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On the Propagation of Chaos for Diffusion Processes with Drift Coefficients Not of Average Form

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Introduction

Let $b^{(n)}[x, u]$ be $\boldsymbol{R}^d$-valued measurable functions defined on $\boldsymbol{R}^d \times \mathscr{P}(\boldsymbol{R}^d)$, where $\mathscr{P}(\boldsymbol{R}^d)$ denotes the space of probability distributions on $\boldsymbol{R}^d$. We consider interacting diffusion processes on $\boldsymbol{R}^d$ described by a system of stochastic differential equations

$$(1) \qquad X_i^{(n)}(t) = X_i^{(n)}(0) + B_i(t) + \int_0^t b^{(n)}[X_i^{(n)}(s), U^{(n)}(s)]ds , \qquad i=1, 2, \cdots, n ,$$

where $U^{(n)}(t) = (1/n) \sum_{i=1}^n \delta_{X_i^{(n)}(t)}$ is the empirical distribution of $(X_1^{(n)}(t), \cdots, X_n^{(n)}(t))$ and $B_i(t)$, $i=1, 2, \cdots, n$, are mutually independent d-dimensional Brownian motions. The initial value $(X_1^{(n)}(0), \cdots, X_n^{(n)}(0))$ is always assumed to be independent of the Brownian motions.

Assuming that the law of large numbers $U^{(n)}(\mathrm{t}) \to u(t)$ and $b^{(n)}[X_1^{(n)}(t), U^{(n)}(t)] \to b[X(t), u(t)]$ hold as $n \to \infty$ and taking the limit formally in (1) for $i=1$, we get the McKean-Vlasov's SDE

$$(2) \qquad X(t) = X(0) + B(t) + \int_0^t b[X(s), u(s)]ds ,$$

where $u(t)$ is the probability distribution of $X(t)$.

The propagation of chaos for the diffusion processes $(X_1^{(n)}(t), \cdots, X_n^{(n)}(t))$ given by (1) states as follows: If the sequence of the initial distributions in (1) is a symmetric u-chaotic family (see §2), then the sequence of the distributions of $(X_1^{(n)}(t), \cdots, X_n^{(n)}(t))$ is also a symmetric $u(t)$-chaotic family, where $u(t)$ is the probability distribution of $X(t)$ in (2) with a u-distributed initial value $X(0)$.

When the drift coefficient $b[x, u] = b^{(n)}[x, u]$, $n \geqq 1$, is of average (or integral) form defined by

$$(3) \qquad b[x, u] = \int b(x, y)u(dy) ,$$

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where $b(x, y)$ is Lipschitz continuous, McKean proved the propagation of chaos in 1967 [3] (see also [1], [9], [10], [11], [12], [13]).

In the investigation of interacting diffusion particles of two segregated groups [8] we have encountered such a system of stochastic differential equations as type (1) with drift coefficients not of average form. Our aim in [8] was to prove the propagation of chaos for a system of two segregated groups and we did this without proving directly the propagation of chaos for SDE's (1). However, if we can prove the propagation of chaos for (1) with drift coefficients which are not of average form, then the propagation of chaos for a system of two segregated groups follows immediately from the lemmas on order statistics in [8]. The purpose of this paper is to give a proof of the propagation of chaos for (1) under certain conditions on $b^{(n)}[x, u]$ which is not of average form; these conditions are satisfied in the case of [8] and consequently the present result yields another proof of Theorem 3 in [8]. Oelschräger [9] discussed the case where both diffusion and drift coefficients are not of average form. However, his assumption on the coefficients is not satisfied in the case of [8].

§1. Chaotic family of symmetric probability distributions.

Let $S=\boldsymbol{R}^d$, $\mathscr{P}(S)$ be the space of all probability distributions on S, and $\mathscr{P}(S^n)$ the space of all symmetric probability distributions on the n-fold product space S^n.

DEFINITION 1. Let $u\in\mathscr{P}(S)$ and $u_n\in\mathscr{P}(S^n)$. $\{u_n; n=1, 2, \cdots\}$ is called a *symmetric u-chaotic family* if, for arbitrary but fixed m and any $f_1, \cdots, f_m\in C_b(S)$,

$$(4)\qquad \lim_{n\to\infty}\langle u_n, f_1\otimes f_2\otimes\cdots\otimes f_m\otimes\underbrace{1\otimes\cdots\otimes 1}_{n-m}\rangle=\prod_{k=1}^{m}\langle u, f_k\rangle\ ,$$

where $C_b(S)$ is the space of bounded continuous functions on S and $\langle u, f\rangle$ denotes the integral of f with respect to the measure u.

LEMMA 1. *Let $u_n\in\mathscr{P}(S^n)$. $\{u_n; n=1, 2, \cdots\}$ is a symmetric u-chaotic family if and only if*

$$(5)\qquad u_n\left[\left\{(x_1, \cdots, x_n)\,;\left|\frac{1}{n}\sum_{i=1}^{n}f(x_i)-\langle u, f\rangle\right|>\varepsilon\right\}\right]\to 0\ ,$$

as $n\to\infty$, for any $\varepsilon>0$ and $f\in C_b(S)$.

