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# Extremal Elliptic Fibrations and Singular $K3$ Surfaces

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## 1. Introduction.

Let $f: X \to \mathbf{P}^1$ be a relatively minimal elliptic fibration having a section (so-called Jacobian fibration) over an algebraically closed field $k$ of characteristic $p \geq 0$. It is well known that for such a non-trivial fibration in characteristic $\neq 2, 3$ the number $s$ of singular fibres is at least 2, and if $f$ is non-isotrivial, then $s$ is at least 3. In characteristic zero a complete list of non-trivial fibrations over $\mathbf{P}^1$ with three or fewer singular fibres together with Kodaira fibre types was given by U. Schmickler-Hirzebruch ([Sc-H]). Her method used the monodromy actions around critical points *à la Kodaira*. It turned out that such a surface is either a rational, or a $K3$ surface. The case of $K3$ surfaces is of interest because of several reasons. In [N1] we discussed a different approach which, as the reader can see easily, is applicable for a similar problem of classifying such fibrations in positive characteristics. *A priori* up to the action of the absolute Frobenius the classification in characteristic $p \neq 2, 3$ should be the same as in characteristic zero. In fact an essentially new idea is to involve the Kodaira-Spencer class, especially the so-called characteristic $p$ function field analogue of Szpiro's conjecture, and the well-known theory of Ogg-Shafarevich (*cf.* [N2]). In this note we first recover the list of [Sc-H] by means of the approach mentioned above (Theorem 2.10 and the first part of theorem 3.4). Next a question arising here is to determine fo which $p$ the Weierstrass equation of a $K3$ surface in the given classification defines a supersingular, and hence (*a priori* modulo Artin's conjecture) unirational, $K3$ surface. To this end it is natural to use the works [P-S] and [In-S] since in characteristic zero $K3$ surfaces with three singular fibres are singular in the sense of [P-S] and [In-S] (the second part of Theorem 3.4). In a sense the note may be thought as a prelude to a complete classification of elliptic pencils with three or fewer singular fibres in positive characteristics (at least,

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$\neq 2, 3$). We remark that the classification problem in characteristics two or three is very difficult because of the presence of "wild" ramification. In these characteristics by means of the methods here one can in fact treat extremal elliptic pencils with non-zero Kodaira-Spencer class (*cf.* [N4]). It should be noted that the idea involving the characteristic $p$ function field analogue of Szpiro's conjecture and the theory of Ogg-Shafarevich allows us to classify also elliptic fibrations over an elliptic base with one singular fibre in characteristic $p \neq 2, 3$. The main result in this case, generalizing Stiller's classification over the complex numbers, says that up to the action of the Frobenius morphism (and up to isomorphism) there are exactly two such fibrations (see [N3]).

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## 2. Rational elliptic surfaces with three or fewer singular fibres.

We are based generically on [K], [Sh1] and [Is-S]. In what follows we shall use Kodaira's notation assuming the known Néron-Kodaira classification of singular fibres for elliptic fibrations (see, for example, [K], [Is-S]). Let $f: X \to C$ be a relatively minimal elliptic fibration having a section (so-called Jacobian) over an algebraically closed field $k$. Throughout the note we shall assume that $\mathrm{char}(k)=0$, although most of facts about elliptic surfaces we derive in this paragraph remains valid in characteristics $\neq 2, 3$. It is well known that then $X$ is birationally isomorphic to a surface in $\mathbf{P}^2 \times C$ given by an equation in the Weierstrass form

$$y^2 = x^3 + \alpha x + \beta \qquad (\alpha, \beta \in k(C)) \tag{2.1}$$

It is easy to see that (2.1) has a singular fibre over $t \in C$ if and only if $\Delta(t)=0$, where $\Delta = 4\alpha^3 + 27\beta^2$ is the discriminant of (2.1).

DEFINITION 2.2. We say that the fibration $f: X \to C$ has potentially good reduction (in the terminology of J. Tate and J.-P. Serre) at $t \in C$ if $X_t := f^{-1}(t)$ becomes non-singular after a suitable base change.

