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A Note on Satake Parameters of Siegel Modular Forms of Degree 2

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Introduction.

For a positive integer k, let S_k be the space of all Siegel cusp forms of weight k on $Sp(2, \boldsymbol{Z})$. Suppose $f \in S_k$ is an eigenform, i.e., a non-zero common eigenfunction of the Hecke algebra. Then we define the spinor L-function attached to f by

$$(0.1)\qquad L(s, f, \underline{\text{spin}}) := \prod_p \{(1-\alpha_{0,p}p^{-s})(1-\alpha_{0,p}\alpha_{1,p}p^{-s})(1-\alpha_{0,p}\alpha_{2,p}p^{-s})(1-\alpha_{0,p}\alpha_{1,p}\alpha_{2,p}p^{-s})\}^{-1}$$

and the standard L-function attached to f by

$$(0.2)\qquad L(s, f, \underline{\text{st}}) := \prod_p \left\{(1-p^{-s})\prod_{j=1}^{2}(1-\alpha_{j,p}^{-1}p^{-s})(1-\alpha_{j,p}p^{-s})\right\}^{-1},$$

where p runs over all prime numbers and $\alpha_{j,p}$ $(0 \leq j \leq 2)$ are the Satake p-parameters of f. The right-hand sides of (0.1) and (0.2) converge absolutely and locally uniformly for $\mathrm{Re}(s)$ sufficiently large.

For an indeterminate t, we put

$$H_p(t, f, \underline{\text{spin}}) := (1-\alpha_{0,p}t)(1-\alpha_{0,p}\alpha_{1,p}t)(1-\alpha_{0,p}\alpha_{2,p}t)(1-\alpha_{0,p}\alpha_{1,p}\alpha_{2,p}t),$$
$$H_p(t, f, \underline{\text{st}}) := (1-t)\prod_{j=1}^{2}(1-\alpha_{j,p}^{-1}t)(1-\alpha_{j,p}t),$$

where $H_p(t, f, \underline{\text{spin}})$, $H_p(t, f, \underline{\text{st}}) \in \boldsymbol{R}[t]$.

DEFINITION. (cf. Kurokawa [9]) We say that $f \in S_k$ satisfies the Ramanujan–Petersson conjecture if the absolute values of the zeros of $H_p(t, f, \underline{\text{spin}})$ are all equal to $p^{-(k-3/2)}$ for all p.

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Since the Satake p-parameters satisfy $\alpha_{0,p}^2\alpha_{1,p}\alpha_{2,p}=p^{2k-3}$, this is equivalent to saying that

$$|\alpha_{1,p}|=|\alpha_{2,p}|=1 \qquad \text{for all } p\,,$$

that is, the absolute values of the zeros of $H_p(t,f,\underline{\mathrm{st}})$ are all equal to 1 for all p, or to saying that

$$\underline{\mathrm{st}}_p(f):=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & a_p & b_p & 0 & 0 \\ 0 & -b_p & a_p & 0 & 0 \\ 0 & 0 & 0 & c_p & d_p \\ 0 & 0 & 0 & -d_p & c_p \end{pmatrix} \in SO(5,\boldsymbol{R}) \qquad \text{for all } p\,,$$

where $a_p=\frac{1}{2}(\alpha_{1,p}+\alpha_{1,p}^{-1})$, $b_p=\frac{1}{2i}(\alpha_{1,p}-\alpha_{1,p}^{-1})$, $c_p=\frac{1}{2}(\alpha_{2,p}+\alpha_{2,p}^{-1})$, $d_p=\frac{1}{2i}(\alpha_{2,p}-\alpha_{2,p}^{-1})$ (cf. Langlands [11]).

For an even integer k, let S_k^* be the Maaß subspace of S_k (cf. Maaß [13, 14, 15], Andrianov [3], Zagier [23]). We know that if f belongs to the Maaß space S_k^*, f doesn't satisfy the Ramanujan–Petersson conjecture. Now, our conjecture takes the following form:

CONJECTURE. (cf. Kurokawa [9, Conjecture 3]) If k is an even integer, any cusp eigenform of weight k in the orthogonal complement of the Maaß space satisfies the Ramanujan–Petersson conjecture. If k is an odd integer, any cusp eigenform of weight k satisfies the Ramanujan–Petersson conjecture.

We will analyze this conjecture from elementary properties of L-functions. Although several authors make numerical researches on our conjecture, so far we don't know even the existence of f which satisfies the Ramanujan–Petersson conjecture (cf. Kurokawa [9], Skoruppa [21]).

NOTATION. 1°. As usual, $\boldsymbol{Z}$ is the ring of rational integers, $\boldsymbol{Q}$ the field of rational numbers, $\boldsymbol{R}$ the field of real numbers, $\boldsymbol{C}$ the field of complex numbers.

2°. Let $m, n\in\boldsymbol{Z}$, $m, n>0$. If A is an $m\times n$-matrix, then we write it also as $A^{(m,n)}$, and as $A^{(m)}$ if $m=n$. The identity matrix of size n is denoted by 1_n.

3°. For $n\in\boldsymbol{Z}$, $n>0$, let $A^{(n)}$ be a diagonal matrix with diagonal entries $a_1,\cdots,a_n$. We denote it by $\mathrm{d}(a_1,\cdots,a_n)$.

