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# A Calculation of the Orbifold Euler Number of the Moduli Space of Curves by a New Cell Decomposition of the Teichmüller Space

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**Abstract.** In [GW], S. B. Giddings and S. A. Wolpert proposed a procedure to obtain a new cell decomposition of the moduli space of curves. In this paper, we work out this procedure in detail. The number of cells in this new cell decomposition is smaller than that in other cell decompositions given in [BE, Ha, P3] and this makes the explicit computations of the orbifold Euler numbers of the moduli spaces for small genera easier. We check in many examples that they coincide with the known value.

## 1. Introduction.

We first recall some basic results of S. B. Giddings and S. A. Wolpert [GW]. Let $(R, \{p_i\}_{i=1}^n)$ be an $n$-pointed Riemann surface of any genus ($n \geq 2$). Given $n$ real numbers $\{r_i\}_{i=1}^n$ satisfying $\sum_{i=1}^n r_i = 0$, we obtain a unique abelian differential of the third kind that has $n$ simple poles at $p_i$ of residue $r_i$ and pure imaginary period for every Riemann surface. This differential has a particular trajectory structure; the sum of real trajectories emanating from zeros forms a finite graph $G$ on $R$. Cutting $R$ along this graph, we obtain finite numbers of components which is biholomorphic to a band of finite width along imaginary axis and infinite length along real axis in the complex plain:

$$R \setminus G = \coprod B_i ,$$
$$B_i \cong \mathbf{R} \times (0, b_i) \subset \mathbf{C} .$$

Thus every Riemann surface can be obtained by pasting such bands and the pattern of pasting is represented by the graph $G$. We show that the set of all marked Riemann surfaces that have the same pasting pattern forms a cell, parametrized by the widths of bands and the distances between zeros. We obtain a cell decomposition of the Teichmüller space by classifying marked Riemann surfaces according to their pasting pattern:

$$\mathcal{T}_g^n = \coprod_{G^*} C(G^*) ,$$

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where $C(G^*)$ is the set of all marked Riemann surfaces that have the same pasting pattern which is represented by the marked graph $G^*$. We denote by $G^*$ a graph with a marking (see §3 for details).

The cell decomposition obtained in this way was introduced by S. B. Giddings and S. A. Wolpert [GW]. Since it is not a cell decomposition of the decorated Teichmüller space [P3] but that of the Teichmüller space itself, the number of cells is smaller than that in other cell decompositions given in [BE, Ha, P3]. This cell decomposition is invariant under the action of the mapping class group and descends to the moduli space. Using this cell decomposition, we calculate the orbifold Euler number of the moduli space by the following formula ([Br]):

$$\chi(\mathcal{M}_g^n) = \sum_G (-1)^{\dim C(G)} \frac{1}{|\mathrm{Aut} G|}.$$

A certain characterization of these graphs enables us to combinatorially enumerate all the graphs, their automorphism groups and the dimensions of their corresponding cells, and thus, to calculate the orbifold Euler numbers. Since the number of cells is smaller than that of the known ones, explicit calculations of the orbifold Euler numbers get much simpler for small genera. We check that, in all of our examples, it coincides with the known value [HZ, P1].

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## 2. Band decomposition.

We first introduce the abelian differential considered in [GW].

PROPOSITION 2.1 (Giddings-Wolpert). *Let $(R, \{p_i\}_{i=1}^n)$ be an n-pointed Riemann surface of any genus $g$ and let $\{r_i\}_{i=1}^n$ be non-zero real numbers satisfying $\sum_{i=1}^n r_i = 0$. Then there exists a unique abelian differential of the third kind on $R$, denoted by $\omega$, which satisfies the following conditions*;

(a) *$\omega$ has simple poles at $p_i$ $(i = 1, \cdots, n)$ of residue $r_i$ and no other poles.*

(b) *$\omega$ has pure imaginary periods.*

REMARK. When $g = 0$ we need only (a) since the condition (b) is always valid.

Because of the condition (b), any imaginary trajectory of $\omega$ is closed and any real trajectory of $\omega$ is not closed; it begins and ends at different poles and zeros (see [GW]). The sum of real trajectories that run through zeros forms a graph $G$ on $R$. Precisely it is defined as follows.

DEFINITION. Every zero or pole of $\omega$ is a vertex of $G$. An edge of $G$ is a real trajectory of $\omega$ that begins or ends at some zero of $\omega$.

A characterization of these graphs for the combinatorial enumeration will be given later in Proposition 2.3. If we cut $R$ along this graph $G$ then we obtain finite numbers of components. Each component is formed by all the real trajectories that are homotopic rel {poles} to each other. In fact, we can see that this component is a band in the following sense.

