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An Existence Theorem for the Steady Navier–Stokes Problem in Higher Dimensions

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Abstract. We extend a well–known result of H. Fujita and H. Morimoto [1] to exterior domains of $\mathbb{R}^m$, with $m \geq 4$.

1. Statement of the theorem

The steady–state Navier–Stokes problem is to find a solution of the equations

$$\begin{aligned} \Delta \boldsymbol{u} - \boldsymbol{u} \cdot \nabla \boldsymbol{u} - \nabla p = \mathbf{0} \quad & \text{in } \Omega\,, \\ \operatorname{div} \boldsymbol{u} = 0 \quad & \text{in } \Omega\,, \\ \boldsymbol{u} = \boldsymbol{a} \quad & \text{on } \partial\Omega \end{aligned} \tag{1}$$

in the unknown $\boldsymbol{u}$ and p.

In a well–known paper of 1997 [1] H. Fujita and H. Morimoto proved that[1] if Ω is a bounded regular domain of $\mathbb{R}^m$ $(m = 2, 3)$ and the boundary datum $\boldsymbol{a}$ is expressed by

$$\boldsymbol{a} = \mu \nabla \xi + \boldsymbol{\gamma}\,, \tag{2}$$

with $\xi \in W^{2,2}(\Omega)$ harmonic function and $\boldsymbol{\gamma}$ satisfying the compatibility condition

$$\int_{\partial\Omega} \boldsymbol{\gamma} \cdot \boldsymbol{n} = 0\,,$$

where $\boldsymbol{n}$ is the outward (with respect to Ω) unit normal to $\partial\Omega$, then there is a discrete at most countable subset G of $\mathbb{R}$ such that for $\mu \notin G$ a positive constant $c_0 = c_0(\mu, \xi, \Omega)$ exists such that if

$$\|\boldsymbol{\gamma}\|_{W^{1/2,2}(\partial\Omega)} \leq c_0\,,$$

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[1] See also [4], [5].

then (1) has a solution

$$(\boldsymbol{u}, p) \in [W^{1,2}_\sigma(\Omega) \cap C^\infty(\Omega)] \times [L^2(\Omega) \cap C^\infty(\Omega)] .$$

A noteworthy consequence of this theorem is that, modulo (at most) a countable set, for every "flux" μ there is a nonpotential solution of (1)[2]. This results has been extended in [6] to boundary data (2) in Lebesgue spaces and to the problem

$$\begin{aligned} \Delta\boldsymbol{u} - \boldsymbol{u}\cdot\nabla\boldsymbol{u} - \nabla p = \boldsymbol{0} &\quad \text{in } \Omega\,, \\ \operatorname{div}\boldsymbol{u} = 0 &\quad \text{in } \Omega\,, \\ \boldsymbol{u} = \boldsymbol{a} &\quad \text{on } \partial\Omega\,, \\ \lim_{|x|\to+\infty} \boldsymbol{u}(x) = \boldsymbol{0}\,, & \end{aligned} \tag{3}$$

where Ω is the exterior Lipschitz domain of $\mathbb{R}^3$

$$\Omega = \mathbb{R}^m \setminus \bigcup_{i=1}^{h} \overline{\Omega}_i, \quad \overline{\Omega}_i \cap \overline{\Omega}_j = \varnothing, \quad i \neq j \tag{4}$$

and Ω_i is a bounded domain with connected Lipschitz boundary. To be precise, it is showed that, if ξ vanishes at infinity, there is a discrete at most countable subset G of $\mathbb{R}$ such that if $\mu \notin G$, then a positive constant c_0 exists such that if $\|\boldsymbol{\gamma}\|_{L^\infty(\partial\Omega)} \le c_0$, then (3) has a C^∞ solution in Ω which decays at infinity as the fundamental solution of the Stokes equations.

