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Generating the Mapping Class Group of a Punctured Surface by Involutions

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Abstract. Let $\Sigma_{g,b}$ denote a closed oriented surface of genus g with b punctures and let $\mathrm{Mod}(\Sigma_{g,b})$ denote its mapping class group. Kassabov showed that $\mathrm{Mod}(\Sigma_{g,b})$ is generated by 4 involutions if $g > 7$ or $g = 7$ and b is even, 5 involutions if $g > 5$ or $g = 5$ and b is even, and 6 involutions if $g > 3$ or $g = 3$ and b is even. We proved that $\mathrm{Mod}(\Sigma_{g,b})$ is generated by 4 involutions if $g = 7$ and b is odd, and 5 involutions if $g = 5$ and b is odd.

1. Introduction

Let $\Sigma_{g,b}$ be an closed oriented surface of genus $g \geq 1$ with arbitrarily chosen b points (which we call punctures). Let $\mathrm{Mod}(\Sigma_{g,b})$ be the mapping class group of $\Sigma_{g,b}$, which is the group of homotopy classes of orientation-preserving homeomorphisms preserving the set of punctures. By $\mathrm{Mod}^0(\Sigma_{g,b})$ we will denote the subgroup of $\mathrm{Mod}(\Sigma_{g,b})$ which fixes the punctures pointwise.

The question of generating mapping class groups by involutions was first investigated by McCarthy and Papadopoulos (see [MP]). In [MP], they proved that $\mathrm{Mod}(\Sigma_{g,0})$ is generated by infinitely many conjugates of a single involution for $g \geq 3$. Luo described the finite set of involutions which generate $\mathrm{Mod}(\Sigma_{g,b})$ for $g \geq 3$ (see [Lu]). He also proved that $\mathrm{Mod}(\Sigma_{g,b})$ is generated by torsion elements in all cases except $g = 2$ and $b = 5k+4$, but this group is not generated by involutions if $g \leq 2$. Brendle and Farb proved that $\mathrm{Mod}(\Sigma_{g,b})$ can be generated by 6 involutions for $g \geq 3$, $b = 0$ and $g \geq 4$, $b \leq 1$ (see [BF]). In [Ka], Kassabov showed that $\mathrm{Mod}(\Sigma_{g,b})$ is generated by 4 involutions if $g > 7$ or $g = 7$ and b is even, 5 involutions if $g > 5$ or $g = 5$ and b is even, and 6 involutions if $g > 3$ or $g = 3$ and b is even.

We show that when b is odd and $g \geq 7$, $\mathrm{Mod}(\Sigma_{g,b})$ is generated by 4 involutions by improving two involutions which are constructed in Section 2 of [Ka]. Furthermore, by using the argument in Section 3.4 of [Ka] we show that when b is odd, $\mathrm{Mod}(\Sigma_{5,b})$ is generated by 5 involutions. We prove these results by the arguments similar to [Ka]. When we combine Kassabov's theorem with these results, we get the following results:

MAIN THEOREM. *For all* $b \geq 0$, $\mathrm{Mod}(\Sigma_{g,b})$ *is generated by*:

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(a) 4 involutions if $g \geq 7$,
(b) 5 involutions if $g \geq 5$.

2. Preliminaries

Let c be a simple closed curve on $\Sigma_{g,b}$. We will denote by T_c the (right handed) Dehn twists about the curve c.

We record the following lemmas.

LEMMA 1. *For all* $h \in \mathrm{Mod}(\Sigma_{g,b})$,

$$T_{h(c)} = hT_c h^{-1} .$$

LEMMA 2. *Let* c *and* d *be two simple closed curves on* $\Sigma_{g,b}$. *If* c *is disjoint from* d, *then*

$$T_c T_d = T_d T_c$$

It is clear that we have the exact sequence:

$$1 \to \mathrm{Mod}^0(\Sigma_{g,b}) \to \mathrm{Mod}(\Sigma_{g,b}) \xrightarrow{\pi} \mathrm{Sym}_b \to 1 .$$

Therefore, we see the following lemma;

LEMMA 3. *Let* H *denote a subgroup of* $\mathrm{Mod}(\Sigma_{g,b})$, *which contains* $\mathrm{Mod}^0(\Sigma_{g,b})$. *If* $\pi(H) = \mathrm{Sym}_b$, *then* H *is equal to* $\mathrm{Mod}(\Sigma_{g,b})$.