Define the $j$-invariant of (2.1) as the function $j = 4\alpha^3/\Delta$ and let $v_t(a)$ denote the order of vanishing at $t$ of $a \in K(C)$. From the classification of singular fibres it is easy to see that the family (2.1) has a potentially good reduction at $t \in C$ if and only if $v_t(j) \geq 0$. Let $r$ denote the (Mordell-Weil) rank of the generic fibre over $k(C)$ and $\rho$ - the Picard number of $X$. We have the following well-known formula ([Sh1], [T]).

$$\rho = 2 + \sum_{t \in C} (n_t - 1) + r \tag{2.3}$$

where $n_t$ denotes the number of components of $X_t$.

**Shioda's formula** ([Sh1]). *For a non-trivial Jacobian fibration $f: X \to C$*

$$r+b_2-\rho=4g(C)-4+2s-s_1 \tag{2.4}$$

*where as usually $b_2$ denotes the second Betti number of $X$, $s$ - the number of singular fibres and $s_1$ is the number of semi-stable singular fibres.*

It should be noted that formula (2.4) may be thought in a different context and a more general situation (cf. [N4]). Now assume $C \simeq \mathbf{P}^1$, then (2.1) becomes

$$y^2=x^3+\alpha(t)x+\beta(t)\,, \qquad t\in\mathbf{P}^1\,,$$

where $\alpha(t)$ and $\beta(t)$ are the polynomials of degrees $4d$ and $6d$ respectively. In this case the surface $X$ has the following invariants: $\chi(\mathcal{O}_X)=d$, $p_g=d-1$ and $K_X \sim (d-2)F$, where $K_X$ and $F$ denote the canonical class and fibre class of $X$.

From formula (2.4) one sees easily that $s$ is at least 2 for a non-trivial fibration $f: X \to \mathbf{P}^1$. Moreover if $s=2$ then $f$ is isotrivial. In fact from (2.3), (2.4) and Kodaira's formula ([K, Theorem 12.2]) it follows that $X$ has potentially good reduction everywhere. Then the standard arguments ([Is-S], 10.1, example 4) show that $f$ is isotrivial. We remark that this fact remains true in characteristics $>3$ (*cf.* [N2]). Also as an immediate consequence of (2.4) if $s=3$ then $X$ is either a rational surface with $r\le 2$ or a $K3$ surface. In the second case $s_1=0$ and $f$ is extremal in the sense of ([M-P]), i.e., $h^{1,1}=\rho$ and $r=0$, in particular, $X$ is a singular $K3$ surface in the sense of [P-S] and [In-S].

**2.5.** We review some facts from the theory of Mordell-Weil lattices essentially due to Shioda we shall need in what follows. As noted in the Introduction with regard to the positive characteristic case one should bear in mind that the theory works in any characteristic (see [Sh2]). Let $\mathcal{S}$ denote the (Mordell-Weil) group of sections with a natural bilinear pairing $\langle\ ,\ \rangle$. Assume that $\sigma_0$ is a section (as zero section) which defines the group structure on each smooth fibre. We derive the following facts (*cf.* [Sh 1-2], [C-Z]).

1) For a section $\sigma$ we have the following relation between $\langle\sigma,\sigma\rangle$ and the intersection number $\sigma.\sigma_0$

$$\langle\sigma,\sigma\rangle=2+2\sigma.\sigma_0-(\text{correction terms}) \tag{2.6}$$

where the correction terms are given in Table 8.16 of [Sh2], or Table 1.19 of [C-Z], *e.g.*, as

$$\sum_{0\le k_i<n_i,i=1}^{s} k_i(n_i-k_i)/n_i$$

for semi-stable case with configuration $(I_{n_1},\cdots,I_{n_s})$.

2) Putting $N=l.c.m.$(exponents of the groups of components of multiplicity one of singular fibres), *e.g.* $N=l.c.m.(n_i)$ in the semi-stable case, we have $N\langle\sigma,\sigma\rangle\in\mathbf{Z}$.

