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Denjoy Systems and Substitutions

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Abstract. We study a way of coding of irrational rotations, by which Denjoy systems are represented as subshifts. First, we state the subshift generated by a coding sequence is conjugate to a Denjoy system. Next, by using an adic model of a Denjoy system we give a sequence of substitutions to generate the coding sequence.

1. Introduction

Let $\mathcal{A} = \{0, 1, \ldots, d\}$ be an alphabet and $\mathcal{A}^*$ be the free monoid over $\mathcal{A}$ with respect to the concatenation, having the empty word (identity element) $\varepsilon_\emptyset$. A *substitution* σ over $\mathcal{A}$ is a map from an alphabet $\mathcal{A}$ to $\mathcal{A}^* \setminus \{\varepsilon_\emptyset\}$. It can be extended to a morphism of $\mathcal{A}^*$ naturally. The *reversal* of a finite word $w = w_1 \cdots w_n$ is the word $\overleftarrow{w} = w_n \cdots w_1$. The *reversal* of a substitution σ is the substitution $\overleftarrow{\sigma}$ defined by

$$\overleftarrow{\sigma}(i) = \overleftarrow{\sigma(i)} \quad (i \in \mathcal{A}) .$$

Notice $\overleftarrow{\sigma(w)} = \overleftarrow{\sigma}(\overleftarrow{w})$. A word v is a *prefix* of a word u if $u = vw$ for some $w \in \mathcal{A}^*$. The set of right infinite (resp. biinfinite) words over $\mathcal{A}$ is denoted by $\mathcal{A}^{\mathbf{Z}_+}$ (resp. $\mathcal{A}^{\mathbf{Z}}$).

Let $(\sigma_1, \sigma_2, \ldots)$ be a sequence of substitutions over $\mathcal{A}$. We say that $(\sigma_n)_{n \in \mathbf{N}}$ *generates* a right infinite word $w = w_0 w_1 \cdots$ if for each n, there exists N such that $w_0 w_1 \cdots w_n$ is a common prefix of $\sigma_1 \sigma_2 \cdots \sigma_N(i)$'s, $i \in \mathcal{A}$: or equivalently,

$$w = \lim_{n \to \infty} \sigma_1 \sigma_2 \cdots \sigma_n(i) \quad \text{for any } \ i \in \mathcal{A} .$$

We say that $(\sigma_n)_{n \in \mathbf{N}}$ *generates* a biinfinite word $\cdots w_{-1}.w_0 w_1 \cdots$ if $(\sigma_n)_{n \in \mathbf{N}}$ generates $w_0 w_1 \cdots$ and $(\overleftarrow{\sigma_n})_{n \in \mathbf{N}}$ generates $w_{-1} w_{-2} \cdots$.

In this paper, we study a coding under an irrational rotation. Take $\alpha \in (0, 1) \setminus \mathbf{Q}$. Let $S^1 = \mathbf{R}/\mathbf{Z}$ and $R_\alpha : S^1 \to S^1$ be the rotation $R_\alpha(\omega) = \omega + \alpha \pmod 1$. Identify $(0, 1]$ with S^1 naturally. Consider a partition $\{t(0), t(1)\}$ of $S^1 = (0, 1]$ where $t(0) = (0, \alpha]$ and $t(1) = (\alpha, 1]$. Define a map $J_\alpha : (0, 1] \to \{0, 1\}^{\mathbf{Z}}$ by $J_\alpha(\omega)_n = i$ if $R_\alpha^n(\omega) \in t(i)$. A

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Sturmian sequence is given by $J_\alpha(\omega)$ for some α and ω. (Precisely, we need to consider another decomposition $[0,1) = [0,\alpha) \cup [\alpha,1)$ to see all Sturmian sequences.) Let $\alpha = [0; a_1, a_2, a_3, \ldots]$ be the simple continued fraction expansion. The following is a folklore theorem.

PROPOSITION 1. *Let* $\sigma_n(0) = 0\underbrace{1\cdots1}_{a_n\ times}$, $\sigma_n(1) = 0\underbrace{1\cdots1}_{a_n-1\ times}$ *for each* $n \in \mathbf{N}$. *Then the sequence* $(\sigma_1, \overleftarrow{\sigma_2}, \sigma_3, \overleftarrow{\sigma_4}, \ldots)$ *generates* $J_\alpha(\alpha)$.

PROOF. Let $u(0) = [0, 1-\alpha)$ and $u(1) = [1-\alpha, 1)$. Usually (for example, refer to [5]), Sturmian sequences are given as $K_\alpha(\omega)$, where $K_\alpha : [0,1) \to \{0,1\}^{\mathbf{Z}}$ is defined by

$$K_\alpha(\omega)_n = \begin{cases} 0 & \text{if } R_\alpha^n(\omega) \in u(0) \\ 1 & \text{if } R_\alpha^n(\omega) \in u(1)\,. \end{cases}$$

It is well-known that (see [4])

$$\begin{array}{c} \text{the sequence } (\eta_1, \overleftarrow{\eta_2}, \eta_3, \overleftarrow{\eta_4}, \cdots) \text{ generates } K_\alpha(0) \\ \text{where } \eta_n(0) = \underbrace{0\cdots0}_{a_n-1 \text{ times}}1 \text{ and } \eta_n(1) = \underbrace{0\cdots0}_{a_n \text{ times}}1. \end{array}$$

Proposition 1 follows this fact immediately, because the following diagram

$$\begin{tikzcd} t(0) \subset \arrow[d] & (0,1] \arrow[r, "R_\alpha"] \arrow[d, "\overline{\cdot}"'] & (0,1] \arrow[d, "\overline{\cdot}"] \\ u(1) \subset & {[0,1)} \arrow[r, "R_\alpha^{-1}"] & {[0,1)} \end{tikzcd}$$

commutes (where $\overline{\cdot} : (0,1] \to [0,1) : x \mapsto \overline{x} := 1 - x$), we see $J_\alpha(\alpha)_n = 1 - K_\alpha(0)_{-n-1}$. $\square$

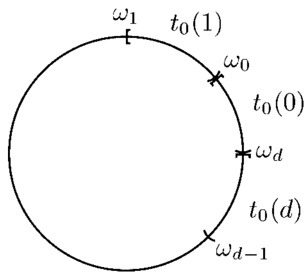
In this paper, we pay attention to a generalization of Proposition 1. For each $\omega \in S^1$, denote by $\mathcal{O}_\omega$ the orbit of ω under R_α, that is,

$$\mathcal{O}_\omega = \{R_\alpha^n(\omega) \mid n \in \mathbf{Z}\}\,.$$

A subset $A \subset S^1$ is said to be *non-coorbital* if $\{\mathcal{O}_\omega \mid \omega \in A\}$ is mutually disjoint. Take a finite non-coorbital subset Λ with $\alpha \in \Lambda$.
Let $\Lambda = \{\omega_0 < \omega_1 < \cdots < \omega_{d-1}\}$ and $\Lambda_1 = \Lambda \cup \{1\}$. So Λ_1 gives a partition of S^1, that is,

$$S^1 = \bigcup_{i \in \mathcal{A}} t_0(i)$$

where $t_0(0) = (0, \omega_0]$ and $t_0(i) = (\omega_{i-1}, \omega_i]$ $(0 < i \le d, \omega_d = 1)$.



