δ by isotopy so to reduce the complexity of Σ as illustrated in Figure 10, which is a contradiction. If $\mathrm{int}\delta^* \cap \mathcal{S}^* \neq \emptyset$, then $\delta^* \cap \mathcal{S}^*$ consists of ρ^*_{pq} and a subarc of an arc which has an end on the interior of ξ^*_{ps} and intersects with the interior of ζ_{qs} from Claims 2.2 and 2.3 and that Σ intersects with K

in two points. In this case, we can reduce the complexity of Σ by 2 using the deformation as illustrated in Figure 10 twice, which is also a contradiction. $\square$

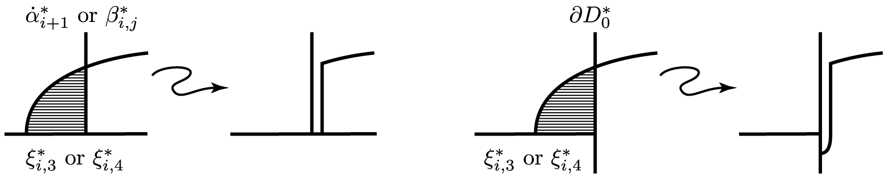


FIGURE 10

CLAIM 2.5. *$\partial\rho^*$ is not contained in $\xi^*_{i,1}$.*

PROOF. Assume otherwise. Then ρ^* bounds a disk δ^* with a subarc μ^* of $\xi^*_{i,1}$ in the subdisk of $D^*_i \cup B^*_i$ bounded by $\xi^*_{i,1}$ and $\dot{\alpha}^*_{i+1}$ from Claim 2.2. From Claim 2.3 and that Σ intersects with K in two points, we have that $\mathrm{int}\delta^* \cap \mathcal{S}^* = \emptyset$, and thus μ is the arc of $K \cap \Omega$, where Ω is the closure of the component of $S^3 - \Sigma$ containing δ. Moreover note that μ^* is in $\mathrm{int}\xi^*_{i,1}$, and thus $\partial\dot{\alpha}^*_{i+1}$ is not on μ^*. Hence δ^* does not contain any ends of α^*_i, since otherwise $\Omega \cap K$ consists of more than one string. Thus μ is a trivial tangle in Ω, which contradicts that Σ is a decomposing sphere for K. $\square$

From Claims 2.4 and 2.5, $\mathcal{S}^* \cap (D^*_i \cup B^*_i)$ consists of at most two arcs each of which has an end on both of $\xi^*_{i,1}$ and $\xi^*_{i,2}$ and arcs whose boundaries are on $\xi^*_{i,2}$. If an arc whose boundary is on $\xi^*_{i,2}$ bounds with a subarc of $\xi^*_{i,2}$ a disk δ^* on $D^*_i \cup B^*_i$ which does not contain an end of α^*_i, then from Claim 2.2, the arc is in the component of $B^*_i - (\dot{\alpha}^*_{i+1} \cup \beta^*_{i,0} \cup \cdots \cup \beta^*_{i,t_i})$ which contains $\xi^*_{i,2}$. However then we can deform Σ along δ by isotopy so to eliminate δ^*, which contradicts that Σ has the minimal complexity. Thus a connected component of $\mathcal{S}^* \cap (D^*_i \cup B^*_i)$ is either

- an arc which has an end on both of $\xi^*_{i,1}$ and $\xi^*_{i,2}$ and which intersects with each of α^*_i, $\dot{\alpha}^*_{i+1}, \beta^*_{i,0}, \dots, \beta^*_{i,t_i}$ once,
- an arc whose boundary is on $\xi^*_{i,2}$ and which intersects with each of $\dot{\alpha}^*_{i+1}, \beta^*_{i,0}, \dots, \beta^*_{i,t_i}$ twice and bounds with a subarc of $\xi^*_{i,2}$ a disk on $D^*_i \cup B^*_i$ containing α^*_i, or
- an arc whose boundary is on $\xi^*_{i,2}$ and which intersects with α^*_i once and intersects with each of $\dot{\alpha}^*_{i+1}, \beta^*_{i,0}, \dots, \beta^*_{i,t_i}$ twice.