PROOF. Let $(X_1, \cdots, X_n)$ be a u_n-distributed random variable. Assume first that $\{u_n\}$ is a symmetric u-chaotic family. Then

$$E\left[\left|\frac{1}{n}\sum_{i=1}^{n} f(X_i)-\langle u, f\rangle\right|^2\right]=\frac{1}{n}E[|f(X_1)-\langle u, f\rangle|^2]$$
$$+\frac{n-1}{n}E[(f(X_1)-\langle u, f\rangle)(f(X_2)-\langle u, f\rangle)]\,,$$

which converges to zero as $n\to\infty$. Therefore (5) follows from Chebischeff inequality. Conversely, assume (5). Then

$$\langle u_n, f_1\otimes\cdots\otimes f_m\otimes\underbrace{1\otimes\cdots\otimes 1}_{n-m}\rangle=E\left[\prod_{i=1}^{m} f_i(X_i)\right]$$
$$=\frac{1}{n!}\sum_{\sigma} E\left[\prod_{i=1}^{m} f_i(X_{\sigma(i)})\right]\,,$$

where the summation runs over all permutations σ of $\{1, 2, \cdots, n\}$. As is in the proof of Lemma 4 in [8], this is equal to

$$\frac{(n-m)!}{n!}\sum_{i_1=1}^{n}\cdots\sum_{i_m=1}^{n} E\left[\prod_{k=1}^{m} f_k(X_{i_k})\right]+o(1)$$
$$=\frac{n^m}{n(m-1)\cdots(n-m+1)}E\left[\prod_{k=1}^{m}\frac{1}{n}\sum_{i=1}^{n} f_k(X_i)\right]+o(1)\,,$$

which converges to $\prod_{k=1}^{m}\langle u, f_k\rangle$ as $n\to\infty$. This completes the proof of Lemma 1.

DEFINITION 2. Let C_R be the space of radial functions g's of the form

$$g(x)=f(|x|)\qquad \text{for } \forall x\in S\,,$$

where f's are of $C^\infty(\boldsymbol{R}^+)$, nonnegative, nondecreasing, constant in neighbourhoods of the origin, and $\lim_{r\to\infty} f(r)=\infty$.

It is clear that if $g\in C_R$ and $c>0$, then

$$P_{g,c}=\{u\in\mathscr{P}(S)\,;\,\langle u, g\rangle\leqq c\} \tag{6}$$

is a compact subset of $\mathscr{P}(S)$ in the topology of the weak convergence if $P_{g,c}\neq\varnothing$.

LEMMA 2. *For a symmetric u-chaotic family* $\{u_n;\ n=1, 2, \cdots\}$ *there exists* $g_0\in C_R$ *such that*

$$\sup_n\langle u_n, g_0\otimes\underbrace{1\otimes\cdots\otimes 1}_{n-1}\rangle<\infty\,, \tag{7}$$

and (5) *holds with the* g_0 *in place of* f, *i.e.*,

$$(8)\qquad u_n\left[\left\{(x_1, \cdots, x_n)\,;\,\left|\frac{1}{n}\sum_{i=1}^{n} g_0(x_i)-\langle u, g_0\rangle\right|>\varepsilon\right\}\right]\to 0\;,$$

as $n\to\infty$, *for any* $\varepsilon>0$.

PROOF. Taking nondecreasing functions $\psi_k\in C^\infty(\boldsymbol{R}^+)$ such that

$$\psi_k(r)=\begin{cases}0 & \text{for}\quad 0\leqq r\leqq k\;,\\ 1 & \text{for}\quad k+1\leqq r\;,\end{cases}$$

set

$$g_k(x)=\psi_k(|x|)\;,\qquad k=1, 2, \cdots\;.$$

Then, since

$$\sup_n\langle u_n, g_k\otimes\underbrace{1\otimes\cdots\otimes 1}_{n-1}\rangle\leqq\sup_n\int 1_{\{|x_1|>k\}}u_n(dx)\to 0\qquad\text{as}\quad k\to\infty\;,$$

we can choose a subsequence $k_1<k_2<\cdots$ such that

$$(9)\qquad \sup_n\langle u_n, g_{k_j}\otimes\underbrace{1\otimes\cdots\otimes 1}_{n-1}\rangle\leqq 2^{-j}\;,\qquad j=1, 2, \cdots\;,$$

and then define

$$(10)\qquad g_0=\sum_{j=1}^{\infty} g_{k_j}\;.$$

We can and do choose g_{k_j} so that

$$(11)\qquad \nabla g_0\quad\text{and}\quad\Delta g_0\quad\text{are bounded}\;.$$

It is clear that $g_0\in C_R$ and satisfies (7) because of (9). To show (8) we divide it into two parts

$$\begin{aligned}
&u_n\left[\left\{(x_1, \cdots, x_n)\,;\,\left|\frac{1}{n}\sum_{i=1}^{n} g_0(x_i)-\langle u, g_0\rangle\right|>\varepsilon\right\}\right]\\
&\leqq u_n\left[\left\{(x_1, \cdots, x_n)\,;\,\left|\sum_{j=1}^{N}\left(\frac{1}{n}\sum_{i=1}^{n} g_{k_j}(x_i)-\langle u, g_{k_j}\rangle\right)\right|>\frac{\varepsilon}{2}\right\}\right]\\
&\quad+u_n\left[\left\{(x_1, \cdots, x_n)\,;\,\left|\sum_{j=N+1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n} g_{k_j}(x_i)-\langle u, g_{k_j}\rangle\right)\right|>\frac{\varepsilon}{2}\right\}\right]\\
&\leqq\sum_{j=1}^{N} u_n\left[\left\{(x_1, \cdots, x_n)\,;\,\left|\frac{1}{n}\sum_{i=1}^{n} g_{k_j}(x_i)-\langle u, g_{k_j}\rangle\right|>\frac{\varepsilon}{2N}\right\}\right]\\
&\quad+\frac{2}{\varepsilon}\sum_{j=N+1}^{\infty}(\langle u_n, g_{k_j}\otimes 1\otimes\cdots\otimes 1\rangle+\langle u, g_{k_j}\rangle)\;,
\end{aligned}$$

where the second summation is bounded by $(4/\varepsilon)\sum_{j=N+1}^{\infty}2^{-j}$, uniformly in n, and for a fixed N the first summation converges to zero as $n\to\infty$ by (5) of Lemma 1, because $\{u_n;\ n=1, 2, \cdots\}$ is a symmetric u-chaotic family and $g_{k_j}\in C_b(S)$. Therefore

$$\overline{\lim_{n\to\infty}}\, u_n\Big[\Big\{(x_1, \cdots, x_n)\ ;\ \Big|\frac{1}{n}\sum_{i=1}^{n} g_0(x_i)-\langle u, g_0\rangle\Big|>\varepsilon\Big\}\Big]$$
$$\leqq\frac{4}{\varepsilon}\sum_{j=N+1}^{\infty}2^{-j}\qquad \text{for } \forall N\,.$$