3) Letting $m_t$ be the number of components of multiplicity one in $X_t$ we have

$$({}^{\#}\mathcal{S}_{\mathrm{tor}})^2 \mid \prod m_t \tag{2.7}$$

and for a basis $\{\sigma_1, \cdots, \sigma_r\}$ of $\mathscr{S}$ modulo torsion

$$\det\langle\sigma_i, \sigma_j\rangle = ({}^{\#}\mathscr{S}_{\text{tor}})^2/\prod m_t . |\det NS(X)| \tag{2.8}$$

It should be noted that in positive characteristic we also have (2.7) under the assumption of unimodularity of $NS(X)$ ([Sh2]). Besides, we shall use the following well-known fact from the general theory (of Néron-Ogg-Shafarevich) cited, *e.g.* in [Sh1], [C-Z].

4) If $X_t$ is not of type $I_n$ then there exists an natural injection

$$\mathscr{S}_{\text{tor}} \hookrightarrow G_t \tag{2.9}$$

into the group $G_t$ of components of multiplicity one.

In the theorem below we list the Kodaira fibre types of non-trivial rational elliptic fibrations $f: X \to \mathbf{P}^1$ having three singular fibres with a description of the corresponding Mordell-Weil group. We remark that configurations of singular fibre types motivate in part notations of the corresponding surfaces. For the other notations, see the proof of Theorem 3.4.

THEOREM 2.10 ($k=\mathbf{C}$). *We have the following possibilities for the case when $X$ is a rational surface with $s=3$.*

A) $r=0$ (*6 cases - in the notation of* [M-P]):
1) $\mathscr{S}_{\text{tor}} \simeq \mathbf{Z}/4$ : $X_{141}(I_1^*, I_4, I_1)$,
2) $\mathscr{S}_{\text{tor}} \simeq (\mathbf{Z}/2)^{\oplus 2}$ : $X_{222}(I_2^*, I_2, I_2)$,
3) $\mathscr{S}_{\text{tor}} \simeq \mathbf{Z}/3$ : $X_{431}(IV^*, I_3, I_1)$,
4) $\mathscr{S}_{\text{tor}} \simeq \mathbf{Z}/2$ : $X_{411}(I_4^*, I_1, I_1)$, $X_{321}(III^*, I_2, I_1)$,
5) $\mathscr{S}_{\text{tor}} \simeq \{0\}$ : $X_{211}(II^*, I_1, I_1)$,

B) $r=1$ (*8 cases*): *In all the cases $\sigma.\sigma_0=0$, where $\sigma$ is a generator of $\mathscr{S}$ modulo torsion. In each case the value $\langle\sigma, \sigma\rangle$ can be computed by* (2.8).
1) $\mathscr{S}_{\text{tor}} \simeq \mathbf{Z}/2$: $X_{321}^2(I_2^*, III, I_1)$, $X_{321}^3(I_1^*, III, I_2)$,
2) $\mathscr{S}_{\text{tor}} \simeq \{0\}$ : $X_{211}^1(I_1^*, IV, I_1)$, $X_{341}^1(III^*, II, I_1)$, $X_{341}^2(IV^*, III, I_1)$, $X_{431}^2(I_3^*, II, I_1)$, $X_{431}^3(I_1^*, I_3, II)$, $X_{442}^1(IV^*, I_2, II)$,

C) $r=2$ (*6 cases*):
1) $\mathscr{S}_{\text{tor}} \simeq \mathbf{Z}/3$: $X_{444}(IV, IV, IV)$,
2) $\mathscr{S}_{\text{tor}} \simeq \mathbf{Z}/2$: $X_{33}^1(I_0^*, III, III)$,
3) $\mathscr{S}_{\text{tor}} \simeq \{0\}$ : $X_{341}^3(I_1^*, III, II)$, $X_{442}^2(I_2^*, II, II)$, $X_{11}^1(0)(I_0^*, IV, II)$, $X_{444}^1(IV^*, II, II)$.