Take a look at the number $\sharp(\mathcal{S}^* \cap \alpha^{k*}_i)$ of intersections of $\mathcal{S}^*$ and α^{k*}_i ($1 \le i \le m_k$). If an arc ρ^* of $\mathcal{S}^* \cap (D^{k*}_i \cup B^{k*}_i)$ is of the first type (resp. last two types), then we have that $\sharp(\rho^* \cap \dot{\alpha}^*_{i+1}) = \sharp(\rho^* \cap \alpha^*_i)$ (resp. $\sharp(\rho^* \cap \dot{\alpha}^*_{i+1}) > \sharp(\rho^* \cap \alpha^*_i)$). Thus we have that $\sharp(\mathcal{S}^* \cap \alpha^*_{i+1}) = \sharp(\mathcal{S}^* \cap \dot{\alpha}^*_{i+1}) \ge \sharp(\mathcal{S}^* \cap \alpha^*_i)$, since $f_{\mathcal{C}}(\dot{\alpha}^*_i) = f_{\mathcal{C}}(\alpha^*_i)$. Here note that we

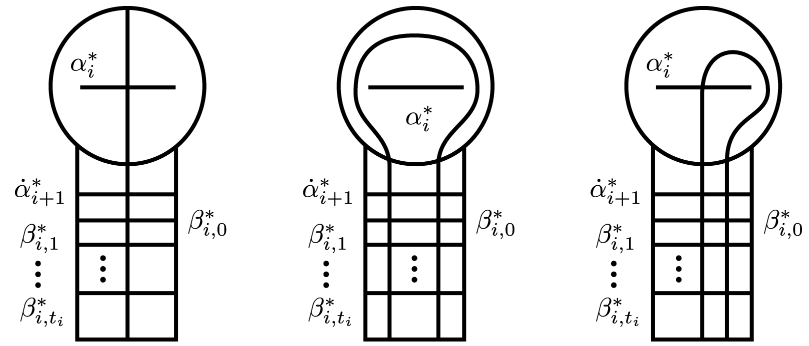


FIGURE 11

have that $\sharp(\mathcal{S}^* \cap \alpha_{i+m_k}^*) = \sharp(\mathcal{S}^* \cap \alpha_i^*)$, since $i + m_k \equiv i$ modulo m_k. Hence we have that $\sharp(\mathcal{S}^* \cap \alpha_{m_k}^*) = \sharp(\mathcal{S}^* \cap \alpha_{m_k-1}^*) = \cdots = \sharp(\mathcal{S}^* \cap \alpha_1^*)$. Therefore $\mathcal{S}^* \cap (\mathcal{D}^* \cup \mathcal{B}^*)$ does not have arcs of the last two types.

Hence $\mathcal{S}^* \cap (\mathcal{D}^* \cup \mathcal{B}^*)$ consists of at most two arcs of the first type of the above, each of which is a component of $\gamma^* \cap (\mathcal{D}^* \cup \mathcal{B}^*)$, since Σ intersects with K in two points. Therefore we have the following five cases with respect to γ^*:

(Case A) $\partial\gamma^*$ is on $\partial D_0^* - \partial\mathcal{B}^*$, and thus γ^* is on D_0^* and $\mathcal{S}^* \cap (\mathcal{D}^* \cup \mathcal{B}^*) = \emptyset$;

(Case B) γ^* has an end on both of ∂D_1^{k*} and ∂D_1^{l*} with $m_k = m_l = 1$;

(Case C) γ^* has an end on both of $\partial D_0^* - \partial\mathcal{B}^*$ and ∂D_1^{k*} with $m_k = 1$;

(Case D) $\partial\gamma^*$ is on ∂D_1^{k*} with $m_k = 1$; or

(Case E) γ^* has an end on both of ∂D_1^{k*} and ∂D_2^{k*} with $m_k = 2$.

