TOKYO J. MATH.
VOL. 3, NO. 1, 1980

On closed subalgebras between A and H^∞

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Introduction

Let D, $\bar{D}$, H^∞ and A be respectively the open unit disc in the complex plane C, the closed unit disc, the algebra of all bounded analytic functions on D and the algebra of all continuous functions on $\bar{D}$ which are analytic in D. The norm of a function f in H^∞ or A is the supremum of $|f|$ on D. Let $L^\infty(T)$ be the algebra of all essentially bounded, Lebesgue measurable functions on the unit circle T in C. The norm of a function f in $L^\infty(T)$ is the essential supremum of $|f|$ on T. We can regard H^∞ as a closed subalgebra of $L^\infty(T)$ by considering the radial limit of a function of H^∞, due to the Fatou's theorem. A convenient reference book for the basic facts about these algebras in Hoffman [2].

It is well known that the Douglas' conjecture about the closed subalgebras between H^∞ and $L^\infty(T)$ was solved affirmatively by Chang [1] and Marshall [2] showing that every closed subalgebra of $L^\infty(T)$ containing H^∞ is unequely determined by its maximal ideal space. However, when we have an intention to characterize the closed subalgebras of H^∞ containing A, the role of the maximal ideal space is no longer so definitive as in the case of the Douglas algebras in $L^\infty(T)$, because of the existence of the two closed subalgebras with the same maximal ideal space.

In this paper we shall construct these algebras, which seem to be never known before. The method of this construction is essentially due to Scheinberg [4]. In [5] we shall show further results, i.e., (i) these algebras have the same Silov-boundary which is coincident with the Silov-boundary of H^∞. (ii) These algebras are not log-modular in $L^\infty(T)$. (iii) Each unit-ball of these algebras is the closed convex hull of its Blaschke products.

Received November 22, 1979

§1. Construction.

For notational convenience, we shall often regard H^∞ as an algebra of continuous functions on the maximal ideal space of H^∞, denoted by $M(H^\infty)$, and we shall as usual denote M_λ the set of those maximal ideals which contain the function $z-\lambda$ for λ in T called a fiber of $M(H^\infty)$ over λ.

By the separability of $\bar{D}$, there exists a countable dense subset, denoted by $\{x_i\}_{i\in N}$ of $\bar{D}$. Let α be a mapping from N into $\bar{D}$ defined by $\alpha(n)=x_n$. By the Čech-compactification βN of N, α can be extended to a unique continuous mapping from βN onto $\bar{D}$, which we also denote by α. Further we know by the compactness of $\beta N-N$, that α maps the growth, $N^*=\beta N-N$, onto $\bar{D}$.

Fix an interpolating sequence $\{z_n\}$, for H^∞, which converges to 1 and identify n in N with z_n in D. We obtain an embedding of βN into $M(H^\infty)$, by which N^* corresponds to a closed subset Y of the fiber M_1. Thus a continuous mapping γ on Y onto $\bar{D}$ is defined and a mapping γ^* of $C(\bar{D})$ into $C(Y)$ is defined naturally by $\gamma^*(f)=f\circ\gamma$. It is clear that γ^* is isometric. We denote by ρ the restriction mapping of H^∞ into $C(Y)$, which is norm-decreasing and which maps H^∞ onto $C(Y)$ because $\{z_n\}$ is interpolating. Define B_A and B_C to be $\rho^{-1}\circ\gamma^*(A)$ and $\rho^{-1}\circ\gamma^*(C(\bar{D}))$ respectively.

We shall show that these subsets B_A and B_C are closed subalgebras satisfying that $A\subsetneqq B_A\subsetneqq B_C\subsetneqq H^\infty$. Since the mapping $(\gamma^*)^{-1}\circ\rho$ is norm-decreasing from B_A (resp. B_C) onto A (resp. $C(\bar{D})$), the completeness of A (resp. $C(\bar{D})$) implies the completeness of B_A (resp. B_C). The image of A by the mapping $(\gamma^*)^{-1}\circ\rho$ is the set of constant functions which is contained properly in A. Therefore B_A contains properly A. The relation $B_A\subsetneqq B_C$ follows immediately from $A\subsetneqq C(\bar{D})$. Since Y is totally disconnected, γ is not injective. Hence $B_C\subsetneqq H^\infty$. Thus we obtain two algebras, closed in H^∞, in which A is contained properly.

Now it has only to show $M(B_A)=M(B_C)$. Let I be the set of all functions of H^∞ which vanish identically on Y. Then I is a closed ideal of H^∞ and at the same time an ideal of B_C and of B_A. Note that I is the kernel of the mapping $(\gamma^*)^{-1}\circ\rho$. Thus we have two isomorphisms;

$$(*)\qquad A\cong B_A/I\,,\qquad C(\bar{D})\cong B_C/I$$

For any φ in $M(B_A)$, either (i) $\varphi^{-1}(0)\supset I$, or (ii) $\varphi^{-1}(0)\not\supset I$. In the first case, a complex homomorphism $\tilde{\varphi}$ from A into C is well-defined by $\varphi=\tilde{\varphi}\circ\psi$, due to (*), where ψ is the canonical mapping of B_A onto B_A/I. It is well known that $\tilde{\varphi}$ corresponds with a suitable point ζ of $\bar{D}$.