Define $J : S^1 \to \mathcal{A}^{\mathbf{Z}}$ by $J(\omega)_n = i$ if $R_\alpha^n(\omega) \in t_0(i)$. Clearly, in the case of $d = 1$, $J(\omega)$ is a Sturmian sequence. In this meaning, we can regard $J(\omega)$ as $d + 1$ letters *Sturmian sequences*. The main goal of this paper is to construct a sequence of substitutions which generates $J(\alpha)$.

First, we state that the subshift generated by $J(\alpha)$ is conjugate to a *Denjoy system*, which is defined as follows. We call $\varphi : S^1 \to S^1$ a Denjoy homeomorphism if φ is an orientation-preserving homeomorphism with irrational rotation number which is not conjugate to a rotation (see [2], §4). A Denjoy system is the unique minimal subsystem of some Denjoy homeomorphism. In [6], an *adic model* (*Bratteli-Vershik system*) of a Denjoy system is concretely constructed. Next, we observe that this adic system naturally corresponds to a sequence of substitutions. We see that this sequence generates $J(\alpha)$.

We consider that Denjoy systems, generalized Sturmian sequences and adic systems have close association each other, but it does not seem to have been clarified yet ([2]). We study a link between them.

In Section 2, we state the main result. In Section 3, we show that a Denjoy system is conjugate to a generalized Sturmian subshift in our sense. Section 4 is devoted to the *natural substitution system* associated with an ordered Bratteli diagram of constant rank. In Section 5, we recall an HPS-adic presentation for a Denjoy system given in [6]. Section 6 is devoted to proof.

We introduce some notations. Denote by $\mathbf{N}$ (resp. $\mathbf{Z}_+$) the set of positive integers (resp. non-negative integers).

For $i \in \mathcal{A}$, denote $\underbrace{i \cdots i}_{a \text{ times}}$ by i^a. Let S_n be a finite set. For $s_* = s_1 s_2 \cdots \in \prod_{n\in\mathbf{N}} S_n$, $s_l s_{l+1} \cdots s_m$ (resp. $s_{l+1}s_{l+2}\cdots$) is denoted by $s_{[l,m]}$ (resp. $s_{(l,\infty)}$) and so on. A subset $A \subset \prod_{n\in\mathbf{N}} S_n$ is said to be *non-cotail* if for any distinct $s_*, t_* \in A$, $s_n \neq t_n$ for infinitely many n. Let S_n be a totally ordered set. Put the total order (lexicographic order) $<_{\text{lex}}$ on $\prod_{n\in\mathbf{N}} S_n$ defined by that $s_* <_{\text{lex}} t_*$ if $s_l < t_l$ where $l = \min\{n \in \mathbf{N} \mid s_n \neq t_n\}$.

For distinct $z, w \in S^1$, denote by $(z, w]$ the left-open right-closed arc between z and w which lies in the positive direction from z. Define the interior of $(z, w]$ as $\text{int}(z, w] = (z, w) = (z, w] \setminus \{w\}$. For an open arc $I = (z, w)$, let $\inf I = z$ and $\sup I = w$.

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2. Main result

Let $\alpha = [0; a_1, a_2, a_3, \ldots]$ be the simple continued fraction expansion, and

$$\begin{bmatrix} p_{-1} & p_0 \\ q_{-1} & q_0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{matrix} p_n = a_n p_{n-1} + p_{n-2} \\ q_n = a_n q_{n-1} + q_{n-2} \end{matrix} \qquad (n \in \mathbf{N}).$$

Now, we introduce the *dual Ostrowski numeration system*. Let

$$M_\alpha = \left\{ x_* = (x_n)_{n\in\mathbf{N}} \in \prod_{n\in\mathbf{N}} \{0, 1, \ldots, a_n\} \;\middle|\; x_n = a_n \Rightarrow x_{n+1} = 0 \right\}.$$

It is well-known ([3]) that for each $\omega \in [0, 1]$, there is $x_* \in M_\alpha$ such that

$$\omega = \sum_{n=1}^{\infty} x_n |q_{n-1}\alpha - p_{n-1}| \text{ (dual Ostrowski expansion of } \omega).$$

For each $x_* \in M_\alpha$, define

$$\nu(x_*) = \sum_{n=1}^{\infty} x_n |q_{n-1}\alpha - p_{n-1}|.$$

OBSERVATION 1. *We can regard* ν *as a map from* M_α *to* S^1 *where* S^1 *is the set* $[0, 1]$ *identifying* 0 *and* 1. *Then the following hold.*

(1) *If* $\omega \in \mathcal{O}_\alpha$, *then* $\nu^{-1}(\omega)$ *is a two-point-set of the form:*

$$\nu^{-1}(\omega) = \{x_{(0,n]}00\cdots,\ x_{(0,n)}(x_n - 1)a_{n+1}0a_{n+3}0\cdots\}.$$

$$(\textit{Especially}, \nu^{-1}(\alpha) = \{100\cdots,\ 0a_20a_4\cdots\} \quad \textit{and} \quad \nu^{-1}(1) = \{00\cdots,\ a_10a_30\cdots\}.)$$

If $\omega \notin \mathcal{O}_\alpha$, *then* $\nu^{-1}(\omega)$ *is a singleton.*

(2) *Let* $\{x_*, x'_*\} \subset M_\alpha$. *If there is* $n \in \mathbf{Z}_+$ *such that* $x_{(n,\infty)} = x'_{(n,\infty)}$, *then* $\nu(x'_*) \in \mathcal{O}_{\nu(x_*)}$. *(Indeed, then* $\nu(x_*) - \nu(x'_*) \in \mathcal{O}_\alpha$.*)*

By Observation 1 (1), for each $\omega \in (0, 1]$, we can choose

$x_*(\omega) := x_1(\omega)x_2(\omega)\cdots \in \nu^{-1}(\omega)$ such that $x_n(\omega) \neq 0$ for infinitely many n

and regard $x_*(\cdot)$ as a map $x_* : (0, 1] \to M_\alpha : \omega \mapsto x_1(\omega)x_2(\omega)\cdots$. So $\nu \circ x_* = \mathrm{id}$. By Observation 1 (2), we see that if $A \subset S^1$ is non-coorbital, then $x_*(A)$ is non-cotail. Especially, we have

$$x_*(\alpha) = 0a_20a_4\cdots, \ x_*(1) = a_10a_30\cdots. \qquad (\sharp)$$

DEFINITION 1 (n-tail and n-th comparison). Define

$$x_{(n,\infty)}(\omega) = x_{n+1}(\omega)x_{n+2}(\omega)\cdots$$

for each $n \in \mathbf{Z}_+$. For each $\omega \in \Lambda_1$, let

$$C_n(\omega) = \#\{\lambda \in \Lambda_1 \mid x_{(n,\infty)}(\lambda) <_{\mathrm{lex}} x_{(n,\infty)}(\omega)\}.$$

We call C_n the n-th comparison.