Now we know that $\mathcal{S}^*$ consists of γ^* and loops on D_0^*. In the followings, we also take a look at the intersections of $\Sigma \cap (\mathcal{C} \cup \partial H)$ on Σ, which consists of γ, the loops of $\Sigma \cap D_0$, and the loops of $\Sigma \cap \partial H$.

CLAIM 2.6. *Each loop of $\Sigma \cap D_0$ and $\Sigma \cap \partial H$ on Σ intersects with γ.*

PROOF. Assume otherwise and take an innermost loop λ of the loops on Σ which do not intersect with γ, and let Σ_λ be the subdisk of Σ bounded by λ which does not contain γ. Thus $(\mathrm{int}\Sigma_\lambda) \cap (\mathcal{C} \cup \partial H) = \emptyset$.

If λ is a loop of $\Sigma \cap D_0$, then let δ be the subdisk of D_0 which λ bounds and B_λ the 3-ball which $\Sigma_\lambda \cup \delta$ bounds in $S^3 - \partial D_0$. Here δ may intersect with $\mathcal{B}$ or Σ. If $\delta \cap \mathcal{B} \neq \emptyset$, then let δ' be a subdisk of δ such that $\delta' \subset \mathrm{int}\,\delta$ and $(\mathrm{int}\,\delta - \delta') \cap (\mathcal{B} \cup \Sigma) = \emptyset$. Let D_0' be the disk obtained from D_0 by replacing δ' with a parallel copy Σ_λ' of Σ_λ such that $\partial\Sigma_\lambda' = \partial\delta'$ and the interior of the 3-ball bounded by Σ_λ, Σ_λ', and $\delta - \mathrm{int}\,\delta'$ does not intersect with $\mathcal{C} \cup \Sigma \cup H$.

Then we obtain another ribbon disk $D_0' \cup \mathcal{D} \cup \mathcal{B}$ such that the number of intersections of $\mathcal{B}$ and D_0' is less than that of $\mathcal{B}$ and D_0, which contradicts the minimality of the number of intersections of $\mathcal{B}$ and D_0. If $\delta \cap \mathcal{B} = \emptyset$, then let λ' be an innermost loop of $\Sigma \cap D_0$ in δ (λ' may be λ) and let δ' the subdisk of D_0 which λ' bounds. Replacing a neighborhood of λ' in Σ with two parallel copies of δ', we obtain two 2-spheres Σ_1 and Σ_2 one of which, say Σ_1, intersects with K twice. Then Σ_1 is another decomposing sphere with less complexity than that of Σ, which contradicts that Σ has the minimal complexity.

If λ is a loop of $\Sigma \cap \partial H$, then λ separates ∂H into two disks δ_1 and δ_2 such that $\delta_1 \cup \delta_2 = \partial H$ and $\delta_1 \cap \delta_2 = \lambda$. If δ_1 (resp. δ_2) does not intersect with $\mathcal{C}$, then replacing a neighborhood of λ in Σ with two parallel copies of δ_1 (resp. δ_2), we obtain two 2-spheres Σ_1 and Σ_2 one of which, say Σ_1, intersects with K twice. Then Σ_1 is another decomposing sphere with less complexity than that of Σ, which contradicts that Σ has the minimal complexity. Thus both of δ_1 and δ_2 intersect with $\mathcal{C}$. We have that Σ_λ is either in H or in $\overline{S^3 - H}$.

In the former case, Σ_λ divide H into two 3-balls, one of which is bounded by Σ_λ and δ_1, say H_1, and the other of which is bounded by Σ_λ and δ_2, say H_2. Since both of δ_1 and δ_2 intersect with $\mathcal{C}$ and $\Sigma_\lambda \cap \mathcal{C} = \emptyset$, both of H_1 and H_2 contain a component of the SR-tangle. However then the SR-tangle is separable, which contradicts the assumption.