4°. For $n\in\boldsymbol{Z}$, $n>0$, let $\Gamma^n:=Sp(n,\boldsymbol{Z})$ be the Siegel modular group of degree n and let $\mathfrak{H}_n$ be the Siegel upper half space of degree n, that is,

$$\mathfrak{H}_n:=\{Z=X+iY\in\boldsymbol{C}^{(n)}\,|\,{}^tZ=Z,\ Y>0\}\,.$$

5°. We put

$$\xi(s):=\Gamma_{\boldsymbol{R}}(s)\zeta(s)=\xi(1-s)\,,$$

$$\Gamma_{\boldsymbol{R}}(s):=\pi^{-s/2}\Gamma(s/2)\,,\quad \Gamma_{\boldsymbol{C}}(s):=2(2\pi)^{-s}\Gamma(s)=\Gamma_{\boldsymbol{R}}(s)\Gamma_{\boldsymbol{R}}(s+1)\,,$$

where $\zeta(s)$ is the Riemann zeta function and $\Gamma(s)$ is the gamma function.

§ 1. Preliminaries.

Let k be a positive integer. A holomorphic function f on $\mathfrak{H}_n$ is called a Siegel modular form of weight k if it satisfies

$$(f\,|\,M)(Z):=\det(CZ+D)^{-k}f((AZ+B)(CZ+D)^{-1})=f(Z)$$

for all $Z\in\mathfrak{H}_n$ and $M=\begin{pmatrix} A^{(n)} & B^{(n)} \\ C^{(n)} & D^{(n)} \end{pmatrix}\in\Gamma^n$ and if it is holomorphic at the cusps when $n=1$. The space of Siegel modular forms of weight k is denoted by M_k^n.

We define the Siegel operator Φ on M_k^n by

$$(\Phi f)(Z):=\lim_{t\to\infty} f\left(\begin{pmatrix} Z & 0 \\ 0 & it \end{pmatrix}\right)$$

for $Z\in\mathfrak{H}_{n-1}$. Then the operator Φ defines the map Φ: $M_k^n\to M_k^{n-1}$. Suppose $f\in M_k^n$. Then it is called a cusp form if it satisfies $\Phi f=0$.

In what follows, we restrict ourselves to the case $n=2$ and we omit subscripts concerning the case $n=2$ when there is no fear of confusion.

We define $G^+:=G^+Sp(2,\boldsymbol{Q})$ by

$$G^+:=\left\{M\in GL(4,\boldsymbol{Q})\,\middle|\,{}^tM\begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix}M=\mu(M)\begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix},\ \mu(M)>0\right\},$$

and for a prime number p, $G_p^+:=G^+\cap GL(4,\boldsymbol{Z}[p^{-1}])$.

Let $\mathscr{H}$ (resp. $\mathscr{H}_p$) be the free $\boldsymbol{C}$-module generated by the double cosets $\Gamma g\Gamma$, $g\in G^+$ (resp. G_p^+). Then $\mathscr{H}$ is a commutative algebra and we call it the Hecke algebra (over $\boldsymbol{C}$). We get $\mathscr{H}=\bigotimes_p\mathscr{H}_p$, where the tensor product is the restricted one. Moreover, the structure of $\mathscr{H}_p$ is known: For $0\le j\le 2$, let w_j be an automorphism of $\boldsymbol{C}[X_0^{\pm1},X_1^{\pm1},X_2^{\pm1}]$ such that

$$\begin{aligned} &w_0(X_0)=X_0, && w_0(X_1)=X_2, && w_0(X_2)=X_1, \\ &w_1(X_0)=X_0X_1, && w_1(X_1)=X_1^{-1}, && w_1(X_2)=X_2, \\ &w_2(X_0)=X_0X_2, && w_2(X_1)=X_1, && w_2(X_2)=X_2^{-1}. \end{aligned}$$

The automorphisms w_j $(0\le j\le 2)$ generate a finite group W. We call it the Weyl group. We get

$$\Psi:\mathscr{H}_p\xrightarrow{\cong}\boldsymbol{C}[X_0^{\pm1},X_1^{\pm1},X_2^{\pm1}]^W,$$

where $\boldsymbol{C}[X_0^{\pm1},X_1^{\pm1},X_2^{\pm1}]^W$ is the W-invariant subalgebra of $\boldsymbol{C}[X_0^{\pm1},X_1^{\pm1},X_2^{\pm1}]$.

For $g\in G^+$, let $\Gamma g\Gamma=\bigcup_{j=1}^r\Gamma g_j$ be a decomposition of the double coset $\Gamma g\Gamma$ into left cosets. For $f\in M_k$ (resp. S_k), we define the Hecke operator $(\Gamma g\Gamma)$ by

$$f|(\Gamma g\Gamma):=\mu(g)^{2k-3}\sum_{j=1}^{r} f|g_j\,.$$

Then we get a homomorphism $\mathscr{H}\to\mathrm{End}(M_k)$ (resp. $\mathrm{End}(S_k)$).

For $\delta\in\boldsymbol{Z}$, $\delta>0$ and a prime number p, we put

$$T(p^\delta):=\sum_{\mu(g)=p^\delta}(\Gamma g\Gamma)\,,$$

where $g=\mathrm{d}(p^{d_1}, p^{d_2}, p^{e_1}, p^{e_2})\in G_p^+$, d_j, $e_j\in\boldsymbol{Z}$ $(j=1, 2)$ and $0\le d_1\le d_2\le e_2\le e_1$.

Suppose $f\in M_k$ is an eigenform. We denote the eigenvalue of $(\Gamma g\Gamma)$ on f by $\lambda_f(\Gamma g\Gamma)$ and that of $T(p^\delta)$ on f by $\lambda_f(p^\delta)$.