Since $x_*(\Lambda_1)$ is non-cotail, C_n is a bijection from Λ_1 to $\mathcal{A}$. By the definition of C_n, we see that $x_{(n,\infty)}(\Lambda_1)$ is arranged in the following way

$$x_{(n,\infty)} \circ C_n^{-1}(0) <_{\text{lex}} x_{(n,\infty)} \circ C_n^{-1}(1) <_{\text{lex}} \cdots <_{\text{lex}} x_{(n,\infty)} \circ C_n^{-1}(d)\,.$$

DEFINITION 2. For each $(c, i) \in \{0, \ldots, a_n\} \times \mathcal{A}$, define

$$\lceil c, i \rceil_n = \#\{\lambda \in \Lambda_1 \mid x_{(n-1,\infty)}(\lambda) <_{\text{lex}} c\, x_{(n,\infty)} \circ C_n^{-1}(i)\}$$

where $c\, x_{(n,\infty)}(\omega) = c\, x_{n+1}(\omega)x_{n+2}(\omega)\cdots$. For each $n \in \mathbf{N}$ and $i \in \mathcal{A}$, define

$$\sigma_n(i) = \begin{cases} \lceil 0, i \rceil_n \lceil 1, i \rceil_n \cdots \lceil a_n, i \rceil_n & \text{if } x_{n+1} \circ C_n^{-1}(i) = 0 \\ \lceil 0, i \rceil_n \lceil 1, i \rceil_n \cdots \lceil a_n - 1, i \rceil_n & \text{otherwise.} \end{cases}$$

Then σ_n is a substitution over $\mathcal{A}$, and the main result is the following:

MAIN THEOREM. *The sequence* $(\sigma_1, \overleftarrow{\sigma_2}, \sigma_3, \overleftarrow{\sigma_4}, \ldots)$ *generates the biinfinite sequence* $J(\alpha)$.

EXAMPLE 1 $(d = 1)$. Let $\Lambda_1 = \{\alpha, 1\}$. Then by $(\sharp)$,

$$(C_n(\alpha), C_n(1)) = \begin{cases} (1, 0) & \text{if } n \text{ is odd} \\ (0, 1) & \text{if } n \text{ is even,} \end{cases} \qquad \sigma_n : \begin{cases} 0 \mapsto 01^{a_n} \\ 1 \mapsto 01^{a_n-1}\,. \end{cases}$$

So Proposition 1 is a special case of Main Theorem.

EXAMPLE 2 $(d = 2)$. Let $\Lambda_1 = \{\alpha, \omega, 1\}$ and $x_n = x_n(\omega)$. Then we have

$$(C_n(\alpha), C_n(\omega), C_n(1)) = \begin{cases} (2, 0, 1) & \text{if } n \text{ is odd and } x_{n+1} = 0 \\ (2, 1, 0) & \text{if } n \text{ is odd and } x_{n+1} > 0 \\ (1, 0, 2) & \text{if } n \text{ is even and } x_{n+1} = 0 \\ (0, 1, 2) & \text{if } n \text{ is even and } x_{n+1} > 0 \end{cases}$$

and

$$\text{if } x_n = x_{n+1} = 0, \text{ then} \quad \sigma_n : \begin{cases} 0 \mapsto 02^{a_n} \\ 1 \mapsto 12^{a_n} \\ 2 \mapsto 12^{a_n-1} \end{cases};$$

$$\text{if } x_n = 0 \text{ and } x_{n+1} > 0, \text{ then} \quad \sigma_n : \begin{cases} 0 \mapsto 02^{a_n} \\ 1 \mapsto 02^{a_n-1} \\ 2 \mapsto 12^{a_n-1} \end{cases};$$

$$\text{if } x_n > 0 \text{ and } x_{n+1} = 0, \text{ then} \quad \sigma_n : \begin{cases} 0 \mapsto 01^{x_n} 2^{a_n - x_n} \\ 1 \mapsto 01^{x_n - 1} 2^{a_n - x_n + 1} \\ 2 \mapsto 01^{x_n - 1} 2^{a_n - x_n} \end{cases} ;$$

$$\text{if } x_n > 0 \text{ and } x_{n+1} > 0, \text{ then} \quad \sigma_n : \begin{cases} 0 \mapsto 01^{x_n} 2^{a_n - x_n} \\ 1 \mapsto 01^{x_n} 2^{a_n - x_n - 1} \\ 2 \mapsto 01^{x_n - 1} 2^{a_n - x_n} \end{cases} .$$

3. Denjoy system and Sturmian subshift

Let $\varphi : S^1 \to S^1$ be a Denjoy homeomorphism, that is, an orientation-preserving homeomorphism with irrational rotation number $\alpha \in (0, 1) \setminus \mathbf{Q}$, which is not conjugate to any rotation. We review Poincare's *rotation number theorem*. There exists a degree 1 map $F : S^1 \to S^1$ satisfying the following:

(1) $R_\alpha \circ F = F \circ \varphi$.

(2) Let $A = \{z \in S^1 \mid \#F^{-1}F(z) = 1\}$ and $X = \operatorname{cl} A$ (the closure of A). Then X is a Cantor set which is the unique minimal set under φ. Moreover $F(X) = S^1$. A connected component of $S^1 \setminus X$ is called a *cutout interval* (indeed, an open arc). The set of endpoints of cutout intervals is $X \setminus A$.

(3) Let F_X be the restriction of F to X. There exists an at most countable non-coorbital subset $\Lambda \subset S^1$ such that

$$F_X(X \setminus A) = \bigcup_{\omega \in \Lambda} \mathcal{O}_\omega .$$

For each cutout interval I, $F(\operatorname{cl} I)$ is a singleton, and $F_X^{-1}(\omega)$ is the set of endpoint of a cutout interval for any $\omega \in F_X(X \setminus A)$. We call $F_X(X \setminus A)$ the *double point set* and Λ a *transversal* of the double point set.

Such F is unique up to rotation. Denote the restriction of φ to X by

$$T : X \to X ,$$

and the subsystem (X, T) is called a *Denjoy system*. Notice that the cardinality $\#\Lambda$ of Λ is independent of the choice of F. We call $\#\Lambda$ the *double orbit number* of (X, T). By choosing appropriate F, we can assume $\alpha \in \Lambda$.

DEFINITION 3. For each $\omega \in F_X(X \setminus A)$, there exists a cutout interval I_ω such that $F^{-1}(\omega) = \operatorname{cl} I_\omega$. Pick $\widetilde{\omega} \in I_\omega$.

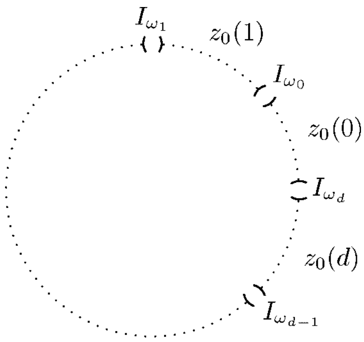
From now on, we consider only the case of finite double orbit number.