In the latter case, Σ_λ divide $\overline{S^3 - H}$ into two 3-balls, one of which is bounded by Σ_λ and δ_1 and the other of which is bounded by Σ_λ and δ_2. This is impossible to occur, since both of δ_1 and δ_2 intersect with $\mathcal{C}$, $\mathcal{C} \cap \overline{S^3 - H}$ is a (singular) disk, and $\Sigma_\lambda \cap \mathcal{C} = \emptyset$. $\square$

(Case A) Since γ is on D_0 and D_0 is in $S^3 - H$, neither a loop of $D_0 \cap \Sigma$ nor a loop of $\partial H \cap \Sigma$ intersects with γ. However this contradicts Claim 2.6. Thus there are no loops on Σ, which induces that $\mathcal{S}^*$ consists of only γ^* and Σ is in $S^3 - H$. Therefore if each component of $\partial D_0^* - \gamma^*$ contains a component of $\partial\mathcal{B}^* \cap \partial D_0^*$, then each component of $S^3 - \Sigma$ contains $D_i^k \cup B_i^k$ for a certain pair of i and k. However, this is impossible, since $\mathcal{D}$ is contained in H and Σ is in $S^3 - H$ and thus a component of $S^3 - \Sigma$ is in $S^3 - H$. Hence one of the two components of $\partial D_0^* - \gamma^*$, say μ^*, does not contain any components of $\partial\mathcal{B}^* \cap \partial D_0^*$. Therefore μ is the arc of $K \cap \Omega$, where Ω is the closure of a component of $S^3 - \Sigma$. Now let δ^* be the subdisk of D_0^* bounded by γ^* and μ^*. Since $\mathcal{S}^*$ consists of only γ^*, we have that $\mathrm{int}\delta^* \cap \mathcal{S}^* = \emptyset$. Thus δ is an embedded disk in Ω. Moreover δ^* does not contain an end of $\dot{\beta}_i^{k*}$ for any pair of i and k, since otherwise $\Omega \cap K$ consists of more than one string. Hence μ is trivial in Ω, which contradicts that Σ is a decomposing sphere for K.

(Case B and C) Let ρ^* be the arc of $\mathcal{S}^* \cap (D_1^{k*} \cup B_1^{k*})$, let A^* (resp. $\dot{A}^*$) the intersection of ρ^* with α_1^{k*} (resp. with $\dot{\alpha}_1^{k*}$), and let ρ_0^* the subarc of ρ^* bounded by A^* and $\dot{A}^*$ (see the leftside of Figure 12). Note that ρ_0 bounds a disk δ on Σ, and that δ does not contain any loop intersections from Claim 2.6. Then we can deform $D_1^k \cup B_1^k$ along δ to eliminate α_1^k by isotopy, which tells us the k-th component of our SR-tangle is trivial. This contradicts that our SR-tangle is not separable from Lemma 1.4