If the homomorphism $\lambda_f\colon \mathscr{H}_p\to\boldsymbol{C}$ coincides with the composite map of the isomorphism Ψ and the evaluation map

$$\boldsymbol{C}[X_0^{\pm1}, X_1^{\pm1}, X_2^{\pm1}]^W\xrightarrow{(X_0, X_1, X_2)\mapsto(\alpha_{0,p}, \alpha_{1,p}, \alpha_{2,p})}\boldsymbol{C}\,,$$

then the numbers $\alpha_{0,p}, \alpha_{1,p}, \alpha_{2,p}\in\boldsymbol{C}^*$, the Satake p-parameters of f, are uniquely determined modulo W. In this case, they are uniquely determined by

$$\lambda_f(\Gamma p1_4\Gamma)=p^{-3}\alpha_{0,p}^2\alpha_{1,p}\alpha_{2,p}\,,\qquad \lambda_f(p)=\alpha_{0,p}(1+\alpha_{1,p})(1+\alpha_{2,p})\,,$$

$$\lambda_f(\Gamma\mathrm{d}(1, p, p^2, p)\Gamma)=p^{-1}\alpha_{0,p}^2(\alpha_{1,p}+\alpha_{2,p})(1+\alpha_{1,p}\alpha_{2,p})+(p^{-1}-p^{-3})\alpha_{0,p}^2\alpha_{1,p}\alpha_{2,p}\,,$$

up to the action of the Weyl group W.

We summarize some facts on Siegel modular forms and on L-functions attached to them. In what follows, we suppose that $f\in S_k$ is an eigenform.

(I) It follows from the hermiteness of Hecke operators $(\Gamma g\Gamma)$, $g\in G^+$, that we have $\lambda_f(\Gamma g\Gamma)\in\boldsymbol{R}$. In fact, by Kurokawa [10], we know that the eigenvalues on f of the Hecke algebra over $\boldsymbol{Q}$ generate a totally real finite extension of $\boldsymbol{Q}$.

(II) We put

$$\Lambda(s, f, \underline{\mathrm{st}}):=\Gamma_{\boldsymbol{R}}(s)\prod_{j=1}^{2}\Gamma_{\boldsymbol{C}}(s+k-j)L(s, f, \underline{\mathrm{st}})\,.$$

Andrianov–Kalinin [4] and Böcherer [6] (cf. Piatetski-Shapiro and Rallis [20]) have discovered that $\Lambda(s, f, \underline{\mathrm{st}})$ has a meromorphic continuation to the whole s-plane and satisfies the functional equation

$$\Lambda(s, f, \underline{\mathrm{st}})=\Lambda(1-s, f, \underline{\mathrm{st}})\,.$$

Moreover, Mizumoto [16] has shown that it is entire.

(III) We put

$$\Lambda(s, f, \underline{\mathrm{spin}}):=\Gamma_{\boldsymbol{C}}(s)\Gamma_{\boldsymbol{C}}(s-k+2)L(s, f, \underline{\mathrm{spin}})\,.$$

Andrianov [1] has shown that $\Lambda(s, f, \underline{\mathrm{spin}})$ has a meromorphic continuation to the whole s-plane and satisfies the functional equation

$$\Lambda(s, f, \underline{\mathrm{spin}}) = (-1)^k \Lambda(2k-2-s, f, \underline{\mathrm{spin}}) .$$

For an odd integer k, it is entire. For an even integer k, Evdokimov [8] and Oda [19] have shown that $\Lambda(s, f, \underline{\mathrm{spin}})$ has a simple pole at $s=k$ (or equivalently, at $s=k-2$) if and only if $f \in S_k^*$. Especially, if $f \in S_k^*$, $L(s, f, \underline{\mathrm{spin}})$ is the following form:

$$(1.1) \qquad L(s, f, \underline{\mathrm{spin}}) = \prod_p \{(1-p^{k-1}p^{-s})(1-p^{k-2}p^{-s})(1-\omega p^{-s})(1-\bar{\omega} p^{-s})\}^{-1} ,$$

where $|\omega| = p^{k-3/2}$.

§2. Results.

First we note that the set $\{\alpha_{1,p}, \alpha_{1,p}^{-1}, \alpha_{2,p}, \alpha_{2,p}^{-1}\}$ is invariant under the action of the Weyl group W.

LEMMA 1. *For an eigenform $f \in S_k$, the set $\{\alpha_{1,p}, \alpha_{1,p}^{-1}, \alpha_{2,p}, \alpha_{2,p}^{-1})$ is one of the following types*:

Type I. $\{p^{a_1}, p^{-a_1}, p^{a_2}, p^{-a_2}\}$ *or* $\{-p^{a_1}, -p^{-a_1}, -p^{a_2}, -p^{-a_2}\}$, *where* $a_1, a_2 \in \boldsymbol{R}$ *and* $0 < a_2 < a_1$.

Type II. $\{e^{i\theta}, e^{-i\theta}, p^a, p^{-a}\}$ *or* $\{-1, -1, -p^a, -p^{-a}\}$, *where* $a \in \boldsymbol{R}$, $0 < a$ *and* $0 \le \theta < 2\pi$.

Type III. $\{p^a e^{i\theta}, p^{-a} e^{-i\theta}, p^a e^{-i\theta}, p^{-a} e^{i\theta}\}$, *where* $a \in \boldsymbol{R}$, $0 < a$ *and* $0 \le \theta < 2\pi$.