Let $\Lambda_1 = \Lambda \cup \{1\} = \{\omega_0 < \omega_1 < \cdots < \omega_d\}$. Then Λ_1 induces a partition of X:

$$X = \bigcup_{i \in \mathcal{A}} z_0(i)$$

where $z_0(0) = (\widetilde{\omega}_d, \widetilde{\omega}_0] \cap X$, $z_0(i) = (\widetilde{\omega}_{i-1}, \widetilde{\omega}_i] \cap X$ $(1 \leq i \leq d)$.

REMARK 1. For each $i \in \mathcal{A}$, $z_0(i) = \mathrm{cl}\, F_X^{-1}(\mathrm{int}\, t_0(i))$.



Notice that $z_0(i)$ is closed and open (*clopen*) in X, and independent of the choice of $\widetilde{\omega}_i$'s.

Define $J_X : X \to \mathcal{A}^{\mathbf{Z}}$ by $J_X(x)_n = i$ if $T^n(x) \in z_0(i)$.

We can see the following relation between $J_X(x)$ and $J(F_X(x))$.

PROPOSITION 2. (1) *If* $x \in X \setminus \{\sup I_{R_\alpha^n(\omega)} \mid n \in \mathbf{Z},\ \omega \in \Lambda\}$, *then*

$$J_X(x) = J(F_X(x)) .$$

Especially, $J_X(\inf I_\alpha) = J(\alpha)$.

(2) *If* $x = \sup I_{R_\alpha^m(\alpha)}$, *then*

$$J_X(x)_{-m-1} = 0,\ \ J(F_X(x))_{-m-1} = d ,$$
$$J_X(x)_{-m} = J(F_X(x))_{-m} + 1 ,$$
$$J_X(x)_n = J(F_X(x))_n \quad (n \neq -m, -m-1) .$$

(3) *If* $x = \sup I_{R_\alpha^m(\omega)}$ *with* $\omega \in \Lambda \setminus \{\alpha\}$, *then*

$$J_X(x)_{-m} = J(F_X(x))_{-m} + 1 ,$$
$$J_X(x)_n = J(F_X(x))_n \quad (n \neq -m) .$$

PROOF. Let $\omega_{-1} := \omega_d$. Notice

$$F_X(z_0(i) \setminus \{\sup I_{\omega_{i-1}}\}) = t_0(i) \quad (i \in \mathcal{A}) .$$

So if $x \in X \setminus \{\sup I_\omega \mid \omega \in \Lambda_1\}$, then $x \in z_0(i)$ and $F_X(x) \in t_0(i)$ for some i.

If $x \in X \setminus \{\sup I_{R_\alpha^n(\omega)} \mid n \in \mathbf{Z},\ \omega \in \Lambda\}$, then $T^n(x) \in X \setminus \{\sup I_\omega \mid \omega \in \Lambda_1\}$ for all n. Hence (1) holds.

Now, we show (2) and (3) in the case of $m = 0$. Let $z_0(d+1) := z_0(0)$. Notice

$$\sup I_{\omega_i} \in z_0(i+1),\ F_X(\sup I_{\omega_i}) = \omega_i \in t_0(i) \quad (i \in \mathcal{A}) .$$

(2) When $x = \sup I_\alpha \in z_0(i+1)$, we have $F_X(x) = \alpha \in t_0(i)$, $T^{-1}(x) = \sup I_{\omega_d} \in z_0(0)$, $R_\alpha^{-1}(F_X(x)) = \omega_d \in t_0(d)$, and $T^n(x) \in X \setminus \{\sup I_\omega \mid \omega \in \Lambda_1\}$ if $n \neq 0, -1$.

(3) When $x = \sup I_\omega \in z_0(i+1)$ for some $\omega \in \Lambda \setminus \{\alpha\}$, we have $F_X(x) \in t_0(i)$ and $T^n(x) \in X \setminus \{\sup I_\omega \mid \omega \in \Lambda_1\}$ if $n \neq 0$. □

PROPOSITION 3. *A Denjoy system (X, T) is conjugate to the subshift $(J_X(X), S)$ via J_X.*

PROOF. Since X is compact and $J_X(X)$ is Hausdorff, it suffices to show that J_X is continuous and one-to-one. For each $x \in X$, the set

$$\bigcap_{n=-l}^{l} T^{-n}\Big(z_0(J_X(x)_n)\Big) = \{y \in X \mid J_X(y)_n = J_X(x)_n \ (-l \leq n \leq l)\}$$

is a neighborhood of x. Hence J_X is continuous.

Let $x, y \in X$ be distinct.

Consider the case of $F(x) \neq F(y)$. Since R_α is a minimal isometry, there exists $n \in \mathbf{Z}$ such that $F(x) \in R_\alpha^n(\operatorname{int} t_0(i))$ and $F(y) \in R_\alpha^n(\operatorname{int} t_0(j))$ with $i \neq j$. This implies $T^{-n}(x) \in z_0(i)$ and $T^{-n}(y) \notin z_0(j)$. So $J_X(x)_{-n} \neq J_X(y)_{-n}$.

Consider the case of $F(x) = F(y)$. Then x, y are the endpoints of some cutout interval, that is, there exists $\omega_i \in \Lambda$ $(0 \leq i < d)$ and $n \in \mathbf{Z}$ such that

$$\{x, y\} = \{\inf I_{R^n(\omega_i)}, \sup I_{R_\alpha^n(\omega_i)}\} .$$

So $T^{-n}(\{x, y\}) = \{\inf I_{\omega_i}, \sup I_{\omega_i}\}$. Since $\inf I_{\omega_i} \in z_0(i)$ and $\sup I_{\omega_i} \in z_0(i+1)$, we have $J_X(x)_{-n} \neq J_X(y)_{-n}$. Anyway, $J_X(x) \neq J_X(y)$. □

By Proposition 3, $J_X(X)$ is the orbit closure of $J_X(x)$ for any $x \in X$.

4. Natural substitution system

In this section, we shall introduce a main tool, that is, a substitution system $(\sigma_n)_{n\in\mathbf{N}}$ via an ordered Bratteli diagram of constant rank.

4.1. Ordered Bratteli diagram. A *Bratteli diagram* is an infinite directed graph $B = (V, E)$, such that the vertex set V and the edge set E can be partitioned into finite sets

$$V = \bigcup_{n\in\mathbf{Z}_+} V_n \quad \text{and} \quad E = \bigcup_{n\in\mathbf{N}} E_n$$

with the following properties: $s(E_n) = V_{n-1}$ and $r(E_n) = V_n$ for all n, where $s : E \to V$ is the *source map* and $r : E \to V$ is the *range map*. For each $n \in \mathbf{Z}_+$, pick a bijection