*Moreover the surfaces with prescribed fibre types above exist and they are in fact unique.*

PROOF. We outline a proof using the theory of Mordell-Weil lattices quoted in 2.5 above and the theory of elliptic modular surfaces with level $(n, m)$ structure ([C-P]), which is also applicable for the classification purpose in positive characteristics $\neq 2, 3$. First remark that for the local Euler numbers one has

$$\sum_t e(X_t) = 12 \tag{2.11}$$

A) All the facts about $\mathscr{S}_{\text{tor}}$ are as a direct consequence of (2.8) and (2.9) except for $X_{141}$ : $\mathscr{S}_{\text{tor}} \simeq \mathbf{Z}/4$ which can be followed from the theory of elliptic modular surfaces with level $(n, m)$ structure (*cf.* [C-P], [C]). Further as in [M-P] we are left with the following cases: $(I_9, II, I_1)$, $(I_8, I_2, II)$, $(I_5, I_5, II)$, $(I_0^*, I_3, I_3)$, $(I_6, I_3, III)$, $(I_6, IV, I_2)$, $(I_8, III, I_1)$.

The first 3 cases are ruled out since we must have $\mathscr{S}_{\text{tor}} \simeq \{0\}$ by (2.9) which contradicts (2.8). Similarly for the next three any torsion has order 2 (resp. 3 for $(I_6, IV, I_2)$) and at the same time $^{\#}\mathscr{S}_{\text{tor}} = 6$ which is impossible. The last case $(I_8, III, I_1)$ is almost obvious since $\mathscr{S}_{\text{tor}} \subseteq \mathbf{Z}/2$ by (2.9) which contradicts (2.8). Thus we are done.

B) In this case we have exactly one semi-stable fibre by Shioda's formula (2.4). Together with (2.11), it leaves us 11 possibilities 8 of which are actually realized. Further (2.9) shows that we have trivial torsion except for the first two cases. Three eliminated possibilities are of types:

1) $(I_2^*, I_2, II)$: with the notation above one has $N=2$, $\prod n_t = 8$. On the other hand $\mathscr{S}_{\text{tor}} \simeq \{0\}$ by (2.9) so that $\langle \sigma, \sigma \rangle = 1/8$ (by (2.8)) which is not possible in view of 2), 2.5.

2) $(I_0^*, IV, I_2)$ and $(I_0^*, I_3, III)$: in both the cases $N=6$, $\prod n_t = 24$ and $\mathscr{S}_{\text{tor}} \subseteq \mathbf{Z}/2$ by (2.9). So either $\langle \sigma, \sigma \rangle = 1/24$, i.e., $N\langle \sigma, \sigma \rangle \notin \mathbf{Z}$ which is again a contradiction, or (2.6) implies

$$1/6 = \langle \sigma, \sigma \rangle = 2 + 2\sigma.\sigma_0 - * - *2/3 - *1/2\,,$$

where $*$ is 0 or 1, which has no solution.

C) This case is much simpler because we have no semi-stable fibre so that we have exactly 6 possibilities all of which are realized with trivial torsion except for $(3IV)$ (resp. $(I_0^*, III, III)$): the arguments here are similar. First by (2.9) one has $\mathscr{S}_{\text{tor}} \subseteq \mathbf{Z}/3$ (resp. $\mathscr{S}_{\text{tor}} \subseteq \mathbf{Z}/2$). If $\mathscr{S}_{\text{tor}} \simeq \{0\}$ then $det\langle\ ,\ \rangle = 1/27$ (resp. 1/16). On the other hand $N=3$ (resp. 2) hence using 2), 2.5 we obtain the desired assertion.