Thus (8) holds. This completes the proof of Lemma 2.

§2. A tightness result.

In $\mathscr{P}(S)$ let us define a metric which is equivalent to the weak convergence by

$$\rho(u, v)=\sum_{k=1}^{\infty}2^{-k}|\langle u-v, f_k\rangle|\ , \tag{12}$$

where $\{f_k;\ k=1, 2, \cdots\}$ is a sequence in $C_K^\infty(S)$ such that (i) the linear hull of $\{f_k;\ k=1, 2, \cdots\}$ is dense in $C_K^\infty(S)$ and (ii) $\|f_k\|$, $\|\nabla f_k\|$, $\|\Delta f_k\|\leqq 1$ for $\forall k\geqq 1$, where $C_K^\infty(S)$ denotes the space of C^∞-functions with compact supports. $\|\cdot\|$ denotes the supremum norm.

Let $b^{(n)}[x, u]$ be S-valued measurable functions on $S\times\mathscr{P}(S)$ and $A_u^{(n)}$, $u\in\mathscr{P}(S)$, be defined by

$$A_u^{(n)}f=\frac{1}{2}\Delta f+b^{(n)}[\cdot, u]\cdot\nabla f\qquad \text{for}\quad f\in C_K^\infty(S)\,. \tag{13}$$

Lemma 3. *Let $U^{(n)}(t)$ be the empirical distribution of $(X_1^{(n)}(t), \cdots, X_n^{(n)}(t))$ which satisfies the SDE (1) with $b^{(n)}[\cdot, u]$ which are bounded uniformly in n and with an initial distribution u_n, where $\{u_n;\ n=1, 2, \cdots\}$ is a symmetric u-chaotic family. Then for $T>0$*

$$\lim_{c\to\infty}\inf_{n} P[U^{(n)}(t)\in P_{g_0,c},\ 0\leqq t\leqq T]=1\ , \tag{14}$$

where g_0 is given in (10).

Proof. The property (14) is equivalent to

$$P[\max_{0\leqq t\leqq T}\langle U^{(n)}(t), g_0\rangle>c]\to 0\ ,\qquad \text{as}\quad c\to\infty\ , \tag{15}$$

uniformly in n. An application of Itô's formula yields

$$\langle U^{(n)}(t), g_0\rangle = \langle U^{(n)}(0), g_0\rangle + \frac{1}{n}\sum_{i=1}^{n}\int_0^t \nabla g_0(X_i^{(n)}(s))\cdot dB_i(s) + \int_0^t \langle U^{(n)}(s), A^{(n)}_{U^{(n)}(s)} g_0\rangle ds\ , \tag{16}$$

where ∇g_0 and Δg_0 are bounded because of (11). Moreover, since $\{u_n; n=1, 2, \cdots\}$ is a symmetric u-chaotic family, we have

$$\frac{1}{n}\sum_{i=1}^{n} g_0(X_i^{(n)}(0)) \to \langle u, g_0\rangle$$

in probability by Lemma 2. Therefore (15) follows from (16). This completes the proof of Lemma 3.

LEMMA 4. *Under the same assumption and notation as in Lemma* 3,

$$E[\rho(U^{(n)}(s), U^{(n)}(t))^4] \leqq const.\,|t-s|^2 \qquad for \quad 0\leqq s\leqq t\leqq T\,. \tag{17}$$

PROOF. By Itô's formula we have for any $f\in C_K^\infty(S)$

$$|\langle U^{(n)}(t)-U^{(n)}(s), f\rangle|^4 \leqq 8\left|\frac{1}{n}\sum_{i=1}^{n}\int_s^t \nabla f(X_i^{(n)}(s))\cdot dB_i(s)\right|^4 + 8\left|\int_s^t \langle U^{(n)}(r), A^{(n)}_{U^{(n)}(r)} f\rangle dr\right|^4$$

and hence

$$\begin{aligned} E[\rho(U^{(n)}(t), U^{(n)}(s))^4] &\leqq E\left[\left\{\sum_{k=1}^{\infty} 2^{-k}\,|\langle U^{(n)}(t)-U^{(n)}(s), f_k\rangle|\right\}^4\right] \\ &\leqq \sum_{k=1}^{\infty} 2^{-k} E[|\langle U^{(n)}(t)-U^{(n)}(s), f_k\rangle|^4] \\ &\leqq \text{const.}\,|t-s|^2\ , \end{aligned}$$

completing the proof.

Combining Lemma 3 with Lemma 4, we have

LEMMA 5. *Let Q_n be the probability measure on $C([0, T], P(S))$ induced by the process $\{U^{(n)}(t); 0\leqq t\leqq T\}$. Then $\{Q_n; n=1, 2, \cdots\}$ is tight.*

§3. The propagation of chaos.

We prove the propagation of chaos for the diffusion processes of (1) under the following three conditions.