$$v_n : \{0, 1, \ldots, c_n - 1\} \to V_n \text{ where } c_n = \#V_n .$$

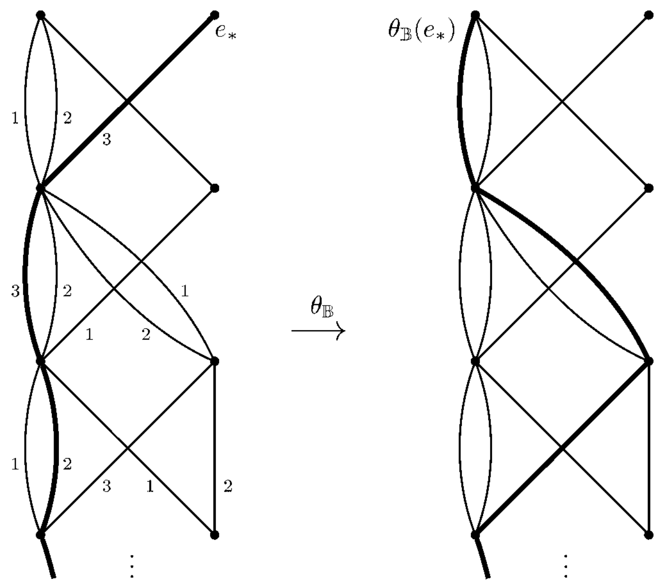
So $V_n = \{v_n(0), v_n(1), \ldots, v_n(c_n - 1)\}$. Let A_n be the $c_n \times c_{n-1}$ matrix defined by

$$(A_n)_{ij} = \#(s^{-1}(v_{n-1}(j)) \cap r^{-1}(v_n(i)))$$

and call A_n the n-th *incidence matrix* of B. Define the *infinite path space* X_B of B by

$$X_B = \left\{ e_* = (e_n)_{n\in\mathbf{N}} \in \prod_{n\in\mathbf{N}} E_n \;\middle|\; r(e_n) = s(e_{n+1}) \text{ for all } n \right\}.$$

An *ordered Bratteli diagram* $\mathbf{B} = (B, \leq)$ is a Bratteli diagram $B = (V, E)$ together with a partial order on E so that edges $e, e' \in E$ are comparable if and only if $r(e) = r(e')$. Then we put the *adic order* on X_B (partial order) by that $e_* < f_*$ if there exists $N \in \mathbf{N}$ such that $e_N < f_N$ and $e_n = f_n$ for all $n > N$, and write $X_{\mathbf{B}} = (X_B, \leq)$.
If there exist a unique minimal path $e_*^{\min}$ and a unique maximal path $e_*^{\max}$, then $\mathbf{B}$ is said to be *properly ordered*.
For a properly ordered Bratteli diagram $\mathbf{B}$, define the *adic transformation* $\theta_{\mathbf{B}} : X_{\mathbf{B}} \to X_{\mathbf{B}}$ as follows: if $e_* \neq e_*^{\max}$, then $\theta_{\mathbf{B}}(e_*) = \min\{f_* \in X_{\mathbf{B}} \mid f_* > e_*\}$; and $\theta_{\mathbf{B}}(e_*^{\max}) = e_*^{\min}$. The system $(X_{\mathbf{B}}, \theta_{\mathbf{B}})$ is called a *Bratteli-Vershik system* or an *adic system*.



4.2. Natural substitution system. When $P = \{p_1 < p_2 < \cdots < p_{\#P}\}$ is a totally ordered finite set, we denote the arrangement of the elements of P in its order, $p_1 p_2 \cdots p_{\#P}$, by $\overrightarrow{P}$. For a map $\eta : P \to Q$, define $\eta(\overrightarrow{P}) = \eta(p_1) \cdots \eta(p_{\#P})$.

A Bratteli diagram is said to be *of constant rank* if $\#V_n$ is independent of $n \in \mathbf{Z}_+$. We call the number $\#V_n$ the *rank* of B, and denote it by $\operatorname{rank}(B)$.

DEFINITION 4. Let $d \in \mathbf{N}$ and $\mathbf{B} = (B, \leq)$ be an ordered Bratteli diagram of $\mathrm{rank}(B) = d + 1$. Define a substitution σ_n by

$$\sigma_n(i) = v_{n-1}^{-1} \circ s(\overrightarrow{r^{-1}(v_n(i))}) .$$

We call the sequence of substitutions $(\sigma_n)_{n\in\mathbf{N}}$ the **natural substitution system** of $\mathbf{B}$.

Clearly, the n-th incidence matrix A_n of B is the “incidence matrix" of σ_n, that is, $(A_n)_{ij}$ is the number of occurrences of j in $\sigma_n(i)$.

Let $\mathbf{B} = (B, \leq)$ be properly ordered of $\mathrm{rank}(B) = d + 1$. Define

$$s : X_B \to V_0 : e_* \mapsto s(e_1)$$

and define a map $J_{\mathbf{B}} : X_{\mathbf{B}} \to \mathcal{A}^{\mathbf{Z}}$ by

$$J_{\mathbf{B}}(e_*)_n = i \quad \text{if } v_0^{-1} \circ s(\theta_{\mathbf{B}}^n(e_*)) = i .$$

Then we have the following.

THEOREM 1. *If $(X_{\mathbf{B}}, \theta_{\mathbf{B}})$ has no periodic points, then the natural substitution system $(\sigma_n)_{n\in\mathbf{N}}$ of $\mathbf{B}$ generates the biinfinite word $J_{\mathbf{B}}(e_*^{\min})$.*

(For its proof, see Subsection 6.1.)

5. HPS-adic presentations of Denjoy systems

5.1. HPS-adic presentation. Let Y be a Cantor set and $U : Y \to Y$ be a homeomorphism. We call (Y, U) a *Cantor system* if U is minimal. For any Cantor system (Y, U), Herman, Putnam and Skau ([1]) had shown that there exists a Bratteli-Vershik system which is conjugate to (Y, U). We shall recall their construction.

A Kakutani-Rokhlin (KR) tower partition of (Y, U) is a partition of the form:

$$\mathcal{P} = \{U^k(Z(j)) \mid 0 \leq j < c,\ 0 \leq k < h(j)\} \quad \text{where } Z(j) \text{ is clopen and } c, h(j) \in \mathbf{N}.$$

Let $\mathcal{P}' = \{U^k(Z'(i)) \mid 0 \leq i < c',\ 0 \leq k < h'(i)\}$ be another KR partition of (Y, U). If $\mathcal{P}'$ is finer than $\mathcal{P}$, then for each $0 \leq i < c'$, $0 \leq j < c$, there exists $H_{ij} \subset [0, h'(i)) \cap \mathbf{Z}$ such that

$$Z(j) = \bigcup_{0\leq i<c'} \bigcup_{\rho\in H_{ij}} U^\rho(Z'(i)) . \qquad (*)$$

To visualize this refinement, it is convenient to consider a graph (W, E) with a partial order $\leq$ on E, where the vertex set $W = V \cup V'$: $V = \{v(0), \ldots, v(c-1)\}$, $V' = \{v'(0), \ldots, v'(c'-1)\}$; and the edge set

$$E = \{(v(j), \rho, v'(i)) \mid 0 \leq j < c,\ 0 \leq i < c',\ \rho \in H_{ij}\}$$

and the partial order $\leq$ on E defined by

$$(v(j), \rho, v'(i)) \leq (v(j'), \rho', v'(i')) \text{ if } i = i' \text{ and } \rho \leq \rho'.$$

Then by $(*)$ there is a correspondence between E and $\{p \in \mathcal{P}' \mid p \subset \bigcup_{0\le j<c} Z(j)\}$ via $(v(j),\rho,v'(i)) \longleftrightarrow p = U^{\rho}(Z'(i))$ with $p \subset Z(j)$. If $(v(j),\rho,v'(i)) \le (v(j'),\rho',v'(i'))$, then $U^{\rho'}(Z'(i'))$ is a forward image of $U^{\rho}(Z'(i))$ (indeed $Z'(i') = Z'(i)$).