PROPOSITION 2.2. *Let $B$ be a component of $R \setminus G$. Fix a point $z_0$ in $B$. The map $f : B \to \mathbf{C}$ defined by*

$$f(z) := \int_{z_0}^{z} \omega$$

*is holomorphic and*

$$Image(f) = \mathbf{R} \times (a, a+b) \subset \mathbf{C}$$

*for some $a$ and $b \in \mathbf{R}$.*

PROOF. We first show that every real trajectory in $B$ has the same endpoints. Suppose that there exist two real trajectories $\gamma_1, \gamma_2$ in $B$ that begin at different poles of $\omega$, say $p_1, p_2$ respectively. Then there must be some zero $z$ of $\omega$ and a real trajectory $\gamma$ in $B$ between $\gamma_1$ and $\gamma_2$ such that $\gamma$ begins at $z$ and $z$ is connected to $p_1$ and $p_2$ by two real trajectories. But this cannot happen because a real trajectory that begins at a zero must be contained in $G$. Hence all the real trajectories in $B$ have the same initial point. Similarly, they must have the same terminal point. Let $p$ be the above initial point of all the real trajectories in $B$. We may choose $z_0$ close enough to $p$ so that there is no zero near $p$ and $z_0$. Let $\delta$ be the imaginary trajectory that runs through $z_0$ and let $z_1$ and $z_2$ be the intersection points of $\delta$ and $\partial \bar{B}$. Notice that there are two intersection points (possibly they are equal). Since the length of $\delta$

$$\int_{\delta} \omega = 2\pi i \mathrm{Res}(\omega, p)$$

is finite, the following lengths $a, b$ are finite:

$$a := \int_{z_0}^{z_1} \omega, \quad b := \int_{z_1}^{z_2} \omega .$$

The paths from $z_0$ to $z_1$ and from $z_1$ to $z_2$ are taken along $\delta$ in the above integrals. We define $f$ by

$$f(z) = \int_{z_0}^{w} \omega + \int_{w}^{z} \omega, \quad z \in B,$$

where $w$ is the intersection point of $\delta$ and the real trajectory that runs through $z$ and the paths from $z_0$ to $w$ and from $w$ to $z$ are taken along the imaginary and the real trajectories respectively (Fig. 1). The image of $f$ is easily seen to be $\mathbf{R} \times (a, a+b) \subset \mathbf{C}$, which is simply-connected. Hence, in fact, the definition of $f$ does not depend on the choice of the path from $z_0$ to $z$ and also the choice of $z_0$ is not essential. □

The graph $G$ derived from the abelian differential of Proposition 2.1 is characterized by the following;

PROPOSITION 2.3. *The graph $G$ derived from the abelian differential of Proposition* 2.1 *satisfies the following conditions.*

(a) *$G$ is a connected finite graph with the orientation on edges.*
(b) *a cyclic order of edges around each vertex is given.*
(c) *$G$ has two types of vertex, called "pole" and "zero".*
(d) *a numbering among poles is given.*

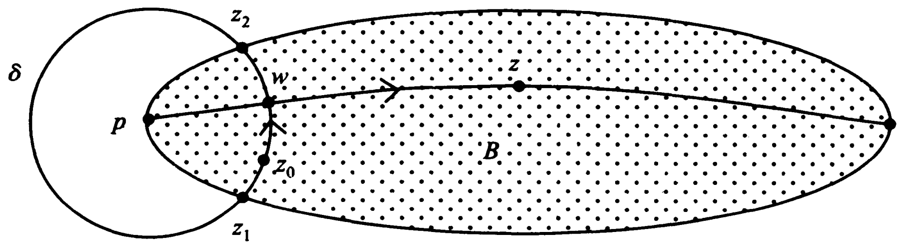


Fig. 1

(e) *edges around a pole are all incoming or all outgoing.*
(f) *a zero has incoming and outgoing edges alternately with respect to the cyclic order.*
(g) *no edge has the same end point and no edge has only poles as its end points.*
(h) *every closed edge-path (see definition below) of $G$ has exactly two poles.*

TERMINOLOGY. Considering $G$ as a ribbon graph, we obtain a surface $F(G)$ with some boundary components. Each boundary component gives a closed chain of edges of $G$ (Fig. 2), called a *closed edge-path* of $G$ (see [P2]).

PROOF OF PROPOSITION 2.3. The orientation on trajectories of an abelian differential can be canonically defined, hence edges of $G$ have the orientations. Numbering among poles comes from the numbering among marked points $\{p_i\}$. The trajectory structure around zeros or poles of general abelian differential leads to the conditions (e) and (f). If some edge has the same end point then the period of $\omega$ with respect to this loop is real, which contradicts to the condition (b) of Proposition 2.1. An edge of $G$ has at least one zero as its end point by definition, hence (g) follows. Condition (h) is a consequence of Proposition 2.2; a closed edge-path $c$ of $G$ corresponds to the boundary of some band $B$, a component of $R \setminus G$. All the real trajectories in $B$ have the same endpoints $p_1$, $p_2$, as mentioned in the proof of Proposition 2.2, hence $c$ has no pole other than $p_1$, $p_2$. □

REMARK. A pole that has all outgoing (resp. incoming) edges is called a *positive* (resp. *negative*) pole. A closed edge-path of $G$ has one positive and one negative pole. An edge-path that begins at positive pole and ends at negative pole is called an *oriented edge-path*. A closed edge-path of $G$ consists of exactly two oriented edge-paths.