Remark 2.12. We have four isotrivial cases $X_{444}^1$, $X_{33}^1$, $X_{444}$ and $X_{11}^1(0)$ which can be easily obtained from $X_{22}(II^*, II)$, $X_{33}(III^*, III)$, $X_{44}(IV^*, IV)$ and $X_{11}(j)(I_0^*, I_0^*)$ ($j$-invariant $=0$ or 1) in [M-P] respectively. In fact as an immediate consequense one can easily infer that together with them the following configurations: $X_{11}(j)(I_0^*, I_0^*)$ $j \in k$; $j=0$: $(I_0^*, 3II)$, $(2IV, 2II)$, $(IV, 4II)$, $(6II)$; $j=1$: $(4III)$ are exhaustive all isotrivial rational fibrations.

## 3. Elliptic $K3$ surfaces with three singular fibres.

**3.1.** We recall some basic facts on singular $K3$ surfaces from [P-S] and [In-S]. For an algebraic $K3$ surface $X$ let us denote by $N_X$ the sublattice of $H_2(X, \mathbf{Z})$ consisting of algebraic cycles (the Néron-Severi lattice) and $T_X := N_X^{\perp}$—the lattice of transcendental cycles (the orthogonal complement of $N_X$ in $H_2(X, \mathbf{Z})$). Let $p_X$ be the period on $T_X$, i.e. the linear functional on $T_X$ defined up to constants by

$$p_X(\gamma) := \int_\gamma \omega_X \qquad (\gamma \in T_X) \tag{3.2}$$

where $\omega_X$ is a non-vanishing holomorphic 2-form on $X$. If $X$ is singular, i.e. the Picard number is 20 ($= h^{1,1}$) - maximal possible, then the lattice $T_X$ has a natural orientation, namely a basis $\{x_1, x_2\}$ of $T_X$ is called *oriented* if the imaginary part $\mathfrak{I}(p_X(x_1)/p_X(x_2))$ is positive. For an oriented basis $\{x_1, x_2\}$ we define

$$Q_X := \begin{bmatrix} x_1^2 & x_1 \cdot x_2 \\ x_1 \cdot x_2 & x_2^2 \end{bmatrix} \tag{3.3}$$

and let

$$\mathcal{Q} := \left\{ Q = \begin{bmatrix} 2a & b \\ b & 2c \end{bmatrix} (a, b, c \in \mathbf{Z},\ a, c > 0,\ b^2 - 4ac < 0) \right\}$$

be the set of $2 \times 2$ positive-definite even integral matrices. We write $Q_1 \sim Q_2$ if and only if $Q_1 = {}^t\delta Q_2 \delta$ for some $\delta \in SL_2(\mathbf{Z})$ and denote by $\{Q\}$ the equivalence class of $Q$. Note that $\{Q_X\}$ is uniquely determined by $X$. The main fact in the theory of singular $K3$ surfaces is that the map $X \mapsto \{Q_X\}$ establishes a *1-1* correspondence from the set of singular $K3$ surfaces onto $\mathcal{Q}/SL_2(\mathbf{Z})$. The injectivity of this correspondence is essentially due to [P-S] (*cf.* [In-S]). The surjectivity follows from the corresponding result on singular abelian surfaces and an explicit construction of certain double coverings of Kummer surfaces coming from products of isogenous elliptic curves with complex multiplications (see [In-S]).

In the following theorem we give a full account of Kodaira fibre types of elliptic $K3$ surfaces $f: X \to \mathbf{P}^1$ with three singular fibres and show that there is an explicit correspondence between these $K3$ surfaces and rational elliptic surfaces with three singular fibres described in Theorem 2.10 that explains our notations of the complete classification.

THEOREM 3.4 ($k = \mathbf{C}$). *In the notation above elliptic $K3$ surfaces with $s = 3$ are of the following types*

A) $\mathcal{S}_{\mathrm{tor}} \simeq \mathbf{Z}/2 \oplus \mathbf{Z}/2$: $X^*_{222}(3I^*_2)$, $Q_X = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$;

B) $\mathcal{S}_{\mathrm{tor}} \simeq \mathbf{Z}/3$: $X_{444*}(3IV^*)$, $Q_X = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$;