Type RP. $\{e^{i\theta_1}, e^{-i\theta_1}, e^{i\theta_2}, e^{-i\theta_2}\}$, *where* $0 \le \theta_1, \theta_2 < 2\pi$.

PROOF. By $H_p(t, f, \underline{\mathrm{st}}) \in \boldsymbol{R}[t]$, we have

$$\{\alpha_{1,p}, \alpha_{1,p}^{-1}, \alpha_{2,p}, \alpha_{2,p}^{-1}\} = \{\overline{\alpha_{1,p}}, \overline{\alpha_{1,p}^{-1}}, \overline{\alpha_{2,p}}, \overline{\alpha_{2,p}^{-1}}\} .$$

From this fact, Lemma 1 is proved except for signatures in type I and in type II.

In type I, it follows from $\lambda_f(p) \in \boldsymbol{R}$ that we have $\alpha_{0,p} \in \boldsymbol{R}$. Combining this with $\alpha_{0,p}^2 \alpha_{1,p} \alpha_{2,p} = p^{2k-3}$, we obtain $\alpha_{1,p}\alpha_{2,p} > 0$. In the same way, we can determine signatures in type II. □

THEOREM 1. *Conjecture holds if any eigenform $f \in S_k$ satisfies the following conditions*:

(A) *For any prime p, the logarithms of the absolute values of the zeros of $H_p(t, f, \underline{\mathrm{spin}})$ (or equivalently, of $H_p(t, f, \underline{\mathrm{st}})$) to the base p are independent of p.*

(B) *For all but a finite number of primes p, $H_p(t, f, \underline{\mathrm{spin}})$ and $H_p(t, f, \underline{\mathrm{st}})$ have no negative real zeros.*

With the use of the Satake parameters, we can replace the condition (A) by the following form:

(A′) *For any primes p and q, by the suitable action of the Weyl group,*

$$\log_p|\alpha_{1,p}|=\log_q|\alpha_{1,q}| \quad \textit{and} \quad \log_p|\alpha_{2,p}|=\log_q|\alpha_{2,q}|$$

hold.

If we note that our L-functions $L(s,f,\underline{\mathrm{spin}})$ and $L(s,f,\underline{\mathrm{st}})$ are unramified at all p in the sense of Langlands [11], we can understand that "any prime" in (A) is "any unramified prime". If so, (A) is true for many L-functions which have the Euler product expansions, e.g., the Riemann zeta-function, the Dirichlet L-functions, the Hasse–Weil L-functions and so on.

PROOF. For a positive integer k, let $f\in S_k$ be an eigenform. We note that, under the condition (A), the types of $\{\alpha_{1,p}, \alpha_{1,p}^{-1}, \alpha_{2,p}, \alpha_{2,p}^{-1}\}$ are the same for all p. If $\{\alpha_{1,p}, \alpha_{1,p}^{-1}, \alpha_{2,p}, \alpha_{2,p}^{-1}\}$ is of type I (resp. type II, type III or type RP) for any prime p, we say that f is of type I (resp. type II, type III or type RP).

In what follows, we assume that f satisfies the condition (B).

If f is of type I, then for almost all p, $H_p(t,f,\underline{\mathrm{st}})$ is the following form:

$$H_p(t,f,\underline{\mathrm{st}})=(1-t)(1-p^{a_1}t)(1-p^{-a_1}t)(1-p^{a_2}t)(1-p^{-a_2}t)\,,$$

where a_1 and a_2 are independent of p. Then $L(s,f,\underline{\mathrm{st}})$ has a pole at $s=1+a_1$. This contradicts the fact (II).

If f is of type II, then for almost all p, $H_p(t,f,\underline{\mathrm{st}})$ is the following form:

$$H_p(t,f,\underline{\mathrm{st}})=(1-t)(1-\mathrm{e}^{i\theta_p}t)(1-\mathrm{e}^{-i\theta_p}t)(1-p^{a}t)(1-p^{-a}t)\,,$$

where $0\le\theta_p<2\pi$ and a is independent of p. Then $L(s,f,\underline{\mathrm{st}})$ has a pole at $s=1+a$. This contradicts the fact (II).

If f is of type III, then for almost all p, $H_p(t,f,\underline{\mathrm{spin}})$ is the following form:

$$H_p(t,f,\underline{\mathrm{spin}})=(1-p^{k-3/2+a}t)(1-p^{k-3/2-a}t)(1-p^{k-3/2}\mathrm{e}^{i\theta_p}t)(1-p^{k-3/2}\mathrm{e}^{-i\theta_p}t)\,,$$

where $0\le\theta_p<2\pi$ and a is independent of p. Then $L(s,f,\underline{\mathrm{spin}})$ has a pole at $s=k-1/2+a$. If k is an odd integer, this contradicts the fact (III). If k is an even integer, we have $f\in S_k^*$ and $a=1/2$. □

For an even integer k, let $S_k^{*\perp}$ be the orthogonal complement of the Maaß space S_k^*, that is, $S_k=S_k^{*\perp}\oplus S_k^*$. For an odd integer k, we put $S_k^*=\{0\}$ when there is no fear of confusion.

Suppose that any eigenform $f\in S_k$ satisfies the condition (A). If f of type I occurs, then there exist infinitely many prime numbers such that $\lambda_f(p)<0$ and $\lambda_f(\Gamma \mathrm{d}(1,p,p^2,p)\Gamma)<0$. If f of type II occurs, then there exist infinitely many prime numbers such that $\lambda_f(p)=0$. If f of type III occurs, then $f\in S_k^*$ or there exist infinitely many prime numbers such that $\lambda_f(p)<0$.