(Case D) Let ρ_1^* and ρ_2^* be the two arcs of $\gamma^* \cap (D_1^{k*} \cup B_1^{k*})$. If ρ_1 and ρ_2 does not intersect each other, then we can obtain a contradiction as the previous case. Thus ρ_1 and ρ_2 intersect in two points $A = f_{\mathcal{C}}(A^*) = f_{\mathcal{C}}(\dot{A}^*)$ and $B = f_{\mathcal{C}}(B^*) = f_{\mathcal{C}}(\dot{B}^*)$, where $A^* = \rho_1^* \cap \alpha_1^{k*}$, $\dot{A}^* = \rho_2^* \cap \dot{\alpha}_1^{k*}$, $B^* = \rho_2^* \cap \alpha_1^{k*}$, and $\dot{B}^* = \rho_1^* \cap \dot{\alpha}_1^{k*}$ (see the rightside of Figure 12). Let δ_1^* be the subdisk of $D_1^{k*} \cup B_1^{k*}$ bounded by the subarc ζ_1^* of ρ_1^* bounded by $A^* \cup \dot{B}^*$, the subarc of $\dot{\alpha}_1^{k*}$ bounded by $\dot{B}^* \cup \dot{A}^*$, the subarc ζ_2^* of ρ_2^* bounded by $\dot{A}^* \cup B^*$, and the subarc of α_1^{k*} bounded by $B^* \cup A^*$. From Claim 2.1, we have that $\mathrm{int}\delta_1 \cap \Sigma = \emptyset$. Thus δ_1 is properly embedded in the closure of the component of $S^3 - \Sigma$. However then, take a subdisk δ_2 of Σ bounded by ζ_1 and ζ_2. Since δ_1 is a Möbius band, $\delta_1 \cup \delta_2$ is a projective plane, which cannot be embedded in S^3. Thus we have a contradiction.

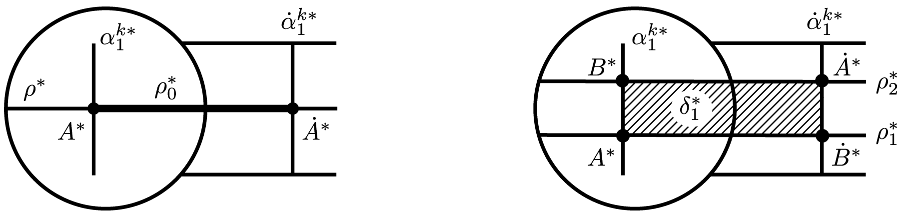


FIGURE 12

In the rest of the paper, we devote ourselves to Case E. We omit the upper index k of D_i^k and B_i^k $(i = 1, 2)$ unless we need to emphasize it.

(Case E) In this case γ^* can be divided into five subarcs as $\gamma^* = \gamma_{D_1}^* \cup \gamma_{B_1}^* \cup \gamma_{D_0}^* \cup \gamma_{B_2}^* \cup \gamma_{D_2}^*$, where γ_X^* is $\gamma^* \cap X^*$. Take a look at $\mathcal{S}^* \cap D_0^*$, which consists of $\gamma_{D_0}^*$ and the pre-images of the loops of $\Sigma \cap D_0$. Then $\gamma_{D_0}^*$ may intersect with $\dot{\beta}_{i,j}^*$, and each loop of $\mathcal{S}^* \cap D_0^*$ intersects with $\dot{\beta}_{i,j}^*$ from Claim 2.6 (see Figure 13 for an example).

Now take a look at the intersections of $\Sigma \cap (\mathcal{C} \cup \partial H)$ on Σ, which consists γ, the loops of $\Sigma \cap D_0$, and the loops of $\Sigma \cap \partial H$. Here note that each of the five subarcs of γ is simple, that γ_{D_i} intersects with $\gamma - \gamma_{D_i}$ only in a point on $\gamma_{B_{i+1}}$, and that γ_{B_i} intersects with $\gamma - \gamma_{B_i}$ in a point on $\gamma_{D_{i+1}}$ and in points on γ_{D_0} $(i = 1, 2)$.

CLAIM 2.7. *We have that* $\mathrm{int}(\gamma_{B_1} \cup \gamma_{B_2}) \cap \mathrm{int}\gamma_{D_0} = \emptyset$.