C) $\mathscr{S}_{\text{tor}} \simeq \mathbf{Z}/2$: 1) $X^*_{411}(I^*_4, 2I^*_1)$, $Q_X = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$;

2) $X^*_{33}(2III^*, I^*_0)$, $Q_X = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$;

3) $X^*_{321}(III^*, I^*_2, I^*_1)$, $Q_X = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$;

D) $\mathscr{S}_{\text{tor}} \simeq \{0\}$: 1) $X^*_{211}(II^*, 2I^*_1)$, $Q_X = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$;

2) $X_{341^*}(III^*, IV^*, I^*_1)$, $\det Q_X = 24$;

3) $X^*_{431}(I^*_3, IV^*, I^*_1)$, $\det Q_X = 48$;

4) $X_{442^*}(2IV^*, I^*_2)$, $Q_X = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$;

5) $X^*_{444}(2II^*, IV)$, $Q_X = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$;

6) $X^*_{11}(0)(II^*, IV^*, I^*_0)$, $Q_X = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

*Moreover there is a correspondence between rational surfaces with* $s = 3$ *and their corresponding K3 surfaces which justifies our notation (see the proof below). In particular, these surfaces are unique. Furthermore*:

*—*$X^*_{222}$, $X^*_{33}$ *are isomorphic;* $X^*_{211}$, $X^*_{411}$ *are isomorphic and together with* $X_{442^*}$ *their Weierstrass equations define supersingular K3 surfaces in characteristics* $p \equiv -1$ $(mod\, 4)$;

*—*$X^*_{444}$, $X^*_{11}(0)$, $X_{444^*}$ *are isomorphic and together with* $X^*_{431}$ *their Weierstrass equations define supersingular K3 surfaces in characteristics* $p \equiv -1$ $(mod\, 3)$;

*—the Weierstrass equation of* $X^*_{321}$ *defines a supersingular K3 surface in characteristics* $p \equiv 5, 7$ $(mod\, 8)$;

*—the Weierstrass equation of* $X_{341^*}$ *defines a supersingular K3 surface in characteristics* $p \equiv 7, 11, 13, 17$ $(mod\, 24)$.

PROOF. Indeed all such $K3$ surfaces are extremal with no semi-stable fibre. So that it is easy to verify that they can be obtained from rational surfaces in Theorem 2.10 via suitable twist transforms, i.e., of the form: $\alpha \to t^2\alpha$, $\beta \to t^3\beta$, *e.g.*, for a bad fibre over $t = 0$ preserving the minimality of (2.1). Explicitly it looks as in the following scheme.

1) $X_{211}, X^1_{211} \to X^*_{211}$: whose defining equation thus is given as

$$y^2 = x^3 - 3(t^2-1)^2 x + 2t(t^2-1)^3\,;$$

2) $X_{411}, X_{141} \to X^*_{411}$: $y^2 = x^3 - 3(t^2-4)^2(t^2-3)x + t(t^2-4)^3(2t^2-9)$;

3) $X^1_{341}, X^2_{341}, X^3_{341} \to X_{341*}$: $y^2 = x^3 - 3t^3(t-1)^2x + 2t^5(t-1)^3$;

4) $X_{321}, X^2_{321}, X^3_{321} \to X^*_{321}$:
$$y^2 = x^3 - t(t-3)^2(4t-3)^2x + (t-1)(t-3)^3(4t-3)^3 ;$$

5) $X_{431}, X^2_{431}, X^3_{431} \to X^*_{431}$:
$$y^2 = x^3 - 3t^2(t-1)^2(8t-9)x + 2t^3(t-1)^3(8t^2-36t+27) ;$$

6) $X^1_{442}, X^2_{442} \to X_{442*}$: $y^2 = x^3 - 3(t^2-1)^3x + 2(t^2-1)^4t$;

7) $X_{222} \to X^*_{222}$:
$$y^2 = x^3 - 3t(t-1)^2(t^2+t+1)^2x - (t-1)^3(t^3+1)(t^2+t+1)^3 ;$$

8) $X_{444} \to X^*_{444}$: $y^2 = x^3 + t^5(t-1)^2$;

9) $X^1_{11}(0) \to X^*_{11}(0)$: $y^2 = x^3 + t^3(t-1)^4$;

10) $X^1_{33} \to X^*_{33}$: $y^2 = x^3 + t^3(t-1)^3x$;

11) $X^1_{444} \to X_{444*}$: $y^2 = x^3 + t^4(t-1)^4$.