3. Proof of Main Theorem

Hereafter, we assume that $g \geq 5$, and that the number of punctures $b = 2l + 1$ is odd.

We will construct two involutions ρ_1, ρ_2 by modifying the involutions ρ_1, ρ_2 which are constructed in Section 2 of [Ka]. We note that we change the action of ρ_1, ρ_2 on punctures and swap the top parts of Figure 1 of [Ka].

Let us embed our surface $\Sigma_{g,b}$ in the Euclidean space in two different ways as shown on Figure 1. (In these pictures we will assume that genus $g = 2k + 1$. In the case of even genus we only have to swap the top parts of the pictures.) In Figure 1 we have also marked the punctures as $x_1, \ldots, x_b$ and we have the curves α_i, β_i, γ_i and δ. The curves α_i, β_i, γ_i are non separating curves and δ is a separating curve.

Let ρ_1 and ρ_2 denote the involutions which are rotation by π about the axises indicated in Figure 1. Then we get the following lemma;

LEMMA 4. *The subgroup of* $\mathrm{Mod}(\Sigma_{g,b})$ *be generated by* ρ_1, ρ_2 *and 3 Dehn twists* T_α, T_β *and* T_γ *around one of the curve in each family contains the subgroup* $\mathrm{Mod}^0(\Sigma_{g,b})$.

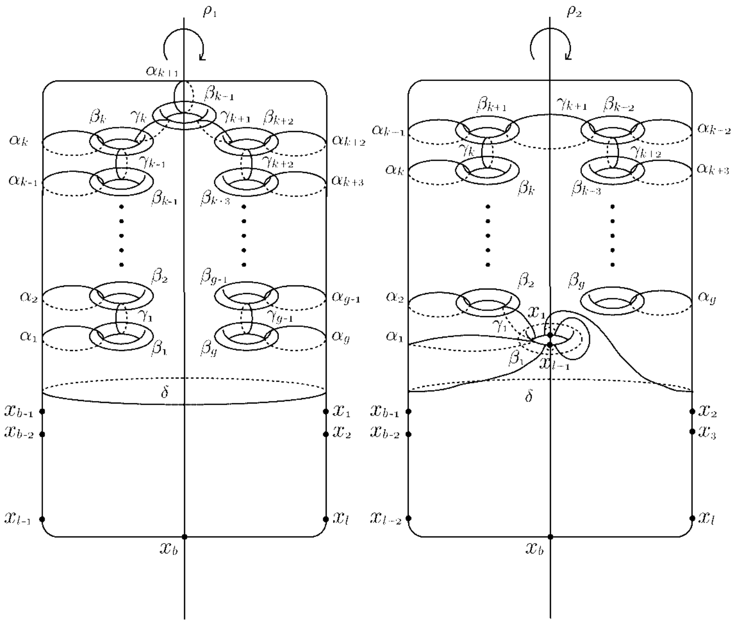

FIGURE 1. The embeddings of the surface $\Sigma_{g,b}$ in Euclidean space used to define the involutions ρ_1 and ρ_2.

We postpone the proof of lemma 4 until Section 4.

Let π be the homomorphism and H be the subgroup of $\mathrm{Mod}(\Sigma_{g,b})$ mentioned in Lemma 3. Showing the surjectivity of π from H to Sym_b is equivalent to showing that the Sym_b can be generated by involutions;

$$r_1 = (1, b-1)(2, b-2)\cdots(l, l+1)(b)$$
$$r_2 = (2, b-1)(3, b-2)\cdots(l, l+2)(1)(l+1)(b)$$
$$r_3 = (1, b)(2, b-1)(3, b-2)\cdots(l, l+2)(l+1)$$

corresponding to 3 involutions in H by π. The group generated by r_i contains the long cycle $r_3r_1 = (1, 2, \ldots, b)$ and transposition $r_3r_2 = (1, b)$. These two elements generate the whole symmetric group, therefore the involutions r_i $(i = 1, 2, 3)$ generate Sym_b. We note that the images of ρ_1 and ρ_2 to Sym_b are r_1 and r_2.