So we have:

COROLLARY 1. *Let $f \in S_k$ be an eigenform. If f satisfies the condition* (A) *and if $\lambda_f(p) > 0$ for almost all p, then f satisfies the Ramanujan–Petersson conjecture or f belongs to the Maaß space S_k^*.*

We consider the spaces $S_{20}^{*\perp} = \mathbf{C}\Upsilon 20$, $S_{22}^{*\perp} = \mathbf{C}\Upsilon 22$, $S_{24}^{*\perp} = \mathbf{C}\Upsilon 24a \oplus \mathbf{C}\Upsilon 24b$ and $S_{26}^{*\perp} = \mathbf{C}\Upsilon 26a \oplus \mathbf{C}\Upsilon 26b$, where $\Upsilon*$ is the same eigenform as that in Skoruppa [21]. Note that $S_k^{*\perp} = \{0\}$ for $k < 20$.

In [21], the first few Euler factors

$$H_p(t, f, \underline{\text{spin}}) = \left(1 - \left(\frac{\lambda_{\Upsilon*}(p)}{2} + \sqrt{d_{\Upsilon*}(p)}\right)t + p^{2k-3}t^2\right)\left(1 - \left(\frac{\lambda_{\Upsilon*}(p)}{2} - \sqrt{d_{\Upsilon*}(p)}\right)t + p^{2k-3}t^2\right),$$

where $d_{\Upsilon*}(p) = -\frac{3}{4}\lambda_{\Upsilon*}(p)^2 + \lambda_{\Upsilon*}(p^2) + p^{2k-4} + 2p^{2k-3}$, have been computed. Using Skoruppa's Table 4 in [21], we compute the first few Euler factors

$$H_p(t, \Upsilon*, \underline{\text{st}}) = (1-t) \times \left(1 - \left(\frac{C_{\Upsilon*}(p)}{2} + \sqrt{D_{\Upsilon*}(p)}\right)\frac{1}{p^{2k-3}}t + t^2\right)\left(1 - \left(\frac{C_{\Upsilon*}(p)}{2} - \sqrt{D_{\Upsilon*}(p)}\right)\frac{1}{p^{2k-3}}t + t^2\right).$$

The resulting values of $C_{\Upsilon*}(p)$ and $D_{\Upsilon*}(p)$ are given in Table below. Within the range of our computations, the Euler factors $H_p(t, \Upsilon*, \underline{\text{spin}})$ and $H_p(t, \Upsilon*, \underline{\text{st}})$ of $\Upsilon 22$ up to $\Upsilon 26b$

TABLE

$\Upsilon*$	p	$C_{\Upsilon*}(p)$	$D_{\Upsilon*}(p)$
$\Upsilon 20$	2	$2^{26}\cdot 3^2\cdot 191$	$2^{50}\cdot 11898121$
	3	$-2^4\cdot 3^{22}\cdot 23\cdot 41413$	$2^2\cdot 3^{44}\cdot 494567\cdot 196757063$
	5	$-2^2\cdot 3^2\cdot 5^{20}\cdot 881\cdot 12576191$	$2^6\cdot 5^{40}\cdot 231611\cdot 253651\cdot 409704728921$
	7	$-2^6\cdot 7^{21}\cdot 709861\cdot 1080713$	$2^2\cdot 3^4\cdot 7^{42}\cdot 53\cdot 1531\cdot 5519\cdot 62131\cdot 149684694787$
$\Upsilon 22$	2	$-2^{25}\cdot 3^2\cdot 6043$	$2^{48}\cdot 7\cdot 687078607$
	3	$-2^4\cdot 3^{23}\cdot 7\cdot 7834259$	$2^2\cdot 3^{46}\cdot 37\cdot 311\cdot 587\cdot 2818358041$
	5	$-2^2\cdot 3^2\cdot 5^{22}\cdot 7\cdot 61\cdot 2458670741$	$2^6\cdot 5^{44}\cdot 11^2\cdot 29\cdot 709\cdot 257371\cdot 1235981\cdot 6006439$
$\Upsilon 24a$	2	$-2^{31}\cdot 3^4\cdot 353$	$2^{60}\cdot 7\cdot 17\cdot 3540071$
	3	$-2^4\cdot 3^{26}\cdot 7^2\cdot 37\cdot 55823$	$2^2\cdot 3^{52}\cdot 11\cdot 1583\cdot 10501\cdot 1380258457$
	5	$-2^2\cdot 3^4\cdot 5^{24}\cdot 7^3\cdot 9463\cdot 1015159$	$2^6\cdot 5^{48}\cdot 31\cdot 41\cdot 216724396286730092854051 1*$
$\Upsilon 24b$	2	$-2^{28}\cdot 3^2\cdot 1019$	$2^{54}\cdot 7\cdot 12641\cdot 2076143$
	3	$-2^4\cdot 3^{25}\cdot 7\cdot 3517\cdot 11987$	$2^2\cdot 3^{50}\cdot 11^3\cdot 23\cdot 167561325051733$
	5	$-2^2\cdot 3^2\cdot 5^{23}\cdot 7\cdot 34913\cdot 235654733$	$2^6\cdot 5^{46}\cdot 29\cdot 31\cdot 241\cdot 2909566712159350025 72891$
$\Upsilon 26a$	2	$-2^{32}\cdot 3^2\cdot 11\cdot 1571$	$2^{62}\cdot 96153491281$
	3	$-2^4\cdot 3^{31}\cdot 751\cdot 14243$	$2^2\cdot 3^{59}\cdot 11\cdot 23\cdot 61\cdot 198676251769691$
	5	$-2^2\cdot 3^2\cdot 5^{28}\cdot 7\cdot 73\cdot 23131\cdot 1652899$	$2^6\cdot 5^{56}\cdot 11\cdot 19\cdot 9719\cdot 3626449\cdot 549711087746599$
$\Upsilon 26b$	2	$-2^{29}\cdot 3^2\cdot 337\cdot 929$	$2^{56}\cdot 41\cdot 21871\cdot 1870919$
	3	$2^4\cdot 3^{29}\cdot 101\cdot 1130863$	$2^2\cdot 3^{58}\cdot 73\cdot 29927101391081233$
	5	$-2^2\cdot 3^2\cdot 5^{28}\cdot 17\cdot 41\cdot 14311199239$	$2^6\cdot 5^{55}\cdot 271\cdot 215309\cdot 5397898373174838980 71$