In the present paper we aim at showing that the Fujita–Morimoto approach can be also used to get existence of a solution of system (3) in higher dimensions for boundary data in natural trace spaces and for large fluxes. Our main purpose is indeed to prove the following

THEOREM 1. *Let Ω be an exterior domain of $\mathbb{R}^m$ $(m \ge 4)$ of class C^2 and let $\boldsymbol{a}$ be expressed by* (2) *with $\xi \in D^{2,q}(\Omega)$ $(q > m/2)$ harmonic function vanishing at infinity. There is a discrete at most countable subset G of $\mathbb{R} \setminus (-\alpha, \alpha)$ with $\alpha = \frac{(m-2)\sqrt{m}}{(m-1)\|\nabla\xi\|_{L^m(\Omega)}}$ such that if $\mu \notin G$, then a positive constant $c_0 = c_0(m, \xi, \mu, \Omega)$ exists such that if $\|\boldsymbol{\gamma}\|_{W^{1-2/m,m/2}(\partial\Omega)} \le c_0$, then* (3) *has a solution*

$$(\boldsymbol{u}, p) \in [D^{1,m/2}_{\sigma,0}(\overline{\Omega}) \cap C^\infty(\Omega)] \times [L^{m/2}(\Omega) \cap C^\infty(\Omega)] .$$

Notation. We use a standard vector notation as, *e.g.*, in [3]. $W^{k,q}(\Omega)$ $(k \in \mathbb{N}_0,\ q \in [1, +\infty])$ is the usual Sobolev space and $W^{k-1/q,q}(\partial\Omega)$ $(k \ge 1)$ is its trace space; $D^{k,q}(\Omega) = \{\varphi \in L^1_{\rm loc}(\Omega) : \|\nabla_k\varphi\|_{L^q(\Omega)} < +\infty\}$, $\nabla_k\varphi = \nabla, \dots, \nabla_{k-\text{times}}\,\varphi$; $D^{k,q}_0(\Omega)$, $D^{k,q}_0(\overline{\Omega})$ are the completions of $C^\infty_0(\Omega)$, $C^\infty_0(\overline{\Omega})$ respectively with respect to $\|\nabla_k\varphi\|_{L^q(\Omega)}$ (for the properties of these spaces see [3] Ch. II); if V is a linear subspace of $L^1_{\rm loc}(\Omega)$, $V_\sigma = \{\boldsymbol{u} \in V : \int_\Omega \boldsymbol{u}\cdot\nabla\varphi = 0, \forall\varphi \in C^\infty_0(\Omega)\}$; $S_R = \{x \in \mathbb{R}^m : |x| < R\}$ and the symbol c stands for a positive constant whose numerical value is unessential to our purposes.

[2]If Ω is an annulus of $\mathbb{R}^2$, then $G = \varnothing$ (see [2], [4]).

2. Preliminary results

Let us recall some well–known results we shall need to prove Theorem 1. In what follows Ω denotes an exterior domain of $\mathbb{R}^m$ $(m \geq 4)$ of class C^2. As is always possible, we assume that $\complement\overline{\Omega}$ contains $\overline{S}_1$.

The Stokes equations

$$\begin{aligned} \Delta \boldsymbol{u} - \nabla p &= \boldsymbol{0}, \\ \operatorname{div} \boldsymbol{u} &= 0\,, \end{aligned} \tag{5}$$

in $\mathbb{R}^m$, $m \geq 4$, admit the fundamental solution (see [3] Ch. IV)

$$\begin{aligned} U_{ij}(x-y) &= -\frac{1}{2|\partial S_1|(m-2)}\left[\frac{\delta_{ij}}{|x-y|^{m-2}} + (m-2)\frac{(x_i-y_i)(x_j-y_j)}{|x-y|^m}\right], \\ q_i(x-y) &= -\frac{1}{|\partial S_1|}\frac{x_i-y_i}{|x-y|^m}\,. \end{aligned}$$

For $\boldsymbol{f} \in L^q(\Omega)$ the pair (volume potential)

$$\begin{aligned} \mathcal{V}[\boldsymbol{f}] &= \int_\Omega \boldsymbol{U}(x-y)\cdot \boldsymbol{f}(y)\mathrm{d}v_y, \\ \mathcal{P}[\boldsymbol{f}] &= \int_\Omega \boldsymbol{q}(x-y)\cdot \boldsymbol{f}(y)\mathrm{d}v_y \end{aligned}$$

is a solution of the (nonhomogeneous) Stokes equations

$$\begin{aligned} \Delta \boldsymbol{u} - \nabla p &= \boldsymbol{f}, \\ \operatorname{div} \boldsymbol{u} &= 0 \end{aligned} \tag{6}$$

in Ω.