Thus every Riemann surface can be decomposed into bands each of which is biholomorphic to $\mathbf{R} \times (0, b)$, by cutting along the graph $G$. Conversely, every Riemann surface can be reconstructed by pasting bands according to the graph $G$, which has the information how many bands we need and how to paste them. Each closed edge-path of $G$ corresponds to some band in this band decomposition; two poles are regarded to be at infinity and two oriented edge-paths correspond to two sides of a band, $\mathbf{R} \times \{0\}$ and $\mathbf{R} \times \{b\}$. If the graph $G$ has $k$ closed edge-paths then we take $k$ bands of the form $\mathbf{R} \times [a, a+b]$ in $\mathbf{C}$. On both sides of bands, we put vertices and edges according to the corresponding oriented edge-paths. Notice that each edge of $G$ corresponds to two edges that are drawn on different sides of bands. Then

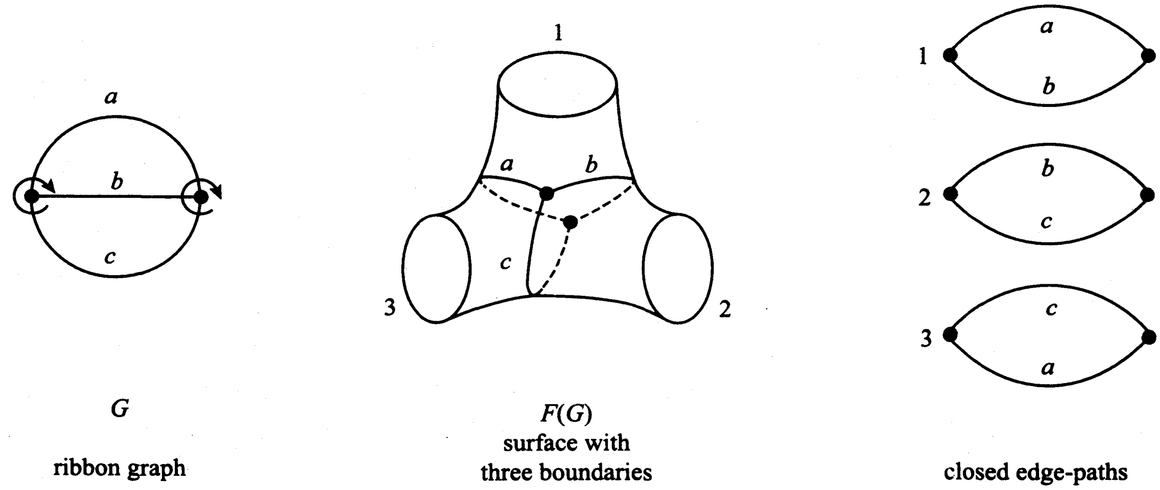


Fig. 2

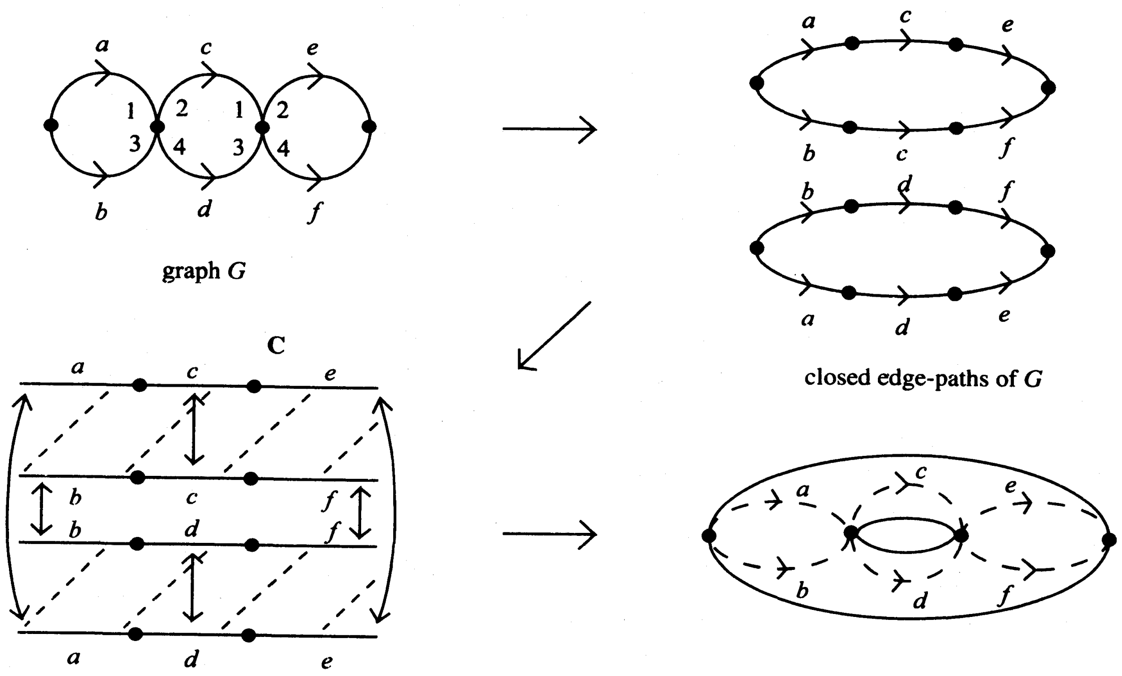


Fig. 3

we identify these two edges by parallel transformation in **C** and add some vertices at infinity to obtain a closed Riemann surface (Fig. 3). Local coordinates around inner points of bands and around edges are defined simply by $z$, the coordinate of **C**. But around vertices that are put on sides of bands, the total angle become larger than $2\pi$ by pasting edges, therfefore we take local coordinate $w = z^{1/n}$ if the vertex has total angle $2n\pi$. Local coordinates around added points are defined by $w = c \log z$. Notice that the abelian differential $dz$ on $\mathbf{C}$ descends to this Riemann surface and its zeros correspond to vertices on sides of bands and its poles correspond to added vertices. This abelian differential satisfies the conditions in Proposition 2.1. Thus we can reconstruct a Riemann surface from a graph characterized by the conditions of Proposition 2.3. In the next section, we classify Riemann surfaces by their corresponding graphs and give a new cell decomposition of the Teichmüller space.