CONDITION B. $b[x, u]$ and $b^{(n)}[x, u]$ are S-valued measurable functions on $S\times\mathscr{P}(S)$ which are bounded uniformly in n.

CONDITION C. If $u_n \in \mathscr{P}(S)$ converges weakly to $u \in \mathscr{P}(S)$ which has a strictly positive density (almost everywhere) with respect to the Lebesgue measure of S, then $\langle u_n, b^{(n)}[\cdot, u_n]f\rangle$ converges to $\langle u, b[\cdot, u]f\rangle$ as $n \to \infty$ for any $f \in C_b(S)$.

CONDITION U. The uniqueness holds for $\mathscr{P}(S)$-valued solution of the initial value problem

$$\begin{aligned} &\frac{d}{dt}\langle u(t), f\rangle = \langle u(t), A_{u(t)}f\rangle \qquad \text{for} \quad \forall f \in C_K^\infty(S)\ , \\ &u(0) = u \in \mathscr{P}(S)\ , \end{aligned} \tag{18}$$

where $A_u f = (1/2)\Delta f + b[\cdot, u]\cdot\nabla f$.

REMARKS. (i) The condition C is weaker than the continuity: $b^{(n)}[x, u_n] \to b[x, u]$ as $n \to \infty$. (ii) The following Amann's lemma gives a sufficient condition for the condition U. Define a non-linear operator G from C_0^* into $(C_0^1)^*$ by

$$\langle G(u), f\rangle = \langle u, b[\cdot, u]\cdot\nabla f\rangle \qquad \text{for} \quad f \in C_0^1\ , \tag{19}$$

where C_0^* (resp. $(C_0^1)^*$) is the dual Banach space of $C_0(S)$ (resp. of $C_0^1(S) = \{f \in C_0(S); \partial_{x_i} f \in C_0(S),\ i = 1, \cdots, d\}$), where $C_0(S)$ denotes the space of continuous functions vanishing at infinity.

LEMMA 6 (Amann, H.). *If G defined by* (19) *satisfies*

$$\| G(u) - G(v) \|_{(C_0^1)^*} \leqq const. \| u - v \|_{C_0^*} \qquad for \quad u, v \in C_0^*\ , \tag{20}$$

then the uniqueness holds for the initial value problem (18) (see [8], Appendix).

Examples which satisfy the conditions B, C and U will be discussed in the next section.

THEOREM. *Given a symmetric u-chaotic family $\{u_n; n = 1, 2, \cdots\}$, let $X^{(n)}(t) = (X_1^{(n)}(t), \cdots, X_n^{(n)}(t))$ be a solution of the SDE* (1) *with an initial value $(X_1^{(n)}(0), \cdots, X_n^{(n)}(0))$ distributed according to u_n. Under the conditions B, C and U the following assertions hold.*

(i) *$U^{(n)}(t)$ converges in probability to some (non-random) limit $u(t)$ which is a solution of the equation* (18).

(ii) *For each $m \leqq n$ the process $(X_1^{(n)}(t), \cdots, X_m^{(n)}(t))$ converges as $n \to \infty$ in law to $(X_1(t), \cdots, X_m(t))$, where $\{X_i(t); i = 1, 2, \cdots, m\}$ are mutually independent and each $X_i(t)$, $i = 1, \cdots, m$, is a copy of the solution $X(t)$ of the*

McKean-Vlasov's SDE

$$X(t)=X(0)+B(t)+\int_0^t b[X(s),\, u(s)]ds\ , \tag{21}$$

where $X(0)$ is u-distributed and $u(s)$ denotes the probability distribution of $X(s)$.

(iii) *$u(t)$ in the first assertion coincides with the one in* (ii).

REMARK. The claim of the existence of a (unique) solution of (18) (resp. (21)) is a part of the theorem.

PROOF OF THEOREM. We first state two lemmas.

LEMMA 7. *Define $Y_k^{(n)}(t)$ by*

$$Y_k^{(n)}(t)=\frac{1}{n}\sum_{i=1}^n\int_0^t \nabla f_k(X_i^{(n)}(s))\cdot dB_i(s)\ , \tag{22}$$

where $f_k\in C_K^\infty(S)$ are those used in the definition (12) *of the distance ρ. Then $Y_k^{(n)}(t)$ converges in law to $Y_k(t)\equiv 0$ as $n\to\infty$, for each $k\geqq 1$.*

PROOF. For any $\varepsilon>0$,

$$\begin{aligned}
&\lim_{n\to\infty} P[\max_{0\leqq t\leqq T}|Y_k^{(n)}(t)|>\varepsilon]\\
&\leqq\lim_{n\to\infty}\frac{1}{\varepsilon^2}E\Big[\Big|\frac{1}{n}\sum_{i=1}^n\int_0^T\nabla f_k(X_i^{(n)}(s))\cdot dB_i(s)\Big|^2\Big]\\
&=\lim_{n\to\infty}\frac{1}{\varepsilon^2 n^2}\sum_{i=1}^n E\Big[\int_0^T|\nabla f_k(X_i^{(n)}(s))|^2 ds\Big]\\
&=\lim_{n\to\infty}\frac{1}{\varepsilon^2 n}E\Big[\int_0^T|\nabla f_k(X_1^{(n)}(s))|^2 ds\Big]\\
&=0\ ,
\end{aligned}$$

where we have used the symmetry of the solution, completing the proof of Lemma 7.