LEMMA 1. *If $q \in (1, m/2)$, then $\mathcal{V}$ maps boundedly $L^q(\Omega)$ into $D^{2,q}_{\sigma,0}(\overline{\Omega})$. Moreover,*

$$\int_\Omega r^\varepsilon |\nabla_2 \mathcal{V}[\boldsymbol{f}]|^{\frac{m}{3}} \leq c\int_\Omega r^\varepsilon |\boldsymbol{f}|^{\frac{m}{3}},$$

for all $\boldsymbol{f}$ such that $r^\varepsilon|\boldsymbol{f}|^{m/3} \in L^1(\Omega)$, with $\varepsilon \in (0,1)$.

LEMMA 2. *If $\varphi \in D^{1,q}_0(\overline{\Omega})$, $q \in [1, m)$, then for large R*

$$\int_{\partial S_1} |\varphi(R,\zeta)|^q \mathrm{d}\zeta \leq cR^{q-m}\int_{\complement S_R} |\nabla\varphi|^q\,. \tag{7}$$

Moreover, there is a positive constant c depending only on q and Ω such that

$$\|\varphi\|_{L^{\frac{mq}{m-q}}(\Omega)} \leq \frac{q(m-1)}{2(m-q)\sqrt{m}}\|\varphi\|_{D^{1,q}(\Omega)}\,, \tag{8}$$

for all $\varphi \in D_0^{1,q}(\overline{\Omega})$.

Lemma 1 is a special version of the classical Calderón–Zygmund theorem (see [7]), while Lemma 2 is the Sobolev's inequality in the space $D_0^{1,q}(\overline{\Omega})$ (see [3] Ch. II).

Consider the Stokes problem

$$\begin{aligned} \Delta \boldsymbol{u} - \nabla p = \boldsymbol{f} &\quad \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 &\quad \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{a} &\quad \text{on } \partial\Omega\,. \end{aligned} \tag{9}$$

LEMMA 3. *If* $\boldsymbol{a} \in W^{1-1/q,q}(\partial\Omega)$, $q \in (\frac{m}{m-1}, m)$, *and* $\boldsymbol{f} \in L^s(\Omega)$ *with* $s = \frac{mq}{m+q}$, $1 < s < \frac{m}{2}$, *then* (9) *admits a unique solution*

$$(\boldsymbol{u}, p) \in [D_{\sigma,0}^{1,q}(\overline{\Omega}) \cap C^\infty(\Omega)] \times [L^q(\Omega) \cap C^\infty(\Omega)]$$

and

$$\|\boldsymbol{u}\|_{D^{1,q}(\Omega)} + \|p\|_{L^q(\Omega)} \le c\{\|\boldsymbol{a}\|_{W^{1-1/q,q}(\partial\Omega)} + \|\boldsymbol{f}\|_{L^s(\Omega)}\}.$$

Lemma 3 is a classical result in the theory of Navier–Stokes equations (see [3] Ch. V).

3. Proof of Theorem 1

Since ξ vanishes at infinity, by well–known properties of harmonic functions

$$|\nabla_k \xi(x)| \le c r^{2-m-k}\,, \tag{10}$$

for large $r = |x|$ and for every nonnegative integer k. Hence for $\boldsymbol{v} \in D_0^{1,m/2}(\overline{\Omega})$ it follows that

$$\nabla\xi \cdot \nabla\boldsymbol{v}, \quad \boldsymbol{v} \cdot \nabla_2\xi \in L^s(\Omega), \quad s \in [1, mq/(m+q)]\,.$$

Therefore, by Lemma 1 the operator

$$\mathcal{K}[\boldsymbol{v}] = \int_\Omega \boldsymbol{U}(x-y) \cdot [\nabla\xi \cdot \nabla\boldsymbol{v} + \boldsymbol{v} \cdot \nabla_2\xi](y)\mathrm{d}v_y$$

maps boundedly $D_{\sigma,0}^{1,m/2}(\overline{\Omega})$ into $D_{\sigma,0}^{2,s}(\overline{\Omega})$ for all $s \in (1, mq/(m+q)]$ and

$$\|\mathcal{K}[\boldsymbol{v}]\|_{D^{2,s}(\Omega)} \le c(s, \xi, \Omega)\|\boldsymbol{v}\|_{D^{1,m/2}(\Omega)}\,. \tag{11}$$

Let $\{\boldsymbol{v}_k\}_{k\in\mathbb{N}}$ be a bounded sequence in $D_{\sigma,0}^{1,m/2}(\overline{\Omega})$. By (11) $\{\mathcal{K}[\boldsymbol{v}_k]\}_{k\in\mathbb{N}}$ is bounded in $D_{\sigma,0}^{2,s}(\overline{\Omega})$ so that, since $q > m/2$, by Rellich's compacteness theorem we can extract a subsequence, still denoted by the same symbol, which converges strongly in $D_{\text{loc}}^{1,m/2}(\overline{\Omega})$.