## 3. Cell decomposition.

Throughout this section, we fix $g \geq 0, n \geq 2$ and a tuple of non-zero real numbers $\{r_i\}_{i=1}^n$ such that $\sum_{i=1}^n r_i = 0$. We denote by $\mathcal{T}_g^n$ the Teichmüller space of genus $g$, $n$-pointed Riemann surfaces. For each marked Riemann surface $R$, we can obtain a marked graph, denoted by $G_R^*$, by using the abelian differential of Proposition 2.1 with marking defined by that of $R$. Here a marking of a graph $G$ means an embedding of $G$ to some fixed surface $S$ and we denote by $G^*$ a graph with a marking. The graph $G_R^*$ satisfies certain conditions. To describe these, we use the following notations for a graph of Proposition 2.3.

NOTATION.
$n(G)$ =the number of poles of $G$,
$k(G)$ =the number of closed edge-paths of $G$,
$\chi(G)$ =the Euler number of $G$,
$p_i$ = the $i$-th pole of $G$ with respect to the numbering among poles of $G$.

We can easily show the following conditions for $G^* = G_R^*$:

$$n(G^*) = n\,, \quad \chi(G^*) + k(G^*) = 2 - 2g\,,$$

$$p_i : \begin{cases} \text{positive} & \text{if} \quad r_i > 0 \\ \text{negative} & \text{if} \quad r_i < 0\,. \end{cases}$$

Define the set of all marked Riemann surfaces that have the given marked graph $G^*$ as their corresponding graph:

$$C(G^*) := \{R \in \mathcal{T}_g^n \mid G_R^* \text{ is isomorphic to } G^*\}\,.$$

We can decompose $\mathcal{T}_g^n$ into these sets:

$$\mathcal{T}_g^n = \coprod_{G^* \in \mathcal{G}} C(G^*)\,,$$

$$\mathcal{G} = \left\{ G^* \text{of Proposition 2.1} \;\middle|\; \begin{array}{l} n(G^*) = n,\ \chi(G^*) + k(G^*) = 2 - 2g, \\ p_i \text{ is positive if } r_i > 0, \text{ negative if } r_i < 0 \end{array} \right\}.$$

Next we show that every $C(G^*)$ is a cell. In the previous section, we considered how to construct a Riemann surface from a given graph, where we did not mention the lengths of edges or the widths of bands. These values are necessary to exactly determine a Riemann surface. We consider below the detailed description of constructing a marked Riemann surface from the given graph $G^*$. Let $\omega$ be the abelian differential derived from a marked Riemann surface $R$. Let $G^* = G_R^*$ be the marked graph obtained from $R$. Suppose that $G^*$ has $k$ closed edge-paths, $c_1, \cdots, c_k$, and $l+1$ zeros, $z_0, \cdots, z_l$. Denote the band that corresponds to the closed edge-path $c_i$ by $B_i$ and its width by $b_i$. Let $d_i$ be the distance between $z_0$ and $z_i$ along the real direction, that is

$$d_i := \operatorname{Re}\left(\int_{z_0}^{z_i} \omega\right).$$

Because of the condition (b) of Proposition 2.1, this integral is well-defined and is called time function in [GW].

In this way, given a marked Riemann surface $R$, we obtain a marked graph $G^*$ and parameters $(b_1, \cdots, b_k, d_1, \cdots, d_l)$. These parameters satisfy the following relations; if a pole $p_i$ is contained in closed edge-paths $c_{i_1}, \cdots, c_{i_j}$ then

$$b_{i_1} + \cdots + b_{i_j} = |r_i|.$$

If there exists an oriented edge-path that connects $z_i$ and $z_j$ in this order then

$$d_i < d_j \quad (d_0 = 0).$$

NOTATION. We denote these relations corresponding to a graph $G$ by $(*)_G$.

Conversely, we define a marked Riemann surface from a given graph $G^*$ and corresponding parameters $(b_1, \cdots, b_k, d_1, \cdots, d_l)$ as follows. Take bands $B_i = \mathbf{R} \times [a_i, a_i + b_i]$ in $\mathbf{C}$, where $a_i$'s are chosen so that $B_i \cap B_j = \emptyset$ $(i \neq j)$. On each side of bands, we put vertices $z_j$, according to the corresponding oriented edge-path, so that their real coordinates in $\mathbf{C}$ are given $d_j$. Identification of edges by parallel transformation and the coordinates around zeros are the same as mentioned in the last section. Coordinate around added point, or pole, $p_i$ is defined by $w = r_i \log z$. The marking of this Riemann surface is cannonically defined by that of $G^*$ since each component of $R \setminus G$ is simply-connected.

DEFINITION. We denote a marked Riemann surface obtained as above by

$$R(G^*; b_1, \cdots, b_k, d_1, \cdots, d_l).$$

REMARK. This definition depends on a choice of the numberings among closed edge-paths of $G^*$ and among zeros of $G^*$.

The graph $G_R^*$ derived from $R = R(G^*; b_1, \cdots, b_k, d_1, \cdots, d_l)$ is isomorphic to the original $G^*$, therefore $R(G^*; b_1, \cdots, b_k, d_1, \cdots, d_l)$ is contained in $C(G^*)$.