3.1. Generating Dehn twists by 4 involutions. By using the arguments similar to Section 3.4 of [Ka] we generate Dehn twists by 4 involutions.

We assume that $g \geq 5$.

Let $S_{0,4}$ be a surface of genus 0 with 4 boundary components. Denote by a_1, a_2, a_3 and a_4 the four boundary curves of the surface $S_{0,4}$ and let the interior curves y_1, y_2 and y_3 be as shown in Figure 2.

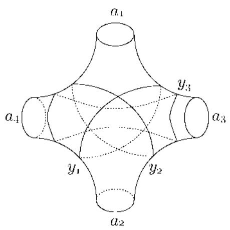

FIGURE 2. Lantern.

The lantern relation is the following relation:

$$T_{y_1}T_{y_2}T_{y_3} = T_{a_1}T_{a_2}T_{a_3}T_{a_4}\,. \tag{1}$$

Notice that the curves a_i do not intersect any other curve and that the Dehn twists T_{a_i} commute with every twists in this relation. Thus we have

$$T_{a_4} = (T_{y_1}T_{a_1}^{-1})(T_{y_2}T_{a_2}^{-1})(T_{y_3}T_{a_3}^{-1})\,. \tag{2}$$

Let R denote the product $\rho_2\rho_1$. By Figure 1 we can see that $R = \rho_2\rho_1$ acts as follows:

$$\begin{aligned} R\alpha_i &= \alpha_{i+1}\,, \quad (1 \le i < g)\\ R\beta_i &= \beta_{i+1}\,, \quad (1 \le i < g)\\ R\gamma_i &= \gamma_{i+1}\,, \quad (1 \le i < g-1). \end{aligned} \tag{3}$$

Let S be a lantern whose boundary components are a_1, a_2, a_3, a_4, and $R^{-2}S$ a lantern whose boundary components are $R^{-2}a_1$, $R^{-2}a_2$, $R^{-2}a_3$, $R^{-2}a_4$. We identify a_1 with $R^{-2}a_2$. Then we obtain a surface S_2 homeomorphic to a sphere with 6 boundary components.

By Figure 3 we see that there exists an involution $\bar{J}$ of S_2 which takes S to $R^{-2}S$. In [Ka] R^2 is used instead of R^{-2}, since g is even in [Ka].

Let us embed the surface S_2 in $\Sigma_{g,b}$ as shown on Figure 4. We note $a_1 = \alpha_{k+1}$, $a_2 = \alpha_{k+3}$, $a_3 = \gamma_{k+2}$, $a_4 = \gamma_{k+1}$, $R^{-2}a_1 = \alpha_{k-1}$, $R^{-2}a_2 = \alpha_{k+1}$, $R^{-2}a_3 = \gamma_k$, $R^{-2}a_4 = \gamma_{k-1}$ and $y_1 = \alpha_{k+2}$. Figure 4 shows the existence of the involution $\tilde{J}$ on the complement of S_2 which is a surface of genus $g-5$ with 6 boundary components. Gluing together $\bar{J}$ and $\tilde{J}$ gives us the involution J of $\Sigma_{g,b}$. By Figure 3 J acts as follows

$$J(a_1) = R^{-2}a_2\,, \quad J(a_3) = R^{-2}a_1\,, \quad J(y_1) = R^{-2}y_2\,, \quad J(y_3) = R^{-2}y_1\,.$$

Therefore, we have

$$\begin{aligned} R^2J(a_1) = a_2\,, &\quad R^2J(y_1) = y_2\\ JR^{-2}(a_1) = a_3\,, &\quad JR^{-2}(y_1) = y_3\,. \end{aligned} \tag{4}$$

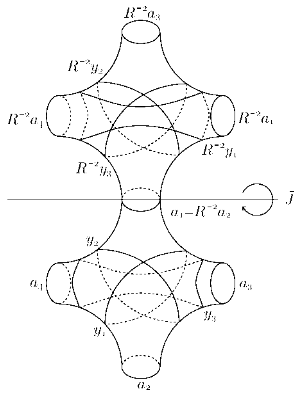

FIGURE 3. S_2 and the involution $\bar{J}$.