Since the family $\{Q_n;\, n=1, 2, \cdots\}$ of probability measures on $C([0, T], \mathscr{P}(S))$ induced by the process $\{U^{(n)}(t);\, 0\leqq t\leqq T\}$ is tight by Lemma 5, we can choose a subsequence $\{Q_{n_k}\}$ which is weakly convergent as $k\to\infty$. For simplicity, fixing the subsequence, we denote it again by Q_n. The Skorokhod's realization theorem of almost sure convergence implies that on $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{P})$, where $\tilde{\Omega}=[0, 1]$, $\tilde{\mathscr{F}}=$ the class of all Borel subsets in [0, 1] and $\tilde{P}=$ the Lebesgue measure on $\tilde{\mathscr{F}}$, we can construct a sequence of processes

$$\{\tilde{U}^{(n)}(t),\ \tilde{Y}_k^{(n)}(t)\ ;\ k=1, 2, \cdots\}\ , \tag{23}$$

which is equivalent to $\{U^{(n)}(t), Y_k^{(n)}; k=1, 2, \cdots\}$ such that the processes $\tilde{U}^{(n)}(t)$ and $\tilde{Y}_k^{(n)}(t)$, $k\geqq 1$, are convergent uniformly in $0\leqq t\leqq T$ almost surely as $n\to\infty$. Let $\tilde{U}(t)$ and $\tilde{Y}_k(t)$ denote the limits of $\tilde{U}^{(n)}(t)$ and $\tilde{Y}_k^{(n)}(t)$, respectively. We write α for an element of $\tilde{\Omega}$, so that

$$\tilde{U}^{(n)}(t)=\tilde{U}^{(n)}(t, \alpha) , \qquad \tilde{Y}_k^{(n)}(t)=\tilde{Y}_k^{(n)}(t, \alpha)$$

and the limits

$$\tilde{U}(t)=\tilde{U}(t, \alpha) , \qquad \tilde{Y}_k(t)=\tilde{Y}_k(t, \alpha) .$$

LEMMA 8. *The limit $\tilde{U}(t, \alpha)$, $t>0$, has a strictly positive density with respect to the Lebesgue measure on S for almost all $\alpha\in[0, 1]$.*

PROOF.*) For each $i=1, 2, \cdots, d$, putting

$$b_n^i(t, x, \alpha) = \text{the } i\text{-th component of } b^{(n)}[x, \tilde{U}^{(n)}(t, \alpha)] , \tag{24}$$

we define a signed measure ν_n^i on $[0, T]\times S\times[0, 1]$ by

$$\nu_n^i(dtdxd\alpha)=b_n^i(t, x, \alpha)dtd\alpha\tilde{U}^{(n)}(t, \alpha, dx) , \tag{25}$$

and also set

$$\begin{aligned}\lambda_n(dtdxd\alpha)&=dtd\alpha\tilde{U}^{(n)}(t, \alpha, dx) ,\\ \lambda(dtdxd\alpha)&=dtd\alpha\tilde{U}(t, \alpha, dx) .\end{aligned} \tag{26}$$

It is clear that, since b_n^i is bounded,

$$-c\lambda_n\leqq\nu_n^i\leqq c\lambda_n \tag{27}$$

with a positive constant c, and that λ_n converges weakly to λ, because $\tilde{U}^{(n)}(t, \alpha)$ converges weakly to $\tilde{U}(t, \alpha)$ for almost all $\alpha\in[0, 1]$, where the exceptional set does not depend on $t\in[0, T]$. On the other hand, since $\{\nu_n^i; n=1, 2, \cdots\}$ is tight, we can choose a subsequence $\nu_{n_k}^i$ which converges to a limit ν^i. Then (27) implies

$$-c\lambda\leqq\nu^i\leqq c\lambda , \tag{28}$$

i.e., ν^i is absolutely continuous with respect to λ. Therefore

$$\begin{aligned}\nu^i(dtdxd\alpha)&=b^i(t, x, \alpha)\lambda(dtdxd\alpha)\\ &=b^i(t, x, \alpha)dtd\alpha\tilde{U}(t, \alpha, dx) ,\end{aligned} \tag{29}$$

where $b^i(t, x, \alpha)$ denotes the Radon-Nikodym derivative $d\nu^i/d\lambda$ which is

*) The idea behind is somewhat similar to that found in pp. 291–292 of [10].

bounded by c. Therefore, writing the subsequence $\{n_k\}$ as $\{n\}$ again, we have, for any $h\in C([0,1])$,

$$\begin{aligned}(30)\qquad &\lim_{n\to\infty}\int_0^1 h(\alpha)d\alpha\int_0^t\langle\tilde{U}^{(n)}(s,\alpha),\, b^{(n)}[\cdot,\,\tilde{U}^{(n)}(s,\alpha)]\cdot\nabla f_k\rangle ds\\ &=\lim_{n\to\infty}\sum_{i=1}^d\left\langle\nu_n^i,\, 1_{[0,t]}h\frac{\partial f_k}{\partial x^i}\right\rangle\\ &=\sum_{i=1}^d\left\langle\nu^i,\, 1_{[0,t]}h\frac{\partial f_k}{\partial x^i}\right\rangle\\ &=\sum_{i=1}^d\int_0^1 h(\alpha)d\alpha\int_0^t\langle\tilde{U}(s,\alpha),\, b(s,\,\cdot,\,\alpha)\cdot\nabla f_k\rangle ds\,,\end{aligned}$$

which implies

$$\begin{aligned}(31)\qquad &\lim_{n\to\infty}\int_0^t\langle\tilde{U}^{(n)}(s,\alpha),\, b^{(n)}[\cdot,\,\tilde{U}^{(n)}(s,\alpha)]\cdot\nabla f_k\rangle ds\\ &=\int_0^t\langle\tilde{U}(s,\alpha),\, b(s,\,\cdot,\,\alpha)\cdot\nabla f_k\rangle ds\qquad\text{for almost all }\alpha\in[0,1]\,.\end{aligned}$$