* The number 2167243962867300928540511 is not a prime number.

have no negative real zeros.

Now we define some L-functions attached to Siegel modular forms. Let sym^2 be the symmetric square representation of $GL(n, \boldsymbol{C})$, i.e.,

$$\mathrm{sym}^2 : GL(n, \boldsymbol{C}) \to GL\left(\frac{n(n+1)}{2}, \boldsymbol{C}\right).$$

For an eigenform $f \in S_k$, we put

$$L(s, f, \mathrm{sym}^2(\underline{\mathrm{st}})) := \prod_p \det(1_{15} - \mathrm{sym}^2(\underline{\mathrm{st}}_p(f)) p^{-s})^{-1}, \tag{2.1}$$

$$L(s, f, \mathrm{sym}^2(\underline{\mathrm{spin}})) := \prod_p \det(1_{10} - \mathrm{sym}^2(\underline{\mathrm{spin}}_p(f)) p^{-(s+2k-3)})^{-1}, \tag{2.2}$$

where $\underline{\mathrm{spin}}_p(f) := \mathrm{d}(\alpha_{0,p}, \alpha_{0,p}\alpha_{1,p}, \alpha_{0,p}\alpha_{1,p}\alpha_{2,p}, \alpha_{0,p}\alpha_{2,p})$. The right-hand sides of (2.1) and (2.2) converge absolutely and locally uniformly for $\mathrm{Re}(s)$ sufficiently large.

For $\mathrm{r} = \mathrm{sym}^2(\underline{\mathrm{st}})$ or $\mathrm{r} = \mathrm{sym}^2(\underline{\mathrm{spin}})$, we put

$$\Lambda(s, f, \mathrm{r}) := \Gamma(s, f, \mathrm{r}) L(s, f, \mathrm{r}),$$

where $\Gamma(s, f, \mathrm{r})$ is the suitable Γ-factor of $L(s, f, \mathrm{r})$.

Then we expect the following:

(C) *Let* $\mathrm{r} = \mathrm{sym}^2(\underline{\mathrm{st}})$ *or* $\mathrm{r} = \mathrm{sym}^2(\underline{\mathrm{spin}})$. *For any eigenform* $f \in S_k$, $\Lambda(s, f, \mathrm{r})$ *has a meromorphic continuation to the whole s-plane and satisfies the functional equation*

$$\Lambda(s, f, \mathrm{r}) = \varepsilon(f, \mathrm{r}) \Lambda(1-s, f, \mathrm{r}), \tag{2.3}$$

where $\varepsilon(f, \mathrm{r})$ *is a constant. Moreover,* $\Gamma(s, f, \mathrm{r})$ *has neither poles nor zeros at* $s = \sigma \in \boldsymbol{R}$, $\sigma > 1$ *and if* $\Lambda(s, f, \mathrm{r})$ *has a pole at* $s = \sigma \in \boldsymbol{R}$, $\sigma > 1$, *then* $f \in S_k^*$.

The following is proved in the same way as Theorem 1.

THEOREM 2. *If the condition* (C) *holds, Conjecture is equivalent to saying that any eigenform* $f \in S_k$ *satisfies the condition* (A).

PROOF. It is clear that any eigenform $f \in S_k$ satisfies the condition (A) if Conjecture holds. So we suppose that any eigenform $f \in S_k$ satisfies the condition (A).

The set of eigenvalues of $\mathrm{sym}^2(\underline{\mathrm{spin}}_p(f)) p^{-(2k-3)}$ is

$$\{1, \alpha_{1,p}, \alpha_{1,p}^{-1}, \alpha_{2,p}, \alpha_{2,p}^{-1}, 1, \alpha_{1,p}\alpha_{2,p}, \alpha_{1,p}\alpha_{2,p}^{-1}, \alpha_{1,p}^{-1}\alpha_{2,p}, \alpha_{1,p}^{-1}\alpha_{2,p}^{-1}\} \tag{2.4}$$

and that of $\mathrm{sym}^2(\underline{\mathrm{st}}_p(f))$ is

$$\begin{aligned}\{1, \alpha_{1,p}, \alpha_{1,p}^{-1}, \alpha_{2,p}, \alpha_{2,p}^{-1}, 1, \alpha_{1,p}^2, \alpha_{1,p}^{-2}, \alpha_{2,p}^2, \alpha_{2,p}^{-2},\\ 1, \alpha_{1,p}\alpha_{2,p}, \alpha_{1,p}\alpha_{2,p}^{-1}, \alpha_{1,p}^{-1}\alpha_{2,p}, \alpha_{1,p}^{-1}\alpha_{2,p}^{-1}\}.\end{aligned} \tag{2.5}$$