Let ρ_3 denote $T_{a_1}\rho_2 T_{a_1}^{-1}$. In [Ka] $T_{a_1}\rho_1 T_{a_1}^{-1}$ is used instead of $T_{a_1}\rho_2 T_{a_1}^{-1}$. By Lemma 1, the relation (4) and $\rho_2(a_1) = \rho_2(\alpha_{k+1}) = \alpha_{k+2} = y_1$, we have

$$\begin{aligned} T_{y_1}T_{a_1}^{-1} &= \rho_2 T_{a_1}\rho_2 T_{a_1}^{-1} = \rho_2\rho_3\,, \\ T_{y_2}T_{a_2}^{-1} &= R^2 J\rho_2\rho_3 J R^{-2}\,, \\ T_{y_3}T_{a_3}^{-1} &= J R^{-2}\rho_2\rho_3 R^2 J\,. \end{aligned} \tag{5}$$

By the relation (2) and (5) we have

$$T_{\gamma_{k+1}} = (\rho_2\rho_3)\,(R^2 J\rho_2\rho_3 J R^{-2})\,(J R^{-2}\rho_2\rho_3 R^2 J)\,. \tag{6}$$

3.2. In the case of genus 5. We assume that $g \geq 5$ and $b = 2l + 1$.

We proof that $\mathrm{Mod}(\Sigma_{g,b})$ is generated by 5 involutions.

The five involutions are ρ_1, ρ_2, ρ_3, J and another involution I which was constructed in Section 3.2 of [Ka]. We note that since we assume that g is odd, I maps α_{k+1} to β_{k+2}.

THEOREM 5. *If $g \geq 5$ and $b = 2l + 1$, the group G_1 generated by ρ_1, ρ_2, ρ_3, I and J is the whole mapping class group* $\mathrm{Mod}(\Sigma_{g,b})$.

PROOF. By the relation (6) we have $T_{\gamma_{k+1}} \in G_1$. Since $J(\alpha_{k-1}) = \gamma_{k+2}$ and $R(\gamma_{k+1}) = \gamma_{k+2}$, we see that $T_{\alpha_{k-1}} = JRT_{\gamma_{k+1}}R^{-1}J^{-1} \in G_1$. Moreover, since $R^2(\alpha_{k-1}) = \alpha_{k+1}$ and $I(\alpha_{k+1}) = \beta_{k+2}$, we have $T_{\beta_{k+2}} \in G_1$. By the construction of J, the image of J to Sym_b is r_3. We note that the images of ρ_1 and ρ_2 to Sym_b are r_1 and r_2. Therefore, there is the surjection from G_1 to Sym_b. By Lemma 3 and 4 we see that G_1 is equal to $\mathrm{Mod}(\Sigma_{g,b})$. □

3.3. In the case of genus 7. We assume that $g \geq 7$ and $b = 2l + 1$.

We will construct the involution J' which acts on the punctures as the involution r_3 by the method similar to Section 3.4 of [Ka]. We note that the action of J' on punctures is different