Now, $\tilde{U}^{(n)}(t,\alpha)$ satisfies

$$\begin{aligned}(32)\qquad \langle\tilde{U}^{(n)}(t,\alpha), f_k\rangle=\langle\tilde{U}^{(n)}(0,\alpha), f_k\rangle+\tilde{Y}_k^{(n)}(t,\alpha)\\ +\int_0^t\langle\tilde{U}^{(n)}(s,\alpha),\, A^{(n)}_{\tilde{U}^{(n)}(s,\alpha)}f_k\rangle ds\,,\end{aligned}$$

for almost all $\alpha\in[0,1]$. Because of (31) and $\lim_{n\to\infty}\tilde{Y}_k^{(n)}(t,\alpha)=0$ for almost all α, we have, passing to the limit $n\to\infty$ in (32),

$$(33)\qquad \langle\tilde{U}(t,\alpha), f_k\rangle=\langle\tilde{U}(0,\alpha), f_k\rangle+\int_0^t\langle\tilde{U}(s,\alpha),\, A(s,\alpha)f_k\rangle ds\,,$$

with

$$(34)\qquad A(s,\alpha)f_k=\frac{1}{2}\Delta f_k+b(s,\,\cdot,\,\alpha)\cdot\nabla f_k\,,$$

for all $k=1,2,\cdots$ and for all $\alpha\in[0,1]-N$, where the exceptional set N depends neither on k nor on $t\in[0,T]$. Since the linear hull of $\{f_k\colon k=1,2,\cdots\}$ is dense in $C_K^\infty(S)$, (33) holds for all $f\in C_K^\infty(S)$, and hence $\tilde{U}(t,\alpha)$ is a $\mathscr{P}(S)$-valued weak solution of the following *linear* parabolic equation

$$(35)\qquad \frac{\partial u}{\partial t}=\frac{1}{2}\Delta u+\nabla\cdot(b(t,\,\cdot,\,\alpha)u)\,.$$

Since $b(t,x,\alpha)$ is bounded, the initial-value problem (35) has a unique solution (Lemma 6). If one writes the solution in terms of the Cameron-Martin-Maruyama density [2] with respect to the Brownian motion on S, it is clear that the weak solution has a strictly positive density with respect to the Lebesgue measure on S for $\forall t>0$. Therefore, $U(t, \alpha)$ has a strictly positive density for $\forall t>0$ and $\forall\alpha\in[0, 1]-N$. This completes the proof of Lemma 8.

Let us complete the proof of the theorem. Since we have shown that the limit $\tilde{U}(t, \alpha)$, $t>0$, has a strictly positive density, we can conclude, applying the condition C, that

$$\begin{aligned}(36)\qquad &\lim_{n\to\infty}\int_0^t\langle\tilde{U}^{(n)}(s, \alpha), b^{(n)}[\cdot, \tilde{U}^{(n)}(s, \alpha)]\cdot\nabla f_k\rangle ds\\ &=\int_0^t\langle\tilde{U}(s, \alpha), b[\cdot, \tilde{U}(s, \alpha)]\cdot\nabla f_k\rangle ds \qquad \text{for } \forall\alpha\in[0, 1]-N .\end{aligned}$$

Therefore $\tilde{U}(t, \alpha)$ satisfies, for $\forall\alpha\in[0, 1]-N$,

$$(37)\qquad \langle\tilde{U}(t, \alpha), f_k\rangle=\langle\tilde{U}(0, \alpha), f_k\rangle+\int_0^t\langle\tilde{U}(s, \alpha), A_{\tilde{U}(s,\alpha)}f_k\rangle ds\ , \qquad k\geqq 1 .$$

This holds for $\forall f\in C_K^\infty(S)$, since the linear hull of $\{f_k;\ k=1, 2, \cdots\}$ is dense in $C_K^\infty(S)$. This means that $\tilde{U}(t, \alpha)$ is a solution of (18). Therefore, by the condition U,

$$(38)\qquad u(t)=\tilde{U}(t, \alpha) \qquad \text{for } \forall\alpha\in[0, 1]-N ,$$

where N is the exceptional set which does not depend on $t\in[0, T]$, and $u(t)$ is the unique solution of the initial value problem (18). At the same time (38) implies that $U^{(n)}(t)$ converges uniformly in $t\in[0, T]$ to the unique solution $u(t)$ in probability. This completes the first assertion (i) of the theorem.

To prove the second assertion (ii), consider an SDE

$$(39)\qquad X(t)=X(0)+B(t)+\int_0^t b(s, X(s))ds$$

where $b(s, x)=b[x, u(s)]$ and $X(0)$ is u-distributed. Since $b(s, x)$ is bounded and measurable in (s, x), SDE (39) has a weak solution by the Cameron-Martin-Maruyama formula and moreover it is a unique pathwise solution by a theorem of Veretennikov [14]. Let $v(t)$ be the probability distribution of the solution $X(t)$ of (39). As is shown in Appendix of [8], $u(t)$ and $v(t)$ satisfy

$$\langle u(t), f\rangle=\langle u, P_t f\rangle+\int_0^t\langle u(s), b[\cdot, u(s)]\cdot\nabla P_{t-s}f\rangle ds\ ,$$

$$\langle v(t), f\rangle = \langle u, P_t f\rangle + \int_0^t \langle v(s), b[\cdot, u(s)]\cdot \nabla P_{t-s} f\rangle ds ,$$

respectively, for $f \in C_0(S)$, where P_t denotes the d-dimensional Brownian semigroup on $C_0(S)$. Therefore

$$\langle u(t)-v(t), f\rangle = \int_0^t \langle u(s)-v(s), b[\cdot, u(s)]\cdot \nabla P_{t-s} f\rangle ds . \tag{40}$$