By (8) and the inequality $(|a|+|b|)^q \le c(|a|^q+|b|^q)$ $(q\ge 1)$

$$\left\{\int_\Omega r^\varepsilon\left|\nabla\mathcal{K}[\boldsymbol{v}_k]\right|^{\frac{m}{2}}\right\}^{\frac{2}{3}} = \left\{\int_\Omega \left|r^{\frac{2\varepsilon}{m}}\nabla\mathcal{K}[\boldsymbol{v}_k]\right|^{\frac{m}{2}}\right\}^{\frac{2}{3}}$$
$$\le c\int_\Omega \left|\nabla(r^{\frac{2\varepsilon}{m}}\nabla\mathcal{K}[\boldsymbol{v}_k])\right|^{\frac{m}{3}} \le c\int_\Omega r^\varepsilon\left|\nabla_2\mathcal{K}[\boldsymbol{v}_k]\right|^{\frac{m}{3}} + c\int_\Omega r^{-\frac{m}{3}+\frac{2\varepsilon}{3}}\left|\nabla\mathcal{K}[\boldsymbol{v}_k]\right|^{\frac{m}{3}},$$

with $0<\varepsilon\ll 1$. Hence by Lemma 1, (11) and the inequality

$$\int_\Omega r^{-\frac{m}{3}+\frac{2\varepsilon}{3}}\left|\nabla\mathcal{K}[\boldsymbol{v}_k]\right|^{\frac{m}{3}} \le \left\{\int_\Omega \left|\nabla\mathcal{K}[\boldsymbol{v}_k]\right|^{\frac{mq}{3}}\right\}^{1/q}\left\{\int_\Omega r^{(\frac{2\varepsilon}{3}-\frac{m}{3})q'}\right\}^{1/q'}$$

for q close to 1, it follows

$$\left\{\int_\Omega r^\varepsilon|\nabla\mathcal{K}[\boldsymbol{v}_k]|^{\frac{m}{2}}\right\}^{\frac{2}{3}} \le c\int_\Omega r^\varepsilon|\nabla\xi\cdot\nabla\boldsymbol{v}_k+\boldsymbol{v}_k\cdot\nabla_2\xi|^{\frac{m}{3}}+c\,. \tag{12}$$

Since by Hölder's inequality and (10)

$$\int_\Omega r^\varepsilon|\nabla\xi\cdot\nabla\boldsymbol{v}_k|^{\frac{m}{3}} \le c\int_\Omega r^{\varepsilon+(1-m)\frac{m}{3}}|\nabla\boldsymbol{v}_k|^{\frac{m}{3}}$$
$$\le c\|\nabla\boldsymbol{v}_k\|^{m/3}_{L^{m/2}(\Omega)}\left\{\int_\Omega r^{(3\varepsilon+(1-m)m}\right\}^{\frac{1}{3}} \le c\|\nabla\boldsymbol{v}_k\|^{m/3}_{L^{m/2}(\Omega)},$$
$$\int_\Omega r^\varepsilon|\boldsymbol{v}_k\cdot\nabla_2\xi|^{\frac{m}{3}} \le c\int_\Omega r^{\varepsilon-\frac{m^2}{3}}|\boldsymbol{v}_k|^{\frac{m}{3}}$$
$$\le c\|\boldsymbol{v}_k\|^{m/3}_{L^m(\Omega)}\left\{\int_\Omega r^{\varepsilon-m^2/2}\right\}^{\frac{2}{3}} \le c\|\nabla\boldsymbol{v}_k\|^{m/3}_{L^{m/2}(\Omega)},$$

(12) and the boundedness of $\{\boldsymbol{v}_k\}$ in $D^{1,m/2}(\Omega)$ imply

$$\int_\Omega r^\varepsilon|\nabla\mathcal{K}[\boldsymbol{v}_k]|^{\frac{m}{2}} \le c\,. \tag{13}$$