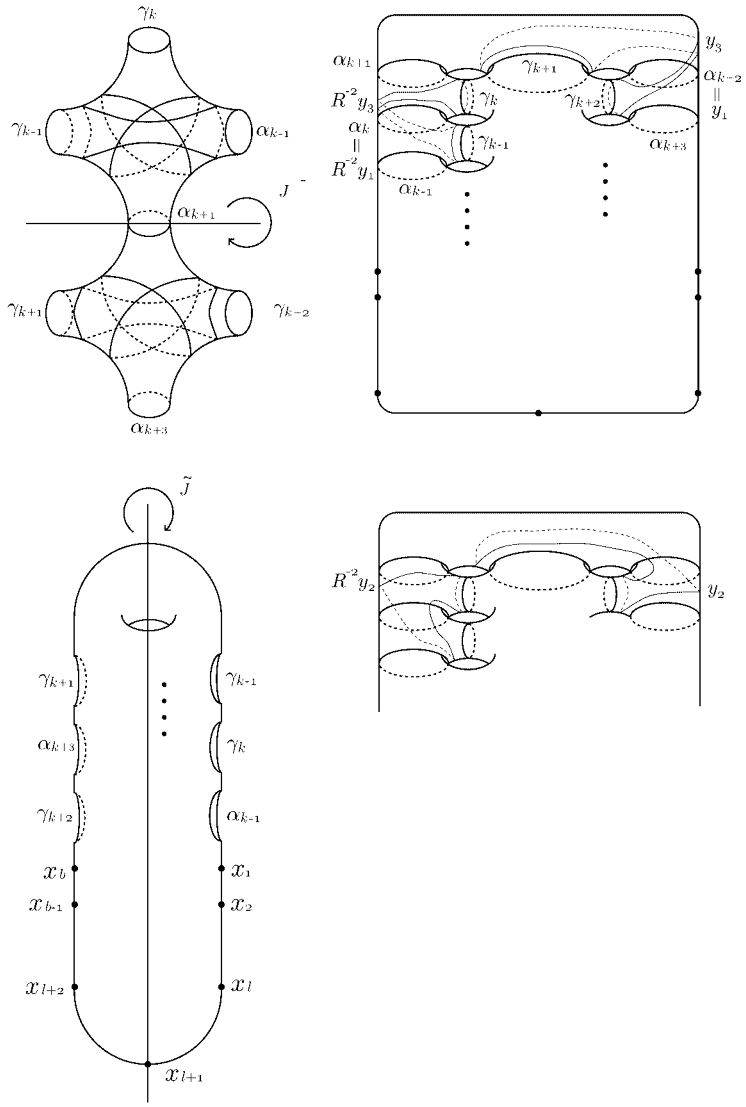

FIGURE 4. The involution J on $\Sigma_{g,b}$.

from that of J which is constructed in Section 3.4 of [Ka].

The S_2 and two pairs of pants have common boundary components $R^{-2}a_1$ and a_3 and their union is a surface S_3 homeomorphic to a sphere with 8 boundary components. Figure 5 shows the existence of the involution $\bar{J}'$ on S_3 which extends the involution $\bar{J}$ on S_2.

Let us embed S_3 in the $\Sigma_{g,b}$ as shown on Figure 5. We note that the embedding of S_2 is