Thus in this sense two rational configurations with $s=3$ are called dual if and only if they are corresponding to one $K3$ surface in the scheme above. So that there is another way to determine defining Weierstrass equations for the rational surfaces in Theorem 2.10 beginning with the Weierstrass equations of corresponding $K3$ surfaces via inverse twist transforms. For illustration we write down the defining Weierstrass equations for four isotrivial cases $X_{444}$, $X^1_{11}(0)$, $X^1_{33}$, $X^1_{444}$ corresponding to $X^*_{444}$, $X^*_{11}(0)$, $X^*_{33}$, $X_{444*}$, respectively: $y^2 = x^3 + t^2(t-1)^2$ $(X_{444})$; $y^2 = x^3 + t^3(t-1)$ $(X^1_{11}(0))$; $y^2 = x^3 + t(t-1)x$ $(X^1_{33})$; $y^2 = x^3 + t(t-1)$ $(X^1_{444})$.

Next for determining $\det Q_X$ and $Q_X$ we use (2.8), (2.9) (since $r=0$ in all the cases) and [In-S]. We just indicate briefly the arguments for each case below.

1) This follows from Theorem 1 of [In-S].

2) In case 2) using explicitly the correspondence above and defining equations we see the following obvious sections (in homogeneous coordinates):
$$[0, 1, 0] ; \quad [(t^2-4)(t^2-2t), 0, 1] ,$$
so that $|\mathscr{S}_{\text{tor}}| = 2$. Further one can exhibit as in the proof of Theorem 1 of [In-S]. So $Q_X$ has the desired form.

3) As for 3) $|\mathscr{S}_{\text{tor}}| = 1$ because of (2.9). Presumably in this case $Q_X = \begin{bmatrix} 6 & 0 \\ 0 & 4 \end{bmatrix}$.

4) The fact that $|\mathscr{S}_{\text{tor}}| = 2$ is because of (2.8) and exhibiting the following sections:
$$[0, 1, 0] , \quad [(t-3)(4t-3)t^{-4}, 0, 1] .$$

It remains to use a known table of positive-definite even integral binary quadratic forms of small discriminants.

5) The torsion subgroup is trivial by (2.9). Presumably $Q_X = \begin{bmatrix} 8 & 4 \\ 4 & 8 \end{bmatrix}$ in this case.

6) This case follows from 7) below since $X^*_{222}$ may be exhibited suitably as a triple covering of $X_{442^*}$.

7) Obviously $\mathcal{S}_{\mathrm{tor}} \simeq \mathbf{Z}/2 \oplus \mathbf{Z}/2$ by showing four sections explicitly:

$$[0, 1, 0]\,, \quad [(1-t^3)(1+t), 0, 1]\,, \quad [(1-t^3)(\rho t+\rho^2), 0, 1]\,, \quad [(1-t^3)(\rho^2 t+\rho), 0, 1]\,,$$

where $\rho^2+\rho+1=0$.

For the remaining cases we refer to Lemma 5.2 of [In-S] and standard arguments using (2.8)–(2.9). The only case with non-trivial torsion subgroup ($\mathcal{S}_{\mathrm{tor}} \simeq \mathbf{Z}/3$) is $X_{444^*}$. Here the three sections are

$$[0, 1, 0]\,, \quad [\pm t^2(t-1)^2, 0, 1]\,.$$

Further since every elliptic curve with complex multiplications is defined over some number field, so it remains to use the well-known result of Deuring on elliptic curves with complex multiplications and standard facts about the Picard number of a Kummer surface to complete the proof of Theorem 3.4.

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