PROPOSITION 3.1. *The map $F$ defined by*

$$F : \{(b_1, \cdots, b_k, d_1, \cdots, d_l) \in \mathbf{R}_+^k \times \mathbf{R}^l \mid \textit{relation}\,(*)_G\} \to C(G^*),$$

$$F(b_1, \cdots, b_k, d_1, \cdots, d_l) := R(G^*; b_1, \cdots, b_k, d_1, \cdots, d_l)$$

*is bijective.*

PROOF. Every marked Riemann surface in $C(G^*)$ is decomposed into bands by Propositions 2.1, 2.2, with parameters $(b_1, \cdots, b_k, d_1, \cdots, d_l)$ satisfying relations $(*)_G$, therefore the map $F$ is surjective. Suppose that

$$F(b_1, \cdots, b_k, d_1, \cdots, d_l) = F(b'_1, \cdots, b'_k, d'_1, \cdots, d'_l),$$

or equally

$$R(G^*; b_1, \cdots, b_k, d_1, \cdots, d_l) = R(G^*; b'_1, \cdots, b'_k, d'_1, \cdots, d'_l).$$

Since the markings of both Riemann surfaces are defined by the same marked graph $G^*$ and the numberings among closed edge-paths of $G^*$ and among zeros of $G^*$ are fixed, the widths of bands and the distances between zeros should be all equal:

$$(b_1, \cdots, b_k, d_1, \cdots, d_l) = (b'_1, \cdots, b'_k, d'_1, \cdots, d'_l). \quad \square$$

There are $n$ relations between $b_1, \cdots, b_k$ among $(*)_G$, which come from the given residues $r_i$ of the pole $p_i$, and these residues satisfy the relation $\sum_{i=1}^n r_i = 0$. Therefore the set $\{(b_1, \cdots, b_k, d_1, \cdots, d_l) \,|\, (*)_G\}$ is a cell of dimension $k + l - n + 1$. Thus we can see that every $C(G^*)$ is a cell. A cell $C(G_1^*)$ is contained in the boundary of a cell $C(G_2^*)$, when $G_1^*$ is obtained from $G_2^*$ by contracting some edges or contracting closed edge-paths. A contraction of a closed edge-path means that we identify two oriented edge-paths, that are contained in this closed edge-path, to form a new oriented edge-path. A contraction of an edge indicates that the distance between zeros tends to zero and a contraction of a closed edge-path indicates that the width of a band tends to zero.

NOTATION. We write $G_1^* < G_2^*$ when $G_1^*$ is obtained by contracting edges or closed edge-paths of $G_2^*$.

With these preparations in hand, we can now formulate the theorem of Giddings and Wolpert [GW] explicitly as follows.

THEOREMM 3.2. *Let $\{r_i\}_{i=1}^n$ be non-zero real numbers satisfying $\sum_{i=1}^n r_i = 0$ and let $p$ be the number of positive numbers among them. There exists a cell decomposition of the Teichmüller space of genus $g$, $n$-pointed Riemann surfaces $\mathcal{T}_g^n$ ($g \geq 0, n \geq 2$):*

$$\mathcal{T}_g^n = \coprod_{G^* \in \mathcal{G}} C(G^*),$$

*where*

$$\mathcal{G} = \left\{ G^* \text{ of Proposition 2.1} \;\middle|\; \begin{array}{l} n(G^*) = n, \ \chi(G^*) + k(G^*) = 2 - 2g, \\ p_i \text{ is positive if } r_i > 0, \text{ negative if } r_i < 0 \end{array} \right\}$$

$$C(G^*) := \{R \in \mathcal{T}_g^n \mid G_R^* \text{ is isomorphic to } G^*\}.$$

*When $G^*$ has $k$ closed edge-paths and $l + 1$ zeros, $C(G^*)$ a cell of dimension $k + l - n + 1$:*

$$C(G^*) \cong \{(b_1, \cdots, b_k, d_1, \cdots, d_l) \in \mathbf{R}_+^k \times \mathbf{R}^l \,|\, (*)_G\}.$$

*If $G_1^* < G_2^*$ then,*

$$C(G_1^*) \subset \partial \overline{C(G_2^*)}.$$

We can compute the orbifold Euler numbers of the moduli space of Riemann surfaces using these cell decompositions since they are invariant with respect to the action of the mapping class group. Let $\chi(\mathcal{M}_g^n)$ be the orbifold Euler number of the moduli space of genus $g$, $n$-pointed Riemann surfaces $\mathcal{M}_g^n$. It is computed as (see [Br])

$$\chi(\mathcal{M}_g^n) = \sum_G (-1)^{\dim C(G)} \frac{1}{|\mathrm{Aut}G|},$$

where the sum is taken over all unmarked graph and Aut$G$ is the automorphism group of $G$ preserving cyclic order and fixing every pole. We give some examples of this calculation below. They are the cases where the number of positive numbers among $\{r_i\}$ is one. For given $(g, n)$, the number of graphs which satisfy the conditions of Proposition 2.3 is smaller than that of the ribbon graphs which give surfaces of type $(g, n)$, so that this cell decomposition is more convenient for the calculations of the orbifold Euler numbers than other cell decompositions.