Now, by (13)

$$\int_\Omega |\nabla\mathcal{K}[\boldsymbol{v}_k-\boldsymbol{v}_h]|^{\frac{m}{2}} \le \int_{\Omega\cap S_R}|\nabla\mathcal{K}[\boldsymbol{v}_k-\boldsymbol{v}_h]|^{\frac{m}{2}}$$
$$+\frac{1}{R^\varepsilon}\int_{\complement S_R} r^\varepsilon|\nabla\mathcal{K}[\boldsymbol{v}_k-\boldsymbol{v}_h]|^{\frac{m}{2}} \le \int_{\Omega\cap S_R}|\nabla\mathcal{K}[\boldsymbol{v}_k-\boldsymbol{v}_h]|^{\frac{m}{2}}+\frac{c}{R^\varepsilon},$$

for large R, so that the sequence $\{\mathcal{K}[\boldsymbol{v}_k]\}_{k\in\mathbb{N}}$ converges strongly in $D^{1,m/2}(\Omega)$ and the operator $\mathcal{K}$ is compact from $D^{1,m/2}_{\sigma,0}(\overline{\Omega})$ into itself.

Let $\mathcal{M}[\boldsymbol{v}]\in D^{1,m/2}_{\sigma,0}(\overline{\Omega})$ be the solution of equations (9) in Ω with boundary datum $-\mathrm{tr}\mathcal{K}[\boldsymbol{v}]_{|\partial\Omega}$. Starting from (11) and repeating the above argument, we see that also $\mathcal{M}$ is compact from $D^{1,m/2}_{\sigma,0}(\overline{\Omega})$ into itself. Then, setting

$$\mathcal{L} = \mathcal{I} - \mu(\mathcal{K} + \mathcal{M}) : D^{1,m/2}_{\sigma,0}(\overline{\Omega}) \to D^{1,m/2}_{\sigma,0}(\overline{\Omega}) ,$$

by classical results we see that $\mathcal{L}$ is invertible for all $\mu \in \mathbb{R} \setminus G$, with G discrete at most countable subset of $\mathbb{R}$.

For $\boldsymbol{u} \in D^{1,m/2}_{\sigma,0}(\overline{\Omega})$ denote by $\mathcal{J}[\boldsymbol{u}]$ the solution of

$$\begin{aligned} \Delta \boldsymbol{v} - \nabla p &= \boldsymbol{u} \cdot \nabla \boldsymbol{u} && \text{in } \Omega, \\ \operatorname{div} \boldsymbol{v} &= 0 && \text{in } \Omega, \\ \boldsymbol{v} &= \boldsymbol{0} && \text{on } \partial\Omega . \end{aligned}$$

By Lemma 1 $\mathcal{J}[\boldsymbol{u}]$ maps $D^{1,m/2}_{\sigma,0}(\overline{\Omega})$ into $D^{2,m/3}_{\sigma,0}(\Omega)$ and it holds

$$\|\mathcal{J}[\boldsymbol{u}]\|_{D^{2,m/3}(\Omega)} \leq c\|\boldsymbol{u}\|^2_{D^{1,m/2}(\Omega)} . \tag{14}$$

Let

$$(\boldsymbol{u}_\gamma, p_\gamma) \in D^{1,m/2}_{\sigma,0}(\overline{\Omega}) \times L^{m/2}(\Omega)$$

be the solution of Stokes' problem with boundary datum $\boldsymbol{\gamma}$ and for $\mu \notin G$ consider the functional equation

$$\boldsymbol{u}' = \overset{-1}{\mathcal{L}}[\boldsymbol{u}_\gamma] + \overset{-1}{\mathcal{L}}\big[\mathcal{J}[\boldsymbol{u}]\big] = \mathcal{S}[\boldsymbol{u}] \tag{15}$$

in $D^{1,m/2}_{\sigma,0}(\overline{\Omega})$. By the continuity of $\overset{-1}{\mathcal{L}}$ and (14)

$$\begin{aligned} &\|\overset{-1}{\mathcal{L}}[\boldsymbol{u}_\gamma]\|_{D^{1,m/2}(\Omega)} \leq c\|\boldsymbol{u}_\gamma\|_{D^{1,m/2}(\Omega)} \leq c_1\|\boldsymbol{\gamma}\|_{W^{1-2/m,m/2}(\partial\Omega)} , \\ &\big\|\overset{-1}{\mathcal{L}}\big[\mathcal{J}[\boldsymbol{u}]\big]\big\|_{D^{1,m/2}(\Omega)} \leq c_2\|\boldsymbol{u}\|^2_{D^{1,m/2}(\Omega)} , \end{aligned}$$

for some positive constants c_1 and c_2 depending only on m, μ, ξ and Ω.