Now, let $(\mathcal{P}_n)_{n\in\mathbf{Z}_+}$ be a refining sequence of KR partitions of (Y,U) where $\mathcal{P}_n = \{U^k(Z_n(i)) \mid 0 \le i < c_n,\ 0 \le k < h_n(i)\}$. Then for each $0 \le i < c_n$ and $0 \le j < c_{n-1}$, there exists $(H_n)_{ij} \subset [0, h_n(i)) \cap \mathbf{Z}$ such that

$$Z_{n-1}(j) = \bigcup_{0\le i<c_n} \bigcup_{\rho\in(H_n)_{ij}} U^{\rho}(Z_n(i)) .$$

We call $\{(H_n)_{ij}\}$ the *hitting time sets* of $(\mathcal{P}_n)_{n\in\mathbf{N}}$.

From $\{(H_n)_{ij}\}$, we construct an ordered Bratteli diagram $\mathbf{B}(\{\mathcal{P}_n\})$ associated with $(\mathcal{P}_n)_{n\in\mathbf{Z}_+}$ as follows:

$$\begin{aligned}
&V_n = \{v_n(0),\dots,v_n(c_n-1)\} ,\\
&E_n = \{(v_{n-1}(j),\rho,v_n(i)) \mid \rho \in (H_n)_{ij}\} ,\\
&(v_{n-1}(j),\rho,v_n(i)) \le (v_{n-1}(j'),\rho',v_n(i')) \text{ if } i = i' \text{ and } \rho \le \rho' ,\\
&s(v_{n-1}(j),\rho,v_n(i)) = v_{n-1}(j) ,\\
&r(v_{n-1}(j),\rho,v_n(i)) = v_n(i) .
\end{aligned}$$

PROPOSITION 4 ([1]). *There exists a refining sequence of KR partitions $(\mathcal{P}_n)_{n\in\mathbf{Z}_+}$, where $\mathcal{P}_n = \{U^k(Z_n(i)) \mid 0 \le i < c_n,\ 0 \le k < h_n(i)\}$ and $c_0 = h_0(0) = 1$, $Z_0(0) = Y$ such that $\mathbf{B} = \mathbf{B}(\{\mathcal{P}_n\})$ is properly ordered, and the corresponding Bratteli-Vershik system $(X_{\mathbf{B}},\theta_{\mathbf{B}})$ is conjugate to (Y,U). A conjugacy ϕ is given by*

$$\{\phi((v_{n-1}(i_{n-1}),\rho_n,v_n(i_n))_{n\in\mathbf{N}})\} = \bigcap_{n\in\mathbf{N}} Z_n\left(i_n, \sum_{l=1}^{n} \rho_l\right) \quad \textit{where } Z_n(i_n,k) := U^k(Z_n(i_n)) .$$

Moreover,

$$\{\phi(e_*^{\min})\} = \bigcap_{n\in\mathbf{N}} \bigcup_{0\le i<c_n} Z_n(i) .$$

We say $(X_{\mathbf{B}},\theta_{\mathbf{B}})$ is an *HPS-adic presentation* of (Y,U), and call ϕ the *natural conjugacy*.

5.2. HPS-adic presentations of Denjoy systems. Consider a Denjoy system (X,T) of rotation number α and double orbit number d. Let Λ be a transversal of its double point set with $\alpha \in \Lambda$, and $\Lambda_1 = \Lambda \cup \{1\}$. In [6], the author, Sugisaki and Yoshida constructed a concrete HPS-adic presentation of (X,T) (based on dual Ostrowski numeration system). We shall introduce its construction.

First, we introduce a modification of dual Ostrowski numeration system. Let

$$\alpha = [0; b_1+1, b_2, b_3, \dots]$$

be the simple continued fraction expansion of α, that is, $b_1 = a_1 - 1$, $b_n = a_n$ $(n \geq 2)$, and

$$M_\alpha^s = \left\{ x_* \in \prod_{n\in\mathbf{N}} \{-1, 0, 1, \dots, b_n\} \;\middle|\; x_n = b_n \Longleftrightarrow x_{n+1} = -1 \right\}.$$

Define the *signed expansion* $x_*^s : (0, 1] \to M_\alpha^s$ by

$$x_n^s(\omega) = \begin{cases} x_n(\omega) - 1 & \text{if } n = 1 \text{ or } x_{n-1}(\omega) = a_{n-1} \\ x_n(\omega) & \text{otherwise}. \end{cases}$$

Next, we define

$$C_n^s : \Lambda_1 \to \mathcal{A} \text{ where } C_n^s(\omega) = \#\{\lambda \in \Lambda_1 \mid x^s_{(n,\infty)}(\lambda) <_{\text{lex}} x^s_{(n,\infty)}(\omega)\}.$$

DEFINITION 5. For each $n \in \mathbf{N}$, let

$$\xi_n = x_n^s \circ (C_{n-1}^s)^{-1} \text{ and } g_n = C_n^s \circ (C_{n-1}^s)^{-1}.$$

Let $(\xi_n, g_n)(j) = (\xi_n(j), g_n(j))$. Associated with (ξ_n, g_n), define

$$D(\xi_n, g_n) = \{(x, i) \in \{-1, 0, \dots, b_n\} \times \mathcal{A} \mid x = b_n \Leftrightarrow i \leq g_n(d)\},$$

and for each $j \in \mathcal{A}$,

$$D(\xi_n, g_n)_j = \begin{cases} \{(x, i) \in D(\xi_n, g_n) \mid (x, i) \leq_{\text{lex}} (\xi_n, g_n)(0)\} & \text{if } j = 0 \\ \{(x, i) \in D(\xi_n, g_n) \mid (\xi_n, g_n)(j-1) <_{\text{lex}} (x, i) \leq_{\text{lex}} (\xi_n, g_n)(j)\} & \text{otherwise} \end{cases}$$

where $(x, i) <_{\text{lex}} (y, j)$ if $x < y$, or $x = y$ and $i < j$.

(The definition of (ξ_n, g_n) is different from the one in [6], but Prop. 8.2 in [6] ensures that both are the same.)
Let

$$\lceil c, i \rceil_n^s = \#\{\lambda \in \Lambda_1 \mid x^s_{(n-1,\infty)}(\lambda) <_{\text{lex}} c\, x^s_{(n,\infty)} \circ (C_n^s)^{-1}(i)\}.$$

LEMMA 1. (1) $D(\xi_n, g_n) = \bigcup_{j\in\mathcal{A}} D(\xi_n, g_n)_j$.

(2) *If* $(c, i) \in D(\xi_n, g_n)$, *then* $(c, i) \in D(\xi_n, g_n)_{\lceil c,i \rceil_n^s}$.