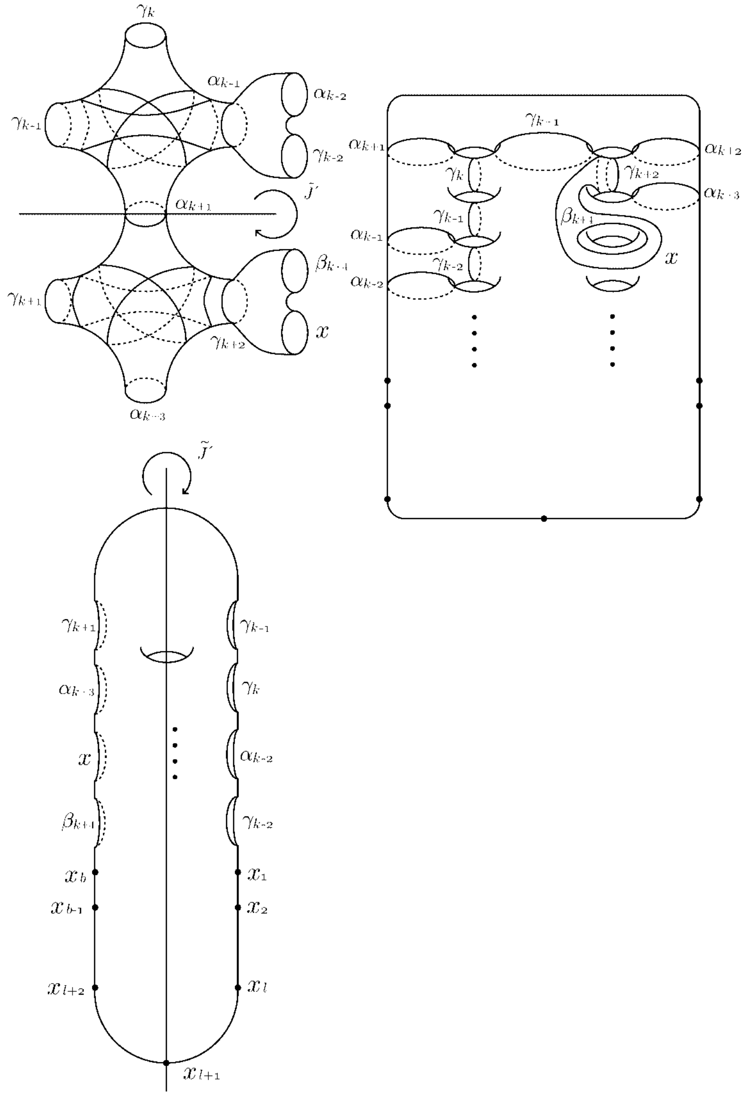

FIGURE 5. The involution J' on $\Sigma_{g,b}$.

similar to that of Section 3.1. From Figure 5 we can find the involution $\tilde{J}'$ of the complement of S_3. Let J' be the involution obtained by gluing together $\bar{J}'$ and $\tilde{J}'$. Moreover, from Figure 5 we find that J' acts on the punctures as the involution r_3.

THEOREM 6. *If $g \geq 7$ and $b = 2l + 1$, the group G_2 generated by ρ_1, ρ_2, ρ_3 and J' is* *the whole mapping class group* $\mathrm{Mod}(\Sigma_{g,b})$.

PROOF. The proof is the argument similar to Section 3.4 of [Ka]. We omit the proof. □

4. The subgroup generated by 2 involutions and 3 Dehn twists, which contains $\mathrm{Mod}^0(\Sigma_{g,b})$

In this section we prove Lemma 4.

Let the subgroup G of $\mathrm{Mod}(\Sigma_{g,b})$ be generated by ρ_1, ρ_2 and 3 Dehn twists T_α, T_β and T_γ around one of the curve in each family. We will show that G contains $\mathrm{Mod}^0(\Sigma_{g,b})$. Let δ', η', δ'', η'', δ_j, η_j ($j = 1, \dots, l-1, l+1, \dots, b-2$) be the curves illustrated in Figure 6. In [Ge] it is shown that $\mathrm{Mod}^0(\Sigma_{g,b})$ is generated by Dehn twists about the curves α_i-es, β_i-es, γ_i-es, δ', δ'' and δ_j-es, for $j = 1, \dots, l-1, l+1, \dots, b-2$.

We recall that $R = \rho_2\rho_1$. By Lemma 1 and the relation (3) we see that $T_{\alpha_i}, T_{\beta_i}, T_{\gamma_i} \in G$ for all i.

From the action of ρ_1 and ρ_2 we can find that $R^{-1}(\delta_j) = \eta_{j-1}$ ($l+2 \le j \le b-1$) and $R^{-1}(\delta_{l+1}) = \eta'$.