Because

$$\| \nabla (P_{t-s} f)\| \leqq \text{const.} \frac{1}{\sqrt{t-s}} \|f\| \qquad \text{for} \quad f \in C_0(S) , \tag{41}$$

it follows from (40) that

$$\| u(t)-v(t) \|_{C_0^*} \leqq \text{const.} \int_0^t \| u(s)-v(s)\|_{C_0^*} \frac{1}{\sqrt{t-s}} ds , \tag{42}$$

which implies $\| u(t)-v(t) \|_{C_0^*}=0$. Therefore the solution $X(t)$ of (39) is actually a unique solution of the non-linear SDE (21). Combining the assertion (i) of the theorem with what we have shown and Lemma 1, one can conclude that the sequence of the probability distributions of $(X_1^{(n)}(t), \cdots, X_n^{(n)}(t))$ is a symmetric $u(t)$-chaotic family and hence $(X_1^{(n)}(t), \cdots, X_m^{(n)}(t))$, m fixed, converges in law to $(X_1(t), \cdots, X_m(t))$ as $n \to \infty$, where $X_i(t)$'s are independent copies of the unique solution of the non-linear SDE (21). (iii) is clear from the above proof of (ii). This completes the proof of the theorem.

§4. Examples.

EXAMPLE 1. Let $b(x, y)$ be an S-valued bounded Borel measurable function on $S \times S$.

CONDITION D. There exists an open subset D in $S \times S$ such that $b(x, y)$ is continuous in D and the $2d$-dimensional Lebesgue measure of D^c is zero.

Let $b[x, u]$ be defined by (3) for $u \in \mathscr{P}(S)$. If $b(x, y)$ satisfies the condition D, then the condition C is fulfilled. In fact, let $u \in \mathscr{P}(S)$ have a strictly positive density with respect to the Lebesgue measure on $S = \boldsymbol{R}^d$, $u_n \in \mathscr{P}(S)$ converge weakly to u and $f \in C_0(S)$. Then, for sufficiently large n, we have

$$\left| \int f(x) b[x, u_n] u_n(dx) - \int f(x) b[x, u] u(dx) \right|$$

$$\leqq \left| \iint_D f(x) b(x, y)\{u_n \otimes u_n(dxdy) - u\otimes u(dxdy)\} \right|$$
$$+ \|f\| \, \|b\| u_n \otimes u_n(D^c)$$
$$\leqq \varepsilon \, .$$

Let G be defined by (19). Then

$$|\langle G(u), f\rangle - \langle G(v), f\rangle| = \left| \iint \nabla f(x) b(x, y)\{u\otimes u(dxdy) - v\otimes v(dxdy)\} \right|$$
$$\leqq 2 \|\nabla f\| \, \|b\| \, \|u-v\|_{C_0^*}$$

and hence the condition U is satisfied by Lemma 6. Therefore, our theorem holds under the condition D for a bounded measurable $b(x, y)$.

REMARK. When $b[x, u]$ is of average form with a bounded measurable $b(x, y)$, the propagation of chaos has been shown already by a different method, see Sznitman [12] and Shiga-Tanaka [11].

EXAMPLE 2. Let $b(x, y, v)$ and $b^{(n)}(x, y, v)$ be S-valued Borel measurable functions on $S\times S\times \mathscr{P}(S)$ which are bounded uniformly in n.

CONDITION E. If $u\in \mathscr{P}(S)$ has a strictly positive density with respect to the Lebesgue measure on S and if $u_n \in \mathscr{P}(S)$ converges weakly to u, then for any $\varepsilon>0$ there exist a compact subset K and a closed subset F of $S\times S$ such that $b(\cdot, \cdot, u)$ is continuous on K, $b^{(n)}(\cdot, \cdot, u_n)$ converges to $b(\cdot, \cdot, u)$ uniformly on K, $u\otimes u(F)\leqq \varepsilon$ and $S\times S = K\cup F$.

Assume the condition E and define $b[x, v]$ (and $b^{(n)}[x, v]$) for $v\in \mathscr{P}(S)$ by

$$b[x, v] = \int b(x, y, v) v(dy) \qquad \left(b^{(n)}[x, v] = \int b^{(n)}(x, y, v) v(dy) \right), \tag{43}$$

then $b^{(n)}[x, v]$ satisfies the condition C. In fact, for $f\in C_0$

$$\left| \int f(x) b^{(n)}[x, u_n] u_n(dx) - \int f(x) b[x, u] u(dx) \right|$$
$$\leqq \left| \iint f(x) b(x, y, u)\{u_n\otimes u_n(dxdy) - u\otimes u(dxdy)\} \right|$$
$$+ \left| \iint f(x)\{b^{(n)}(x, y, u_n) - b(x, y, u)\} u_n\otimes u_n(dxdy) \right|$$
$$= I + II \, ,$$

where $I\leqq \varepsilon$ for sufficiently large n as is shown in Example 1, and II can be estimated as follows:

$$
\begin{aligned}
II &\leqq \sup_K |b^{(n)}(x, y, u_n) - b(x, y, u)| \iint_K |f(x)| \, u_n \otimes u_n(dxdy) \\
&\quad + \| f \| (\| b \| + \| b^{(n)} \|) u_n \otimes u_n(F) \\
&\leqq \varepsilon
\end{aligned}
$$

for sufficiently large n, if we choose K and F so that $u \otimes u(F)$ is dominated by $\varepsilon\{1+\|f\|(\|b\|+\sup_n\|b^{(n)}\|)\}^{-1}$. Thus the condition C is satisfied.