PROOF. First, we claim that

$$(\xi_n, g_n)(j) <_{\text{lex}} (\xi_n, g_n)(j+1).$$

Indeed, notice that $x^s_{(n,\infty)} \circ (C_n^s)^{-1}(i) <_{\text{lex}} x^s_{(n,\infty)} \circ (C_n^s)^{-1}(j) \Longleftrightarrow i < j$. So by the definition of (ξ_n, g_n), we have

$$x^s_{(n-1,\infty)} \circ (C_{n-1}^s)^{-1}(j) <_{\text{lex}} x^s_{(n-1,\infty)} \circ (C_{n-1}^s)^{-1}(j+1)$$
$$\Longleftrightarrow \xi_n(j) < \xi_n(j+1), \text{ or } \xi_n(j) = \xi_n(j+1) \text{ and}$$

$$\begin{aligned}&x^s_{(n,\infty)} \circ (C^s_n)^{-1}(g_n(j)) <_{\rm lex} x^s_{(n,\infty)} \circ (C^s_n)^{-1}(g_n(j+1))\\ &\qquad \Longleftrightarrow (\xi_n, g_n)(j) <_{\rm lex} (\xi_n, g_n)(j+1)\,.\end{aligned}$$

(1) By the definition of $D(\xi_n, g_n)_j$, clearly $D(\xi_n, g_n) \supset \bigcup_j D(\xi_n, g_n)_j$. It suffices to show that $(\xi_n, g_n)(d) = (b_n, g_n(d))$, that is, $\xi_n(d) = b_n$. By the above claim, $\xi_n(d) = \max\{\xi_n(i) \mid i \in \mathcal{A}\} = \max\{x^s_n(\omega) \mid \omega \in \Lambda_1\}$. By ($\sharp$) in Section 2, we see that $\max\{x^s_n(\omega) \mid \omega \in \Lambda_1\} = b_n$.

(2) Let $j = \lceil c, i \rceil^s_n$. By the definition of $\lceil c, i\rceil^s_n$,

$$x^s_{(n-1,\infty)} \circ (C^s_{n-1})^{-1}(j-1) <_{\rm lex} c\, x^s_{(n,\infty)} \circ (C^s_n)^{-1}(i) \leq_{\rm lex} x^s_{(n-1,\infty)} \circ (C^s_{n-1})^{-1}(j)\,.$$

Notice that

$$\begin{aligned}&c\, x^s_{(n,\infty)} \circ (C^s_n)^{-1}(i) \leq_{\rm lex} x^s_{(n-1,\infty)} \circ (C^s_{n-1})^{-1}(j)\\ &\qquad \Longleftrightarrow c < \xi_n(j),\ \text{or } c = \xi_n(j) \text{ and } x^s_{(n,\infty)} \circ (C^s_n)^{-1}(i) \leq_{\rm lex} x^s_{(n,\infty)} \circ (C^s_n)^{-1}(g_n(j))\\ &\qquad \Longleftrightarrow (c, i) \leq_{\rm lex} (\xi_n, g_n)(j)\,,\end{aligned}$$

Similarly, we can see that

$$x^s_{(n-1,\infty)} \circ (C^s_{n-1})^{-1}(j-1) <_{\rm lex} c\, x^s_{(n,\infty)} \circ (C^s_n)^{-1}(i) \Longleftrightarrow (\xi_n, g_n)(j-1) <_{\rm lex} (c, i)\,.$$

□

For each $n \in \mathbf{Z}$, let $n_+ = \max\{n, 0\}$ and

$$\varepsilon_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even}\,. \end{cases}$$

Define

$$s_n(x) = \begin{cases} (-\varepsilon_{n+1}(-1)^n q_n + (x - b_n + \varepsilon_n)(-1)^{n-1} q_{n-1})_+ & \text{if } -1 \leq x < b_n \\ 0 & \text{if } x = b_n\,. \end{cases}$$

Hence we see that

$$\begin{aligned}&\text{if } n \text{ is odd, then } s_n(-1) < s_n(0) < \cdots < s_n(b_n - 1)\,;\\ &\text{if } n \text{ is even, then } s_n(-1) > s_n(0) > \cdots > s_n(b_n - 1)\,.\end{aligned}$$

The next proposition follows Th. 5.1, Lem. 7.1, Cor. 7.1 and Cor. 7.2 in [6].

PROPOSITION 5. *For a Denjoy system (X, T) of finite double orbit number d and rotation number α, there exists a refining sequence of KR partitions $(\mathcal{P}^X_n)_{n \in \mathbf{Z}_+}$ whose form is $\mathcal{P}^X_n = \{T^k(z_n(i)) \mid i \in \mathcal{A},\ 0 \leq k < h_n(i)\}$ $(n \geq 1)$, with hitting time sets given by*

$$\begin{aligned}(H_1)_{i0} &= \{s_1(x) \mid (x, i) \in D(\xi_1, g_1)\}\,,\\ (H_n)_{ij} &= \{s_n(x) \mid (x, i) \in D(\xi_n, g_n)_j\} \quad (n \geq 2)\,,\end{aligned}$$

such that $\mathbf{B} = \mathbf{B}(\{\mathcal{P}_n^X\})$ *is properly ordered, and the associated Bratteli-Vershik system* $(X_{\mathbf{B}}, \theta_{\mathbf{B}})$ *is conjugate to* (Y, S). *Moreover* $\phi(e_*^{\min}) = \inf I_\alpha$ *where* ϕ *is the natural conjugacy, and*

$$z_0(j) = \bigcup_{i \in \mathcal{A}} \bigcup_{(x,i) \in D(\xi_1, g_1)_j} T^{s_1(x)}(z_1(i)) .$$

($z_0(j)$*'s are defined in Section* 3.)

6. Proof

6.1. Proof of Theorem 1. First, we prepare some lemmas to prove Theorem 1. Let $\mathbf{B}$ be an ordered Bratteli diagram. Let $P_{[l,m]}(v)$ $(1 \leq l \leq m,\ v \in V_m)$ be the set of finite paths from V_{l-1} to v, that is,

$$P_{[l,m]}(v) = \left\{ e_{[l,m]} \in \prod_{n=l}^{m} E_n \mid r(e_n) = s(e_{n+1})\ (l \leq n < m),\ r(e_m) = v \right\} .$$

Naturally the order of Bratteli diagram induces a total order on $P_{[l,m]}(v)$: $e_{[l,m]} < f_{[l,m]}$ if there exists $l \leq N \leq m$ such that $e_N < f_N$ and $e_n = f_n$ for all $N < n \leq m$.
For each $e_{[l,m]} \in P_{[l,m]}(v)$, define $s(e_{[l,m]}) = s(e_l)$.

LEMMA 2. (1) $X_{\mathbf{B}}$ *has a unique minimal path if and only if for each* $n \geq 2$, *there exists* $N \geq n$ *such that* $s(\min P_{[n,N]}(v))$ *is independent of* v. *In this case*, $s(\min P_{[n,N]}(v)) = s(e_n^{\min})$.