EXAMPLE 1. $(g, n) = (0, 3)$.
There is only one graph in this case (Fig. 4 (a)) with

$$\dim C(G) = 0, \quad |\mathrm{Aut}G| = 1,$$

therefore the orbifold Euler number of $\mathcal{M}_0^3$ is 1:

$$\chi(\mathcal{M}_0^3) = 1.$$

EXAMPLE 2. $(g, n) = (0, 4)$.
The graph (b) in Fig. 4 has naively six numberings among poles but, in fact, there are three different ways to give numbering because of the symmetry that the graph has. The dimension of the corresponding cell is two. Similarly, the graph (c) has six different numberings and the dimension of its cell is one and the graph (d) has two numberings and the dimension of its cell is zero. The automorphism groups of these graphs are all trivial. Thus,

$$\chi(\mathcal{M}_0^4) = 3 - 6 + 2 = -1.$$

EXAMPLE 3. $(g, n) = (1, 2)$.
There are four graphs in this case (Fig. 4(e), (f), (g), (h)). The dimensions of the cells and the orders of the automorphism groups are as follows.

| graph | (e) | (f) | (g) | (h) |
|---|---|---|---|---|
| $\dim C(G)$ | 4 | 3 | 2 | 2 |
| Aut $G$ | 4 | 1 | 3 | 2 |

Then

$$\chi(\mathcal{M}_1^2) = \frac{1}{4} - 1 + \frac{1}{3} + \frac{1}{2} + \frac{1}{12}.$$

We calculated the orbifold Euler numbers of $\mathcal{M}_0^3$, $\mathcal{M}_0^4$, $\mathcal{M}_0^5$, $\mathcal{M}_0^6$, $\mathcal{M}_0^7$, $\mathcal{M}_0^8$, $\mathcal{M}_1^2$, $\mathcal{M}_1^3$, $\mathcal{M}_1^4$, $\mathcal{M}_1^5$, $\mathcal{M}_1^6$, $\mathcal{M}_2^2$, $\mathcal{M}_2^3$, $\mathcal{M}_2^4$, $\mathcal{M}_3^2$, which coincided with the values already computed in [HZ, P1]. We enumerated all the graphs, the orders of their automorphism groups and the dimensions of the corresponding cells, which are shown on the following tables. As in the