If f is of type I, then the set (2.4) is

$$\{1,\ \pm p^{a_1},\ \pm p^{-a_1},\ \pm p^{a_2},\ \pm p^{-a_2},\ 1,\ p^{a_1+a_2},\ p^{a_1-a_2},\ p^{-a_1+a_2},\ p^{-a_1-a_2}\}\,,$$

where a_1 and a_2 are independent of p. So $L(s, f, \mathrm{sym}^2(\underline{\mathrm{spin}}))$ has a pole at $1+a_1+a_2$ and $f \in S_k^*$ by the condition (C). But this is impossible because of the form of (1.1).

If f is of type II, then the set (2.5) is

$$\{1, \mathrm{e}^{i\theta}, \mathrm{e}^{-i\theta}, \pm p^a, \pm p^{-a}, 1, \mathrm{e}^{2i\theta}, \mathrm{e}^{-2i\theta}, p^{2a}, p^{-2a},\\ 1, \pm p^a \mathrm{e}^{i\theta}, \pm p^{-a}\mathrm{e}^{i\theta}, \pm p^a \mathrm{e}^{-i\theta}, \pm p^{-a}\mathrm{e}^{-i\theta}\}\,,$$

where a is independent of p. So $L(s, f, \mathrm{sym}^2(\underline{\mathrm{st}}))$ has a pole at $1+2a$ and $f \in S_k^*$ by the condition (C). But this is impossible because of the form of (1.1).

If f is of type III, then the set (2.4) is

$$\{1, p^a \mathrm{e}^{i\theta}, p^{-a}\mathrm{e}^{-i\theta}, p^a \mathrm{e}^{-i\theta}, p^{-a}\mathrm{e}^{i\theta}, 1, p^{2a}, \mathrm{e}^{2i\theta}, \mathrm{e}^{-2i\theta}, p^{-2a}\}\,,$$

where a is independent of p. So $L(s, f, \mathrm{sym}^2(\underline{\mathrm{spin}}))$ has a pole at $1+2a$ and $f \in S_k^*$ by the condition (C). Thus Theorem 2 is proved. □

REMARKS. (i) If $f \in S_k^*$, then $L(s, f, \mathrm{sym}^2(\underline{\mathrm{spin}}))$ diverges at $s=2$.

(ii) In (C), we don't assert absolute convergence of $L(s, f, \mathrm{r})$ for $\mathrm{Re}(s) > 1$.

At the referee's suggestion we study the sign of $\lambda_f(p)$. For this, we need the following condition (cf. §1 (III)).

(D) *Let $f \in S_k$ be an eigenform. Then the Γ-factor of $L(s, f, \mathrm{sym}^2(\underline{\mathrm{spin}}))$ is*

$$\Gamma(s, f, \mathrm{sym}^2(\underline{\mathrm{spin}})) = \prod_{j=1}^{10} \Gamma_{\mathbf{R}}(s+\gamma_j) \qquad (\gamma_j \in \tfrac{1}{2}\mathbf{Z})\,.$$

The function $\Lambda(s, f, \mathrm{sym}^2(\underline{\mathrm{spin}}))$ has a meromorphic continuation to the whole s-plane and satisfies the functional equation (2.3). *In any vertical strip, there exists a constant $K > 0$ such that*

$$L(s, f, \mathrm{sym}^2(\underline{\mathrm{spin}})) = O(\mathrm{e}^{t^K}) \qquad (s = \sigma + it)$$

as $t \to \infty$. The function $\Lambda(s, f, \mathrm{sym}^2(\underline{\mathrm{spin}}))$ has a pole if and only if $f \in S_k^*$. Moreover, if $f \notin S_k^*$, $L(1, f, \underline{\mathrm{st}}) \neq 0$ and $L(1, f, \mathrm{sym}^2(\underline{\mathrm{spin}})) \neq 0$.

Then we have:

THEOREM 3. *Let $f \in S_k$ be an eigenform. Suppose that the condition* (D) *holds. If f satisfies the Ramanujan–Petersson conjecture, then $\lambda_f(p)$ changes sign infinitely often.*

Then Corollary 1 takes the following form.

COROLLARY 2. *Let $f \in S_k$ be an eigenform. Under the conditions* (A) *and* (D), *f belongs to the Maaß space S_k^* if and only if $\lambda_f(p) > 0$ for almost all p.*

For our purposes we consider the Dirichlet series

$$\varphi_1(s)=\sum_{n=1}^{\infty}\frac{a_n}{n^s}:=L(s+k-3/2,f,\underline{\mathrm{spin}})\,,$$

$$\varphi_2(s)=\sum_{n=1}^{\infty}\frac{b_n}{n^s}=L(s,f,\underline{\mathrm{spin}}^{\otimes 2}):=\zeta(s)L(s,f,\underline{\mathrm{st}})L(s,f,\mathrm{sym}^2(\underline{\mathrm{spin}}))\,,$$

where $s=\sigma+it$. Note that $b_p=a_p^2=\lambda_f(p)^2p^{-(2k-3)}$ for all p.