Consider the ball of $D^{1,m/2}_{\sigma,0}(\overline{\Omega})$

$$\mathcal{B} = \left\{ \boldsymbol{u} \in D^{1,m/2}_{\sigma,0}(\overline{\Omega}) : \|\boldsymbol{u}\|_{D^{1,m/2}(\Omega)} \leq \frac{1}{(2+\varepsilon)c_2} \right\}$$

for some positive ε, and assume that

$$\|\boldsymbol{\gamma}\|_{W^{1-2/m,m/2}(\partial\Omega)} < \frac{1}{(2+\varepsilon)^2 c_1 c_2} .$$

Of course, $\mathcal{S}$ maps $\mathcal{B}$ into itself and it is not difficult to see that

$$\begin{aligned}\|\mathcal{S}[\boldsymbol{u}]-\mathcal{S}[\boldsymbol{v}]\|_{\mathcal{B}} &\le c_2\|\boldsymbol{u}-\boldsymbol{v}\|_{D^{1,m/2}(\Omega)}\left\{\|\boldsymbol{u}\|_{D^{1,m/2}(\Omega)}+\|\boldsymbol{v}\|_{D^{1,m/2}(\Omega)}\right\}\\ &\le \frac{2}{2+\varepsilon}\|\boldsymbol{u}-\boldsymbol{v}\|_{D^{1,m/2}(\Omega)}\,.\end{aligned}$$

Therefore, $\mathcal{S}$ is a contraction in $\mathcal{B}$ so that (15) has a fixed point $\boldsymbol{w}$

$$\boldsymbol{w}=\overset{-1}{\mathcal{L}}[\boldsymbol{u}_\gamma]+\overset{-1}{\mathcal{L}}\big[\mathcal{J}[\boldsymbol{w}]\big]$$

and coming back to $\mathcal{L}$, we have that $\boldsymbol{w}$ satisfies the equation

$$\boldsymbol{w}-\mu(\mathcal{K}+\mathcal{M})[\boldsymbol{w}]=\boldsymbol{u}_\gamma+\mathcal{J}[\boldsymbol{w}]\,.$$

Hence, taking the Stokes operator, it follows that $\boldsymbol{w}$ is a solution of

$$\begin{aligned}\Delta\boldsymbol{w}-\mu\nabla\xi\cdot\nabla\boldsymbol{w}-\mu\boldsymbol{w}\cdot\nabla_2\xi-\boldsymbol{w}\cdot\nabla\boldsymbol{w}-\nabla Q=\boldsymbol{0}&\quad\text{in } \Omega\,,\\ \operatorname{div}\boldsymbol{w}=0&\quad\text{in } \Omega\,,\\ \boldsymbol{w}=\boldsymbol{\gamma}&\quad\text{on } \partial\Omega\,,\end{aligned}$$

for a suitable pressure field Q. Since $(\mu\nabla\xi,-\frac{1}{2}|\mu\nabla\xi|^2)$ is a solution of $(3)_{1,2}$ with boundary datum $\mu\nabla\xi$, the pair

$$(\boldsymbol{w}+\mu\nabla\xi,\,Q-\tfrac{1}{2}|\mu\nabla\xi|^2)$$

gives the desired solution of (3). The regularity of $(\boldsymbol{w},Q)$ in Ω is proved by well–known arguments (see [3] Section VIII.5) and the condition $(3)_4$ is satisfied in the sense of (7).

It remains to show that if $\mu\in(-\alpha,\alpha)$, then $\operatorname{Kern}\mathcal{L}=\{\boldsymbol{0}\}$. If $\boldsymbol{u}\in\operatorname{Kern}\mathcal{L}$, then $\boldsymbol{u}$ satisfies the equations

$$\begin{aligned}\Delta\boldsymbol{u}-\mu\nabla\xi\cdot\nabla\boldsymbol{u}-\mu\boldsymbol{u}\cdot\nabla_2\xi-\nabla p=\boldsymbol{0}&\quad\text{in } \Omega\,,\\ \operatorname{div}\boldsymbol{u}=0&\quad\text{in } \Omega\,,\\ \boldsymbol{u}=\boldsymbol{0}&\quad\text{on } \partial\Omega\end{aligned}\tag{16}$$