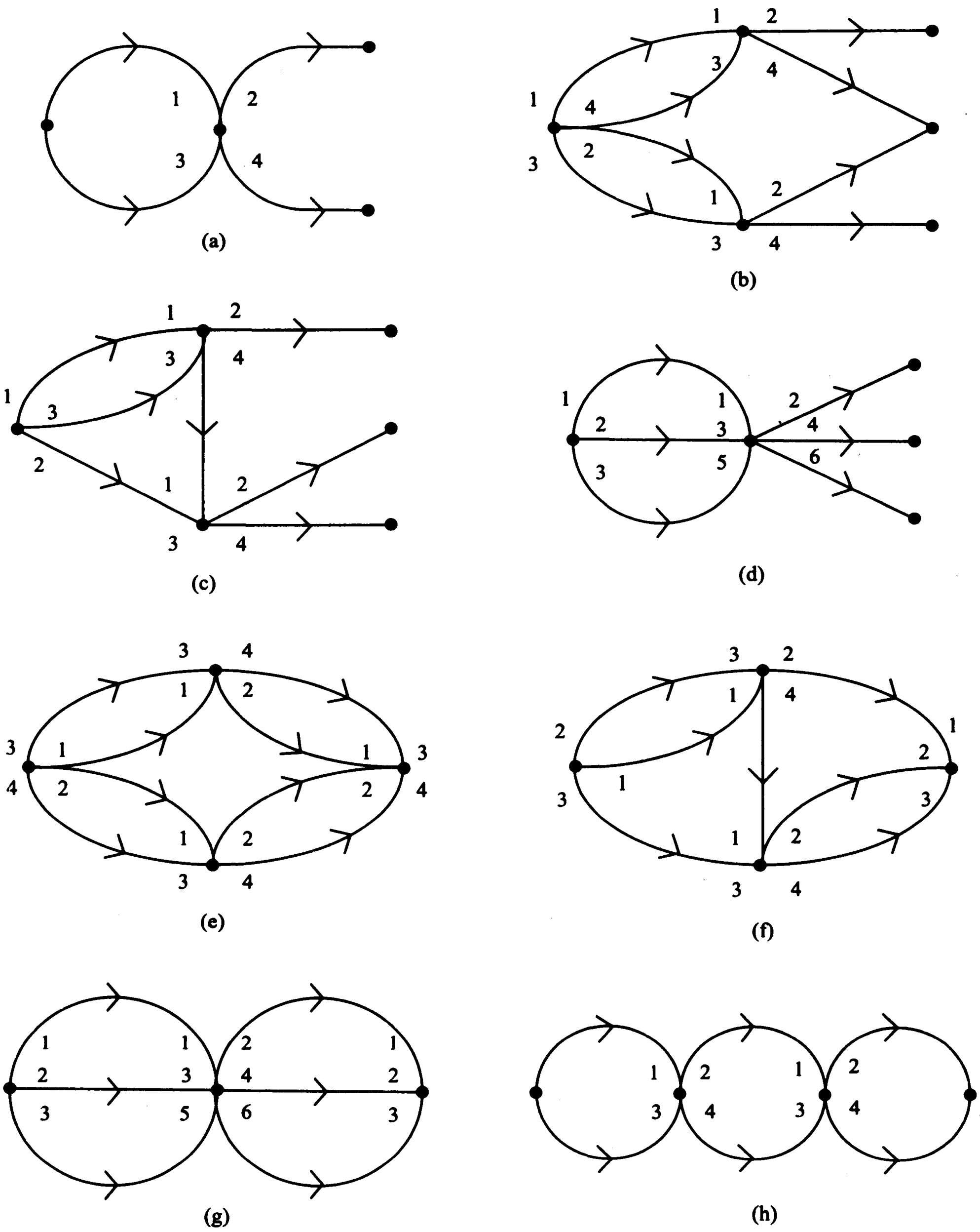


Fig. 4

above examples, these are the cases where the number of positive numbers among $\{r_i\}$ is one. Moreover, we allow the automorphism group of a graph not to fix poles except the positive pole. It is because we take the different definitions of the moduli spaces in these cases. That is, two $n$-pointed Riemann surfaces are biholomorphic if and only if there exists a biholomorphism between these Riemann surfaces which fixes the first point (the positive pole) and fixes other points (negative poles) setwise. Therefore, the orbifold Euler numbers of these moduli spaces differ from those of usual ones (i.e. fix $n$ points point-wise) by the factor $1/(n-1)!$.

EXAMPLE 4. $(g, n) = (0, 5)$.

In the following tables, the bracketed number shows the order of the automorphism groups and the number of the graphs is shown next to it. The orbifold Euler number of $\mathcal{M}_0^5$ is computed as

$$\chi(\mathcal{M}_0^5) = (-3+3-2) + \frac{1}{2}(1+2) + \frac{1}{3} + \frac{1}{4} = \frac{1}{12}.$$

The orbifold Euler numbers for other $g$ and $n$ are computed in the same way, and are shown in the tables.

TABLE 1. $(g, n) = (0, 5)$ $\chi(\mathcal{M}_0^5) = \frac{1}{12}$

| dim | 4 | 3 | 2 | 1 | 0 |
|---|---|---|---|---|---|
| $\lvert\mathrm{Aut}G\rvert = 1$ | 0 | 3 | 3 | 2 | 0 |
| $\lvert\mathrm{Aut}G\rvert \neq 1$ | [2]1<br>[3]1 | | [2]2 | | [4]1 |
| total | 2 | 3 | 5 | 2 | 1 |

TABLE 2. $(g, n) = (0, 6)$ $\chi(\mathcal{M}_0^6) = -\frac{1}{20}$

| dim | 6 | 5 | 4 | 3 | 2 | 1 | 0 |
|---|---|---|---|---|---|---|---|
| $\lvert\mathrm{Aut}G\rvert = 1$ | 1 | 9 | 18 | 22 | 12 | 3 | 0 |
| $\lvert\mathrm{Aut}G\rvert \neq 1$ | [2]1<br>[4]1 | | [2]3 | | [2]1 | | [5]1 |
| total | 3 | 9 | 21 | 22 | 13 | 3 | 1 |

TABLE 3. $(g, n) = (0, 7)$ $\chi(\mathcal{M}_0^7) = \frac{1}{30}$

| dim | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 |
|---|---|---|---|---|---|---|---|---|---|
| $\lvert\mathrm{Aut}G\rvert = 1$ | 3 | 28 | 79 | 140 | 139 | 86 | 26 | 4 | 0 |
| $\lvert\mathrm{Aut}G\rvert \neq 1$ | [2]2<br>[5]1 | | [2]7<br>[3]1 | | [2]9 | [3]2 | [2]4 | | [6]1 |
| total | 6 | 28 | 87 | 140 | 148 | 88 | 30 | 4 | 1 |

TABLE 4. $(g, n) = (0, 8)$ $\chi(\mathcal{M}_0^8) = -\frac{1}{42}$

| dim | 10 | 9 | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\lvert\mathrm{Aut}G\rvert = 1$ | 9 | 90 | 330 | 747 | 1076 | 1044 | 649 | 250 | 52 | 5 | 0 |
| $\lvert\mathrm{Aut}G\rvert \neq 1$ | [2]3<br>[3]1<br>[6]1 | | [2]12 | [3]3 | [2]16 | | [2]8<br>[3]1 | | [2]1 | | [7]1 |
| total | 14 | 90 | 342 | 750 | 1092 | 1044 | 658 | 250 | 53 | 5 | 1 |

TABLE 5. $(g, n) = (1, 3)$ $\chi(\mathcal{M}_1^3) = -\frac{1}{12}$

| dim | 6 | 5 | 4 | 3 | 2 |
|---|---|---|---|---|---|
| $\lvert \mathrm{Aut}G \rvert = 1$ | 1 | 9 | 18 | 18 | 5 |
| $\lvert \mathrm{Aut}G \rvert \neq 1$ | [2]1<br>[6]1 | | [2]3 | | [2]1<br>[4]1 |
| total | 3 | 9 | 21 | 18 | 7 |

TABLE 6. $(g, n) = (1, 4)$ $\chi(\mathcal{M}_1^4) = \frac{1}{12}$

| dim | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 |
|---|---|---|---|---|---|---|---|---|---|
| $\lvert \mathrm{Aut}G \rvert = 1$ | 7 | 60 | 177 | 308 | 283 | 138 | 24 | 0 | 0 |
| $\lvert \mathrm{Aut}G \rvert \neq 1$ | [2]3<br>[4]1 | | [2]11 | | [2]12<br>[6] 1 | | [2]4 | [3]1 | |
| total | 11 | 60 | 188 | 308 | 296 | 138 | 28 | 1 | 0 |