CONDITION L.

$$
|b(x, y, u) - b(x, y, v)| \leqq c \| u - v \|_{C_0^*}
$$

for $u, v \in \mathscr{P}(S)$, where c is a positive constant.

Under the condition L the condition U is satisfied. In fact, for G defined by (19) we have

$$
\begin{aligned}
|\langle G(u), f\rangle - \langle G(v), f\rangle| &\leqq \left| \iint \nabla f(x) b(x, y, u)\{u \otimes u(dxdy) - v \otimes v(dxdy)\} \right| \\
&\quad + \left| \iint \nabla f(x)\{b(x, y, u) - b(x, y, v)\} v \otimes v(dxdy) \right| \\
&\leqq \text{const.} \| \nabla f \| \, \| u - v \|_{C_0^*}
\end{aligned}
$$

and hence the condition U follows from Lemma 6.

Therefore, our theorem holds under the conditions E and L for a bounded measurable $b(x, y, v)$.

§5. An application to a system of interacting coloured particles.

Given a bounded continuous functions $\{b_{ij}(x, y); i, j=1, 2\}$ on $\boldsymbol{R}^2$ we define

$$
b(x, y, u) = \begin{cases} b_{11}(x, y)\dfrac{1}{\theta}\mathbf{1}_{(-\infty,\gamma(u)]}(y) + b_{12}(x, y)\dfrac{1}{1-\theta}\mathbf{1}_{(\gamma(u),\infty)}(y) , & x \leqq \gamma(u) , \\ b_{21}(x, y)\dfrac{1}{\theta}\mathbf{1}_{(-\infty,\gamma(u)]}(y) + b_{22}(x, y)\dfrac{1}{1-\theta}\mathbf{1}_{(\gamma(u),\infty)}(y) , & x > \gamma(u) , \end{cases} \tag{44}
$$

where $0<\theta<1$ and $\gamma(u)$ is the segregating front of the distribution $u \in \mathscr{P}(\boldsymbol{R}^1)$ defined by

$$
\gamma(u) = \gamma(u, \theta) = \min\{x \,;\, u((-\infty, x)) \leqq \theta \leqq u((-\infty, x])\} . \tag{45}
$$

It is clear that $b(x, y, u)$ satisfies the condition E and moreover the condition L (see Appendix of [8]). Thus the conditions B, C and U are fulfilled.

Let $(X_1, X_2, \cdots, X_{m+n})$ be the solution of the SDE (1) with $b^{(n)}[x, u]$ defined by (43) in terms of $b(x, y, u)$ given in (44), where $\theta=\theta(m, n)=m/(m+n)$, and moreover let the sequence of initial distributions be a symmetric u-chaotic family. Applying the order statistics (Lemma 2 of [8]) to $(X_1, \cdots, X_{m+n})$, we obtain the reflected processes of two segregated groups $(X_1, \cdots, X_m, Y_1, \cdots, Y_n)$ which satisfy the following SDE's

$$
(46)\qquad \begin{cases} X_i(t)=X_i(0)+B_i^-(t)+\dfrac{1}{m}\displaystyle\sum_{k=1}^{m}\int_0^t b_{11}(X_i(s), X_k(s))ds \\ \qquad +\dfrac{1}{n}\displaystyle\sum_{k=1}^{n}\int_0^t b_{12}(X_i(s), Y_k(s))ds-\sum_{k=1}^{n}\Phi_{ik}(t)\ , \qquad 1\leqq i\leqq m\ , \\ Y_j(t)=Y_j(0)+B_j^+(t)+\dfrac{1}{m}\displaystyle\sum_{k=1}^{m}\int_0^t b_{21}(Y_j(s), X_k(s))ds \\ \qquad +\dfrac{1}{n}\displaystyle\sum_{k=1}^{n}\int_0^t b_{22}(Y_j(s), Y_k(s))ds+\sum_{k=1}^{m}\Phi_{kj}(t)\ , \qquad 1\leqq j\leqq n\ , \end{cases}
$$

where

$$
(47)\qquad \max_{1\leqq i\leqq m} X_i(t)\leqq \min_{1\leqq j\leqq n} Y_j(t) \qquad \text{for } \forall t\geqq 0\ ,
$$

$$
(48)\qquad \begin{array}{l} \Phi_{ij}(t) \text{ is continuous, monotone nondecreasing}\ , \\ \Phi_{ij}(0)=0 \quad \text{and} \quad \mathrm{supp}(d\Phi_{ij})\subset\{t\geqq 0 : X_i(t)=Y_j(t)\}\ , \end{array}
$$

and $\{B_i^-(t), B_j^+(t); 1\leqq i\leqq m, 1\leqq j\leqq n\}$ are independent one dimensional Brownian motions starting at 0, which are independent of the initial values. Under the assumption that the distribution u satisfies the positivity condition (see (20) of [8]), the sequence of the distributions of the initial values $(X_1(0), \cdots, X_m(0), Y_1(0), \cdots, Y_n(0))$ is (u_l, u_r)-chaotic (Definition 3 of [8]) by Lemma 4 of [8].

By our theorem in §4, the propagation of chaos holds for the diffusion processes $(X_1, X_2, \cdots, X_{m+n})$, i.e., the sequence of the distributions of $(X_1(t), X_2(t), \cdots, X_{m+n}(t))$ is a symmetric $u(t)$-chaotic family, where $u(t)$ is the unique solution of the non-linear equation (18). Therefore, the sequence of the distributions of $(X_1(t), \cdots, X_m(t), Y_1(t), \cdots, Y_n(t))$ is $(u(t)_l, u(t)_r)$-chaotic by Lemma 4 of [8]. That is, the propagation of chaos holds for the diffusion processes of two segregated groups. This is another proof of Theorem 3 in [8].

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