For $r=1, 2$, define the function

$$N_{\varphi_r}(\sigma, T):=\#\{\rho=\beta+i\gamma\,|\,\varphi_r(\rho)=0 \text{ with } \beta\geq\sigma \text{ and } |\gamma|\leq T\}\,.$$

LEMMA 2. *Let $f\in S_k$ be an eigenform. Suppose that the condition* (D) *holds. If f satisfies the Ramanujan–Petersson conjecture, then $\varphi_r(s)$ $(r=1,2)$ has the following four properties*:

(1)
$$\sum_{p\leq x}a_p\log p=-\sum_{|\gamma|\leq T}\frac{x^\rho}{\rho}+O\left(\frac{x(\log x)^2}{T}\right),$$

$$\sum_{p\leq x}b_p\log p=x-\sum_{|\gamma|\leq T}\frac{x^\rho}{\rho}+O\left(\frac{x(\log x)^2}{T}\right),$$

where the sum on the left-hand side is over primes and that on the right-hand side is over the zeros $\rho=\beta+i\gamma$ of $\varphi_r(s)$ with $|\gamma|\leq T\leq x^{1/2}$ and $\beta\geq 0$.

(2)
$$\varphi_r(s)\neq 0 \text{ in the region } \sigma\geq 1-\frac{A}{\log(2+|t|)}\,,\qquad A>0\,.$$

(3)
$$N_{\varphi_r}(\sigma, T)\ll T^{c(1-\sigma)}\,,\qquad c>0\,,$$

uniformly for $1/2\leq\sigma\leq 1$, $T\to\infty$.

(4)
$$N_{\varphi_r}(0, T)\ll T\log T\,.$$

We say that $\varphi_r(s)$ has the Hoheisel property if the four properties above hold.

PROOF. By an application of Perron's formula to the function $-\varphi_r'(s)/\varphi_r(s)$, we have

$$\frac{1}{2\pi i}\int_{1+\delta-iT}^{1+\delta+iT}-\frac{\varphi_r'(s)}{\varphi_r(s)}\frac{x^s}{s}ds+O\left(\frac{x(\log x)^2}{T}\right)=\begin{cases}\sum_{p\leq x}a_p\log p\,, & \text{if } r=1\,,\\ \sum_{p\leq x}b_p\log p\,, & \text{if } r=2\,,\end{cases}\tag{2.6}$$

where $\delta>0$. The integral on the left-hand side of (2.6) is

$$\frac{1}{2\pi i}\int_{1+\delta-iT}^{1+\delta+iT}-\frac{\varphi_r'(s)}{\varphi_r(s)}\frac{x^s}{s}ds=\frac{1}{2\pi i}\int_C+\frac{1}{2\pi i}\left(\int_{-u+iT}^{1+\delta+iT}+\int_{-u-iT}^{-u+iT}+\int_{1+\delta-iT}^{-u-iT}\right),$$

where the contour C consists of four lines joining the points $1+\delta-iT$, $1+\delta+iT$, $-u+iT$, $-u-iT$ and $1+\delta-iT$. Then we have

$$\frac{1}{2\pi i}\int_C = \begin{cases} -\sum_{|\gamma|\le T}\dfrac{x^\rho}{\rho}+O(1), & \text{if } r=1, \\ x-\sum_{|\gamma|\le T}\dfrac{x^\rho}{\rho}+O(1), & \text{if } r=2, \end{cases}$$

$$\frac{1}{2\pi i}\left(\int_{-u+iT}^{1+\delta+iT}+\int_{-u-iT}^{-u+iT}+\int_{1+\delta-iT}^{-u-iT}\right)=O\left(\frac{x(\log x)^2}{T}\right)$$

as $u\to\infty$ (cf. Chandrasekharan [7, Chapter V Theorem 1]). Thus $\varphi_r(s)$ has the property (1).

If we apply the same method as in the proof of Murty's Theorem 3 in [18] to $f(s)=\varphi_2(s)$ and $g(s)=\zeta(s)\varphi_1(s)^2\varphi_2(s)$, we have $\varphi_1(1+it)\neq 0$ and $\varphi_2(1+it)\neq 0$. Then the same method as in the proof of Titchmarsh's Theorem 3.5 in [22, Chapter III] gives a zero-free region of the type (2) for $f(s)$ and $g(s)$. Thus $\varphi_r(s)$ has the property (2).

The property (3) is proved in the same way as that in Chandrasekharan [7, Chapter V Theorem 3]. The property (4) has been shown by Berndt [5, Theorem 10]. □

Proof of Theorem 3. By Lemma 2, we know that $\varphi_r(s)$ has the Hoheisel property. Then, by Moreno [17] (cf. Murty [18, Lemma 4]), there is a $v>0$, such that if $h=x^\theta$, $v<\theta<1$, we have

(i) $$\sum_{x\le p\le x+h} a_p=o(h),$$

(ii) $$\sum_{x\le p\le x+h} b_p\gg h.$$

Suppose that $\lambda_f(p)=a_p p^{k-3/2}$ doesn't change sign in $x\le p\le x+h$, for $h=x^\theta$, where θ satisfies the condition above. Then we have

$$\left|\sum_{x\le p\le x+h} a_p\right|=\sum_{x\le p\le x+h}|a_p|\gg\sum_{x\le p\le x+h}a_p^2=\sum_{x\le p\le x+h}b_p,$$

since f satisfies the Ramanujan–Petersson conjecture. But this is a contradiction to (i) and (ii). Therefore, $\lambda_f(p)$ changes sign in the interval $x\le p\le x+h$. This completes the proof of Theorem 3. □

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