TABLE 7. $(g, n) = (1, 5)$ $\chi(\mathcal{M}_1^5) = -\frac{1}{12}$

| dim | 10 | 9 | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\lvert \mathrm{Aut}G \rvert = 1$ | 38 | 350 | 1326 | 3065 | 4462 | 4179 | 2347 | 727 | 93 | 2 | 0 |
| $\lvert \mathrm{Aut}G \rvert \neq 1$ | [2]8 | | [2]30<br>[3] 2 | | [2]43<br>[4] 1 | [3]4 | [2]27 | | [2]5<br>[3]1<br>[4]2 | | |
| total | 46 | 350 | 1358 | 3065 | 4506 | 4183 | 2374 | 727 | 101 | 2 | 0 |

TABLE 8. $(g, n) = (1, 6)$ $\chi(\mathcal{M}_1^6) = \frac{1}{12}$

| dim | 12 | 11 | 10 | 9 | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\lvert \mathrm{Aut}G \rvert = 1$ | 183 | 1890 | 8652 | 24583 | 46862 | 62409 | 57352 | 35407 | 13743 | 3047 | 304 | 7 |
| $\lvert \mathrm{Aut}G \rvert \neq 1$ | [2]17<br>[3] 1<br>[4] 2<br>[6] 1 | | [2]84 | [3]3 | [2]158<br>[4] 4 | | [2]155<br>[3] 1 | | [2]68<br>[4] 1 | | [2]9 | |
| total | 204 | 1890 | 8736 | 24586 | 47024 | 62409 | 57508 | 35407 | 13812 | 3047 | 313 | 7 |

TABLE 9. $(g, n) = (2, 2)$ $\chi(\mathcal{M}_2^2) = -\frac{1}{40}$

| dim | 10 | 9 | 8 | 7 | 6 | 5 | 4 | 3 | 2 |
|---|---|---|---|---|---|---|---|---|---|
| $\lvert \mathrm{Aut}G \rvert = 1$ | 2 | 21 | 73 | 154 | 175 | 107 | 26 | 0 | 0 |
| $\lvert \mathrm{Aut}G \rvert \neq 1$ | [2]1<br>[8]1 | | [2]5 | | [2]4<br>[3]1<br>[4]1<br>[6]1 | | [2]1<br>[5]3 | [3]3 | |
| total | 4 | 21 | 78 | 154 | 182 | 107 | 30 | 3 | 0 |

TABLE 10. $(g, n) = (2, 3)$ $\chi(\mathcal{M}_2^3) = \frac{1}{20}$

| dim | 12 | 11 | 10 | 9 | 8 | 7 | 6 | 5 | 4 | 3 | 2 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\lvert\mathrm{Aut}G\rvert = 1$ | 45 | 462 | 2042 | 5555 | 9772 | 11259 | 8035 | 3293 | 635 | 34 | 0 |
| $\lvert\mathrm{Aut}G\rvert \neq 1$ | [2]6<br>[5]1<br>[10]1 | | [2]30<br>[3] 3 | | [2]56<br>[4] 1 | [3]6 | [2]45 | | [2]10<br>[3] 1<br>[4] 2<br>[6] 1 | | |
| total | 53 | 462 | 2075 | 5555 | 9829 | 11265 | 8080 | 3293 | 649 | 34 | 0 |

TABLE 11. $(g, n) = (2, 4)$ $\chi(\mathcal{M}_2^4) = -\frac{1}{12}$

| dim | 14 | 13 | 12 | 11 | 10 | 9 | 8 | 7 | 6 | 5 | 4 | 3 | 2 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\lvert\mathrm{Aut}G\rvert = 1$ | 527 | 5985 | 31417 | 103151 | 231883 | 370421 | 418265 | 325358 | 165941 | 51359 | 8315 | 506 | 4 |
| $\lvert\mathrm{Aut}G\rvert \neq 1$ | [2]22<br>[3] 1<br>[4] 2<br>[6] 1 | | [2]124 | [3]3 | [2]269<br>[4] 4 | | [2]281<br>[3] 2 | | [2]133<br>[4] 1 | [3]3 | [2]23 | | [3]3 |
| total | 553 | 5985 | 31541 | 103154 | 232156 | 370421 | 418548 | 325358 | 166075 | 51362 | 8338 | 506 | 7 |

TABLE 12. $(g, n) = (3, 2)$ $\chi(\mathcal{M}_3^2) = \frac{5}{252}$

| dim | 16 | 15 | 14 | 13 | 12 | 11 | 10 | 9 | 8 | 7 | 6 | 5 | 4 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\lvert\mathrm{Aut}G\rvert = 1$ | 118 | 1485 | 8443 | 30273 | 74646 | 131856 | 165644 | 144441 | 83290 | 29635 | 5681 | 447 | 4 |
| $\lvert\mathrm{Aut}G\rvert \neq 1$ | [2]10<br>[3] 2<br>[12] 1 | | [2]61 | [3]9 | [2]156<br>[4] 1 | | [2]211<br>[3] 8<br>[6] 1<br>[9] 2 | | [2]140<br>[4] 2<br>[8] 2 | [3]5 | [2]37<br>[7] 5 | | [3]5<br>[4]4 |
| total | 131 | 1485 | 8504 | 30282 | 74803 | 131856 | 165866 | 144441 | 83434 | 29640 | 5723 | 447 | 13 |

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