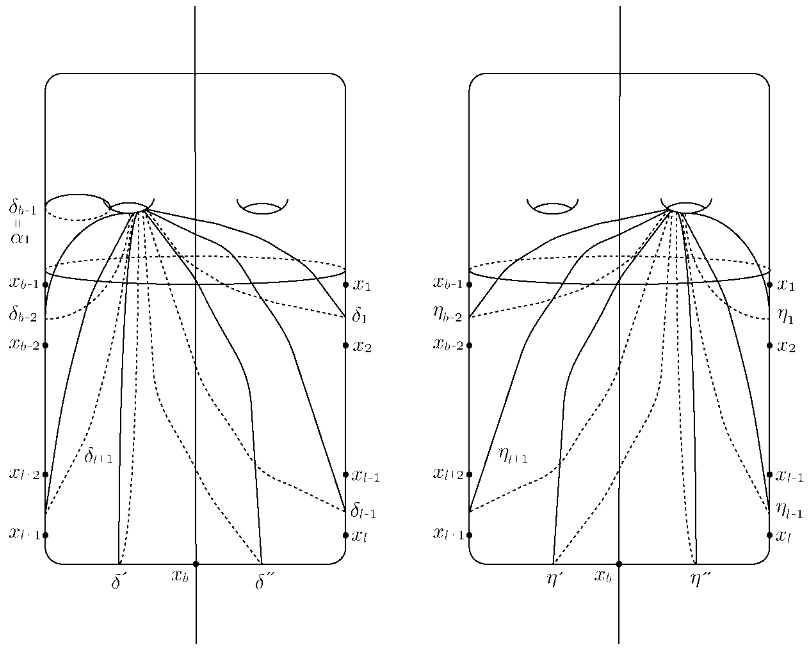

FIGURE 6. The curves δ_i-es, η_i-es.

LEMMA 7. $T_{\delta_j}, T_{\delta'}, T_{\delta''} \in G$ $(j = 1, \dots, l-1, l+1, \dots, b-2)$.

PROOF. We will prove $T_{\delta_j} \in G$ $(j = l+1, \dots, b-1)$ by induction on j and $T_{\delta'} \in G$.

The base case, $j = b-1$, is clear because G contains $T_{\delta_{b-1}} = T_{\alpha_1}$. Suppose that G contains the twist T_{δ_j}. By $R^{-1}(\delta_j) = \eta_{j-1}$ we have

$$T_{\eta_{j-1}} = R^{-1} T_{\delta_j} R \in G\,.$$

Let $U \in G$ denote the product

$$U = T_{\beta_1}^{-1} T_{\gamma_1}^{-1} T_{\beta_2}^{-1} \cdots T_{\beta_{g-1}}^{-1} T_{\gamma_{g-1}}^{-1} T_{\beta_g}^{-1} T_{\alpha_g}^{-1} T_{\alpha_1} T_{\beta_1} T_{\gamma_1} T_{\beta_2} \cdots T_{\beta_{g-1}} T_{\gamma_{g-1}} T_{\beta_g}\,.$$

We find that

$$\begin{aligned} U(\eta') &= \delta' \\ U(\eta'') &= \delta'' \\ U(\eta_j) &= \delta_j \quad (j = 1, \dots, l-1, l+1, \dots, b-2)\,. \end{aligned} \tag{7}$$

Therefore, we see that $T_{\delta_{j-1}} = U T_{\eta_{j-1}} U^{-1} \in G$ $(j = l+2, \dots, b-1)$. Moreover, since $R^{-1}(\delta_{l+1}) = \eta'$ and $U(\eta') = \delta'$, we have that $T_{\delta'} \in G$.

We will prove that $T_{\delta''}, T_{\delta_j} \in G$ $(j = 1, \dots, l-1)$.

By Figure 6, we find that $\rho_1(\delta'') = \eta'$, $\rho_1(\delta_j) = \eta_{b-1-j}$ $(1 \le j \le l-1)$. Therefore, we see that $T_{\delta_j} = \rho_1^{-1} T_{\eta_{b-1-j}} \rho_1$, $T_{\delta''} = \rho_1^{-1} T_{\eta'} \rho_1 \in G$. We finished proving Lemma 7. □

COROLLARY 8. *The group G contains the subgroup* $\mathrm{Mod}^0(\Sigma_{g,b})$.

Therefore, we can prove Lemma 4.

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References

[BF] T. E. BRENDLE and B. FARB, Every mapping class group is generated by 3 torsion elements and by 6 involutions, J. Algebra **278** (2004), 187–198.

[Ge] S. GERVAIS, A finite presentation of the mapping class group of a punctured surface, Topology **40** (2001), No. 4, 703–725.

[Ka] M. KASSABOV, Generating Mapping Class Groups by Involutions, arXiv:math.GT/0311455 v1 25 Nov 2003.

[Lu] N. LU, On the mapping class groups of the closed orientable surfaces, Topology Proc. **13** (1988), 293–324.

[MP] J. MACCARTHY and A. PAPADOPOULOS, Involutions in surface mapping class groups, Enseign. Math. **33** (1987), 275–290.

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