Consequently for any f in B_A holds $\varphi(f) = ((\gamma^*)^{-1}\circ\rho(f))(\zeta)$. In the second case, there is an element f in I such that $\varphi(f)\neq 0$. We may assume $\varphi(f)=1$ and define a mapping $\hat{\varphi}_f$ from H^∞ into C by $\hat{\varphi}_f(g)=\varphi(fg)$. This mapping $\hat{\varphi}_f$ is well defined because of $H^\infty\cdot f\subset I$. Further $\hat{\varphi}_f$ is a complex homomorphism. In fact, for elements g and h of H^∞, $\hat{\varphi}_f(gh)=\varphi(fgh)=\varphi(f)\cdot\varphi(fgh)=\varphi(fg)\cdot\varphi(fh)=\hat{\varphi}_f(g)\cdot\hat{\varphi}_f(h)$. So $\hat{\varphi}_f$ is an element of $M(H^\infty)$ which is an extension of φ of $M(B_A)$. Note that $\hat{\varphi}_f$ is in $M(H^\infty)-Y$.

One can get the same result as above if one changes B_A to B_C.

To any element φ of $M(B_A)$, in the case of (i), we correspond an element Φ of $M(B_C)$ defined by $\Phi(g)=((\gamma^*)^{-1}\circ\rho(g))(\zeta)$, where ζ is a point of $\bar{D}$ induced by φ, and in the case of (ii), we correspond an element Φ defined by $\Phi(g)=\hat{\varphi}_f(g)$. In either case Φ is always an extension of φ. Hence $M(B_A)$ is injectively mapped into $M(B_C)$. On the other hand, the restriction mapping of any element Φ of $M(B_C)$ to B_A, denoted by $\Phi|_{B_A}$, is also injective. In fact, for two different elements Φ_1 and Φ_2 of $M(B_C)$, there are three cases;

(a) $\Phi_1^{-1}(0)\supset I$ and $\Phi_2^{-1}(0)\supset I$,

(b) $\Phi_1^{-1}(0)\not\supset I$ and $\Phi_2^{-1}(0)\supset I$,

(c) $\Phi_1^{-1}(0)\not\supset I$ and $\Phi_2^{-1}(0)\not\supset I$.

In the case of (a), to Φ_1 and Φ_2, two different points ζ_1 and ζ_2 in $\bar{D}$ correspond respectively; for any g in B_C, we have $\Phi_i(g)=((\gamma^*)^{-1}\circ\rho(g))(\zeta_i)$, $i=1$ or 2. There is a function f in A such that $f(\zeta_1)\neq f(\zeta_2)$. We can find a function $\tilde{f}$ in B_A with $f=(\gamma^*)^{-1}\circ\rho(\tilde{f})$. For this function $\tilde{f}$ holds $\Phi_1(\tilde{f})\neq\Phi_2(\tilde{f})$. Hence $\Phi_1|_{B_A}\neq\Phi_2|_{B_A}$. In the case of (b), there is a function f in I such that $\Phi_1(f)\neq 0$ and $\Phi_2(f)=0$. Because of $I\subset B_A$, we have $\Phi_1|_{B_A}\neq\Phi_2|_{B_A}$. In the case of (c), there is a function f in I with $\Phi_1(f)\neq 0$. If $\Phi_1(f)\neq\Phi_2(f)$, this case is reduced to (b). Thus we have only to consider the case where $\Phi_1(f)=\Phi_2(f)$. Since $\Phi_1\neq\Phi_2$, there is a function g in B_C with $\Phi_1(g)\neq\Phi_2(g)$. The function fg is in I and satisfies that $\Phi_1(fg)\neq\Phi_2(fg)$, whence $\Phi_1|_{B_A}\neq\Phi_2|_{B_A}$. Consequently we have $M(B_A)=M(B_C)$.

§2. Remark.

When we study the properties of subalgebras between A and H^∞, the method in §1 seems to be useful. The essential parts of this construction is as follows; first, we fix a separable compact Hausdorff space X and have two different uniform algebras B_1 and B_2 on X ready beforehand, with some convenient relations or properties. Next we fix an interpolating sequence $\{z_n\}$ in D, converging to a point of T, and fix a mapping from the growth, i.e., $Y=\beta N-N$, of the Čech-compactification of $\{z_n\}$ onto X.

Then we can construct two subalgebras $\tilde{B}_1$ and $\tilde{B}_2$ in the same way as in §1, satisfying with $A \subsetneqq \tilde{B}_1$, $\tilde{B}_2 \subset H^\infty$ and that the given relations between B_1 and B_2 are translated into relations between $\tilde{B}_1$ and $\tilde{B}_2$. In §1, we take X for the closed unit disc $\bar{D}$, B_1 for the disc algebra A and B_2 for $C(\bar{D})$. By using that $C \subsetneqq A \subsetneqq C(\bar{D})$ and that $M(A) = M(C(\bar{D})) = \bar{D}$, we obtain the subalgebras B_A and B_C with $A \subsetneqq B_A \subsetneqq B_C \subsetneqq H^\infty$ and $M(B_A) = M(B_C)$.

If we take X for the unit circle T, B_1 for the disc algebra A restricted to T and B_2 for $C(T)$, then, since by the Wermer's maximality theorem, *there is no proper closed subalgebra between A and $C(T)$*, we can obtain closed subalgebras B'_A and B'_C with $A \subsetneqq B'_A \subsetneqq B'_C \subsetneqq H^\infty$, between which there is no proper closed algebra.

If we take X for the topological boundary of an open ball or polydisc, denoted by O, in C^n, and B for the algebra of all continuous functions on $O \cup X$, analytic in O, then we obtain the closed subalgebra $\tilde{B}$ of H^∞ containing properly A, for which the corona theorem fails. This fact is due to Scheinberg [4].

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