PROOF. Assume otherwise. Then γ_{B_i} has a subarc ζ which bounds a disk δ_ζ on Σ with a subarc of γ_{D_0} $(i = 1, 2)$, where we may assume that δ_ζ does not contain any subarcs of γ_{B_1} and of γ_{B_2}. Here δ_ζ may intersect with a loop of $\Sigma \cap D_0$ in an arc whose ends are on ζ. However then, we can eliminate the intersections from an outermost one by deforming $\mathcal{B}$ along the subdisk of δ_ζ bounded by the intersection and a subarc of ζ by isotopy, which contradicts the minimality of the number of intersections of $\mathcal{B} \cap D_0$. Hence $\mathrm{int}\delta_\zeta \cap \mathcal{C} = \emptyset$. Now we have two cases that an end of ζ is on $\partial\gamma_{B_i} \cap \partial\gamma_{D_0}$ or not. In either case, we can

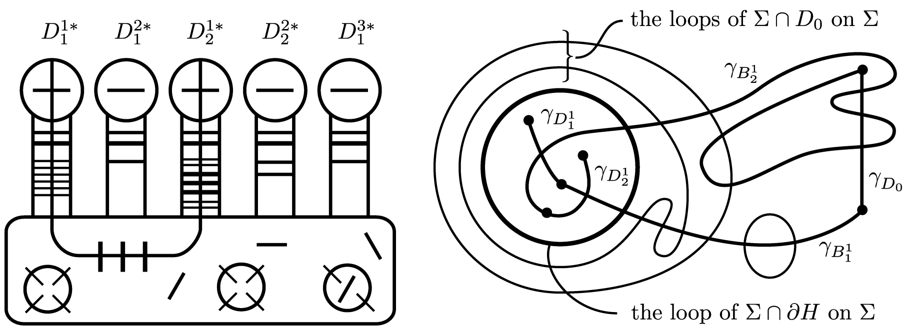


FIGURE 13

deform B_i along δ_ζ by isotopy so to eliminate the intersection(s) of intγ_{B_i} and intγ_{D_0} (an end or the ends of ζ). However this also contradicts the minimality of the number of intersections of $\mathcal{B} \cap D_0$. □

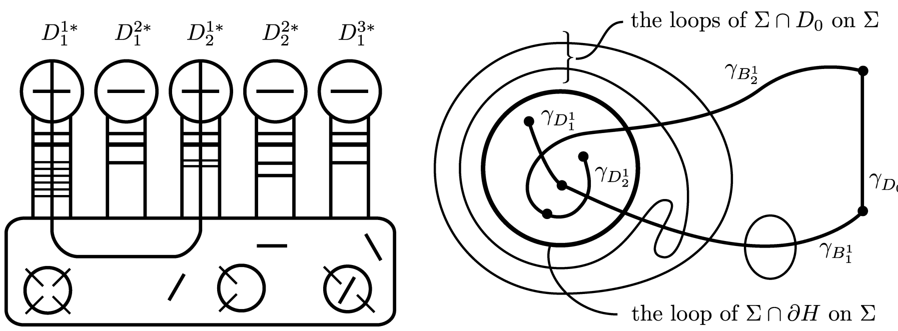


FIGURE 14

CLAIM 2.8. *Each loop on Σ intersects with γ exactly in two points $\beta_{1,j} \cap \Sigma$ and $\beta_{2,j} \cap \Sigma$ $(j = 0, 1, \dots, t_1 = t_2)$.*

PROOF. Let B_1' (resp. B_2') be the closure of the component of $B_1 - \alpha_2$ (resp. $B_2 - \alpha_1$) which intersects with ∂H, and let λ a loop on Σ. Note that λ intersects with γ from Claim 2.6, moreover only in $\gamma_{B_1'}$ or $\gamma_{B_2'}$, since $\beta_{i,j}$ is on B_i' $(i = 1, 2, j = 0, 1, \dots, t_i)$. First we claim that λ intersects with $\gamma_{B_i'}$ at most once $(i = 1, 2)$. If λ is of $\Sigma \cap \partial H$, then it is clear, since each band of $\mathcal{B}$ intersects with ∂H only once. Assume that λ is of $\Sigma \cap D_0$ and intersects with $\gamma_{B_i'}$ in more than once. Such a loop has a subarc which bounds a disk on Σ with a subarc of $\gamma_{B_i'}$ $(i = 1, 2)$. Let δ be an innermost disk among such disks. We may assume that δ is bounded by a subarc of λ and a subarc of $\gamma_{B_1'}$. Then we can deform B_1 along δ by isotopy so to eliminate the two intersections. However this contradicts the minimality of the number of intersections of $\mathcal{B} \cap D_0$.