for a suitable pressure field p. By what we said about the operator $\mathcal{L}$, $(\boldsymbol{u},p)$ is regular and $(\boldsymbol{u},p)\in D^{1,q}_{\sigma,0}(\Omega)\times L^q(\Omega)$, $q\in[2,m/2]$. Let g be a regular function in $\mathbb{R}^m$, equal to 1 in S_R, vanishing in $\complement S_{2R}$ and such that $|\nabla g|\le cR^{-1}$. Multiplying (16) scalarly by $g\boldsymbol{u}$ and integrating over Ω, we have

$$\begin{aligned}\int_\Omega g|\nabla\boldsymbol{u}|^2=\mu\int_\Omega g\boldsymbol{u}\cdot\nabla\boldsymbol{u}\cdot\nabla\xi+\mu\int_\Omega[\tfrac{1}{2}|\boldsymbol{u}|^2\nabla\xi+(\boldsymbol{u}\cdot\nabla\xi)\boldsymbol{u}]\cdot\nabla g\\ -\int_\Omega[\nabla\boldsymbol{u}\cdot\boldsymbol{u}-p\boldsymbol{u}]\cdot\nabla g\,.\end{aligned}\tag{17}$$

Since by Hölder inequality

$$\left|\int_\Omega |\boldsymbol{u}|^2 \nabla\xi \cdot \nabla g\right| \le \frac{c}{R^m}\int_{S_{2R}\setminus S_R} |\boldsymbol{u}|^2 \le \frac{c}{R^2}\left\{\int_\Omega |\boldsymbol{u}|^m\right\}^{2/m},$$

$$\left|\int_\Omega (\boldsymbol{u}\cdot\nabla\xi)\boldsymbol{u}\cdot\nabla g\right| \le \frac{c}{R^2}\left\{\int_\Omega |\boldsymbol{u}|^m\right\}^{2/m},$$

$$\begin{aligned}\left|\int_\Omega \nabla g\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{u}\right| &\le \|\nabla g\|_{L^m(S_{2R}\setminus S_R)}\|\boldsymbol{u}\|_{L^{\frac{2m}{m-2}}(S_{2R}\setminus S_R)}\|\nabla\boldsymbol{u}\|_{L^2(S_{2R}\setminus S_R)}\\ &\le c\|\boldsymbol{u}\|_{L^{\frac{2m}{m-2}}(S_{2R}\setminus S_R)}\|\nabla\boldsymbol{u}\|_{L^2(S_{2R}\setminus S_R)},\end{aligned}$$

$$\left|\int_\Omega p\boldsymbol{u}\cdot\nabla g\right| \le c\|\boldsymbol{u}\|_{L^{\frac{2m}{m-2}}(S_{2R}\setminus S_R)}\|p\|_{L^2(S_{2R}\setminus S_R)}$$

and $\boldsymbol{u}\cdot\nabla\boldsymbol{u}\cdot\nabla\xi \in L^1(\Omega)$, we are allowed to let $R\to+\infty$ in (17) to get

$$\int_\Omega |\nabla\boldsymbol{u}|^2 = \mu\int_\Omega \boldsymbol{u}\cdot\nabla\boldsymbol{u}\cdot\nabla\xi\,. \tag{18}$$

Therefore, taking into account that by Sobolev's inequality (8) $\|\boldsymbol{u}\|_{L^{\frac{2m}{m-2}}(\Omega)} \le \frac{(m-1)}{(m-2)\sqrt{m}}\|\nabla\boldsymbol{u}\|_{L^2(\Omega)}$,

$$\begin{aligned}\left|\int_\Omega \boldsymbol{u}\cdot\nabla\boldsymbol{u}\cdot\nabla\xi\right| &\le \|\boldsymbol{u}\|_{L^{\frac{2m}{m-2}}(\Omega)}\|\nabla\boldsymbol{u}\|_{L^2(\Omega)}\|\nabla\xi\|_{L^m(\Omega)}\\ &\le \frac{(m-1)}{(m-2)\sqrt{m}}\|\nabla\xi\|_{L^m(\Omega)}\|\nabla\boldsymbol{u}\|^2_{L^2(\Omega)}\end{aligned}$$

and (18) yields

$$\left[1-\frac{|\mu|(m-1)}{(m-2)\sqrt{m}}\|\nabla\xi\|_{L^m(\Omega)}\right]\int_\Omega |\nabla\boldsymbol{u}|^2 \le 0\,.$$

Hence the desired result follows at once. □

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