Therefore we complete the proof, since $\gamma_{B_1} \cup \gamma_{D_0} \cup \gamma_{B_2'}$ and a subarc of γ_{D_1} form a cycle on Σ. $\square$

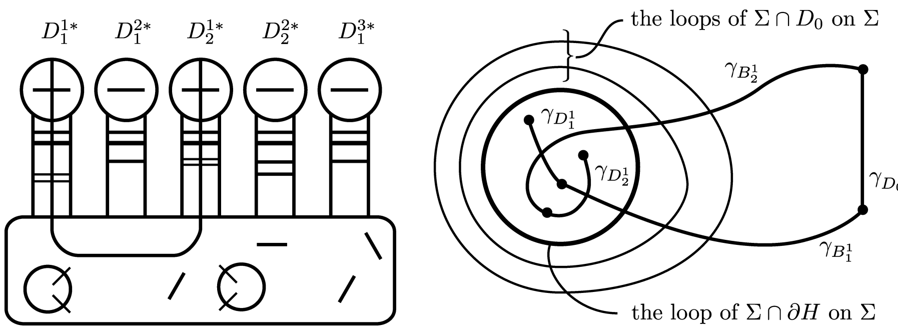


FIGURE 15

From Claim 2.8, we have that $\Sigma \cap \partial H$ consists of only one loop, i.e., the loop of $\Sigma \cap \partial H$ which intersects with γ_{B_1} is the loop of $\Sigma \cap \partial H$ which intersects with γ_{B_2}, and thus $\Sigma \cap H$ is a disk Σ_H. Note that $\Sigma_H \cap \mathcal{C} = \Sigma_H \cap (\mathcal{B}^k \cup \mathcal{D}^k)$. Therefore the SR-tangle consists of only one component, since otherwise we can take a disk $\Sigma_H \times \{1\}$ or $\Sigma_H \times \{-1\}$ to separate the k-th component from another component.

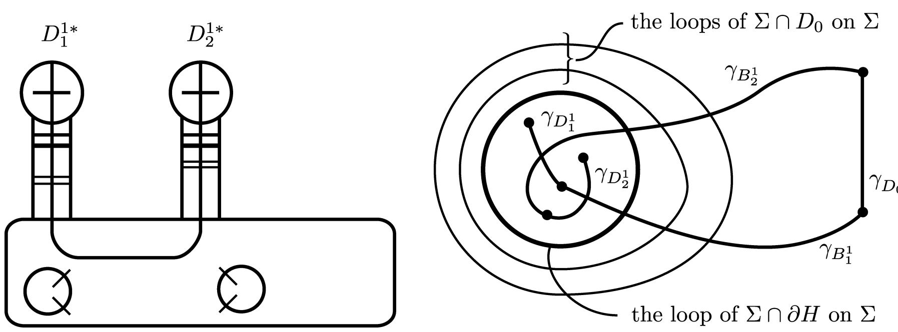


FIGURE 16

CLAIM 2.9. *There do not exist loops of* $\Sigma \cap D_0$.

PROOF. Assume otherwise. Then take an innermost one, say λ on Σ, i.e., a loop which bounds a disk on Σ that contains the loop of $\Sigma \cap \partial H$ but does not contain any other loops of $\Sigma \cap D_0$. Let A_λ be the annulus on Σ bounded by λ and the loop of $\Sigma \cap \partial H$ and let $B_{i,1}$

the subband of B_i bounded by $\beta_{i,0}$ and $\beta_{i,1}$ $(i = 1, 2)$, where note that $\beta_{i,1}$ intersects with λ. Then we have that $(H \cup A_\lambda \cup B_{1,1} \cup B_{2,1}) \cap D_0 = \lambda \cup \beta_{1,1} \cup \beta_{2,1}$.

Now let δ_λ be the subdisk of D_0 bounded by λ and take a subdisk δ of D_0 such that $\delta \cap (\mathcal{B} \cup \Sigma) = (\delta_\lambda \cup \beta_{1,1} \cup \beta_{2,1}) \cap (\mathcal{B} \cup \Sigma)$. Then take a disk δ' with $\partial\delta' = \partial\delta$ and $\mathrm{int}\delta' \cap (\mathcal{C} \cup \Sigma \cup H) = \emptyset$ which bounds a 3-ball with δ containing $H \cup A_\lambda \cup B_{1,1} \cup B_{2,1}$. Let $D_0' = (D_0 - \delta) \cup \delta'$, and then $(\mathcal{B} \cup \mathcal{D}) \cup D_0'$ is another ribbon disk for K such that the number of intersections of $\mathcal{B}$ and D_0' is less than that of $\mathcal{B}$ and D_0, which is a contradiction. $\square$

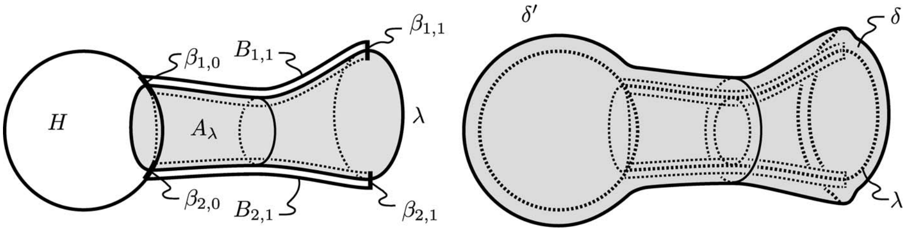


FIGURE 17

Therefore we have that $\mathcal{B} \cap D_0 = \emptyset$ and $\Sigma \cap \mathcal{C} = \gamma$, and thus $\mathcal{C} \cup H \cup \Sigma$ is as illustrated in Figure 18. Then we know that K is the square knot, which contradicts the assumption.

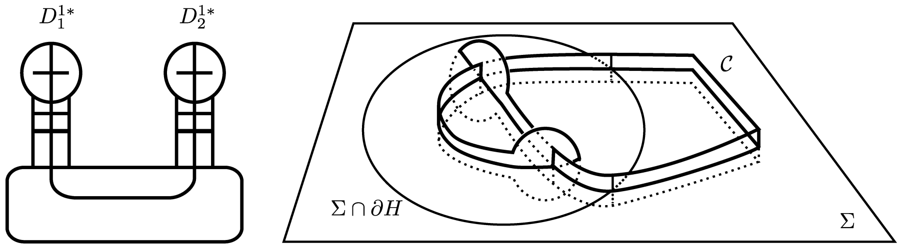


FIGURE 18

Hence we can conclude that there does not exist a composite knot which is not the square knot and can be transformed into the trivial knot by a single SR^--move whose associated tangle is not separable. This completes the proof. $\square$

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Present Addresses:
TETSUO SHIBUYA
DEPARTMENT OF MATHEMATICS,
OSAKA INSTITUTE OF TECHNOLOGY,
ASAHI, OSAKA, 535–8585 JAPAN.
e-mail: shibuya@ge.oit.ac.jp

TATSUYA TSUKAMOTO
DEPARTMENT OF MATHEMATICS,
OSAKA INSTITUTE OF TECHNOLOGY,
ASAHI, OSAKA, 535–8585 JAPAN.
e-mail: tsukamoto@ge.oit.ac.jp