(2) $X_{\mathbf{B}}$ *has a unique maximal path if and only if for each* $n \geq 2$, *there exists* $N \geq n$ *such that* $s(\max P_{[n,N]}(v))$ *is independent of* v. *In this case*, $s(\max P_{[n,N]}(v)) = s(e_n^{\max})$.

PROOF. We prove (1). Note $X_{\mathbf{B}}$ is compact. For each $n \in \mathbf{N}$, let

$$X_n = \{e_* \in X_{\mathbf{B}} \mid e_{[1,n]} = \min P_{[1,n]}(r(e_n))\} .$$

Then $\bigcap_n X_n$ is the set of minimal paths. Observe $X_n \supset X_{n+1}$ and X_n is non-empty closed. Therefore $\bigcap_n X_n$ is non-empty, that is, there exist minimal paths.

Suppose $e_*^{\min}$ is the unique minimal path of $X_{\mathbf{B}}$, that is, $\bigcap_n X_n = \{e_*^{\min}\}$. Then for each $n \geq 2$,

$$\bigcap_{N \in \mathbf{N}} X_N \cap \{e_* \in X_{\mathbf{B}} \mid e_{[1,n-1]} \neq e_{[1,n-1]}^{\min}\} = \emptyset ,$$

Therefore for any $n \geq 2$, there exists $N \geq n$ such that

$$X_N \subset \{e_* \in X_{\mathbf{B}} \mid e_{[1,n-1]} = e_{[1,n-1]}^{\min}\} .$$

This implies that $s\big(\min P_{[n,N]}(v)\big) = r(e_{n-1}^{\min}) = s(e_n^{\min})$ for any $v \in V_N$.

Conversely, suppose that for each $n \geq 2$, there exists $N \geq n$ such that $s(\min P_{[n,N]}(v))$ is independent of v. Let e_*, f_* be minimal paths and $n \geq 2$. Since $e_*, f_* \in X_N$, we see that $s(e_n) = s(f_n) = s(\min P_{[n,N]}(v))$ and

$$e_{[1,n-1]} = f_{[1,n-1]} = \min P_{[1,n-1]}(s(\min P_{[n,N]}(v)))\,.$$

Since n is arbitrary, this implies $e_* = f_*$. $\square$

Let $\mathbf{B}$ be a properly ordered Bratteli diagram. By the definition of $\theta_{\mathbf{B}}$, we have the following.

REMARK 2. If $e_* \neq e_*^{\max}$, then there exists N such that for any $n > N$, $e_n = \theta_{\mathbf{B}}(e_*)_n$.

LEMMA 3. *Let $\mathbf{B}$ be a properly ordered Bratteli diagram. Then $(X_{\mathbf{B}}, \theta_{\mathbf{B}})$ has a periodic point if and only if*

$$\lim_{n\to\infty} \min_{v \in V_n} \#P_{[1,n]}(v) < \infty\,.$$

PROOF. Notice that $\min_{v\in V_n} \#P_{[1,n]}(v) \leq \min_{v \in V_{n+1}} \#P_{[1,n+1]}(v)$.

Suppose $\theta_{\mathbf{B}}^p(e_*) = e_*$ $(p \in \mathbf{N})$. If $e_* \neq e_*^{\max}$ then $\theta_{\mathbf{B}}(e_*) > e_*$. So there exists $0 \leq q < p$ such that $\theta_{\mathbf{B}}^q(e_*) = e_*^{\max}$. Then $e_*^{\max} = \theta_{\mathbf{B}}^{p-1}(e_*^{\min})$. By Remark 2, there exists N such that for all $n > N$, $e_n^{\max} = e_n^{\min}$. Let $n > N$. Since $\min r^{-1}(r(e_n^{\min})) = e_n^{\min} = e_n^{\max} = \max r^{-1}(r(e_n^{\max}))$, we see $r^{-1}(r(e_n^{\min})) = \{e_n^{\min}\}$. Then

$$\#P_{[1,n]}(r(e_n^{\min})) = \#P_{[1,n-1]}(r(e_{n-1}^{\min}))\,.$$

Therefore $\lim_{n\to\infty} \min_{v\in V_n} \#P_{[1,n]}(v) < \infty$.

Conversely, suppose $\lim_{n\to\infty} \min_{v\in V_n} \#P_{[1,n]}(v) < \infty$. There exist $N, p \in \mathbf{N}$ such that

$$\min_{v\in V_n} \#P_{[1,n]}(v) = p \quad \text{for any } n \geq N\,.$$

To prove the existence of a periodic point, we show the following claims.

Claim 1. *There exists $f_* \in X_{\mathbf{B}}$ such that $r^{-1}(r(f_n)) = \{f_n\}$ for all $n > N$.*

For each $n \geq N$, let $Y_n = \{e_* \in X_{\mathbf{B}} \mid \#P_{[1,n]}(r(e_n)) = p\}$. For $e_* \in Y_{n+1}$, we have

$$p = \#P_{[1,n+1]}(r(e_{n+1})) = \sum_{e \in r^{-1}(r(e_{n+1}))} \#P_{[1,n]}(s(e)) \geq \#P_{[1,n]}(r(e_n)) \geq p\,,$$

hence $r^{-1}(r(e_{n+1})) = \{e_{n+1}\}$ and $e_* \in Y_n$. In particular, $Y_{n+1} \subset Y_n$. Since Y_n is non-empty and closed, $\bigcap_{n\geq N} Y_n$ is non-empty. Let $f_* \in \bigcap_{n \geq N} Y_n$, then f_* is the desired one.

Claim 2. *Let $Z = \{e_* \in X_{\mathbf{B}} \mid e_n = f_n$ for all $n > N\}$. Then $\theta_{\mathbf{B}}(Z) \subset Z$.*

For each $n > N$, let $Z_n = \{e_* \in X_{\mathbf{B}} \mid e_n = f_n\}$. So $Z = \bigcap_{n>N} Z_n$. For $e_* \in Z_{n+1}$, $e_n \in r^{-1}(s(e_{n+1})) = r^{-1}(s(f_{n+1})) = r^{-1}(r(f_n)) = \{f_n\}$. Hence $Z_{n+1} \subset Z_n$.

For each $n \in \mathbf{N}$, let $X_n = \{e_* \in X_{\mathbf{B}} \mid e_{[1,n]} = \min P_{[1,n]}(r(e_n))\}$. Then $X_{n+1} \subset X_n$ and $\bigcap_{n\in\mathbf{N}} X_n = \{e_*^{\min}\}$. For each $n > N$, $\min P_{[1,n]}(r(f_n))f_{(n,\infty)} \in X_n \cap Z_n$ since $r^{-1}(r(f_n)) = \{f_n\}$. So $X_n \cap Z_n$ is non-empty and closed. Clearly $X_{n+1} \cap Z_{n+1} \subset X_n \cap Z_n$. Therefore

$$\emptyset \neq \bigcap_{n>N} X_n \cap Z_n \subset \bigcap_{n>N} X_n = \{e_*^{\min}\}\,.$$

Thus $e_*^{\min} \in Z$.

Let $e_* \in Z$. If $e_* = e_*^{\max}$, then $\theta_{\mathbf{B}}(e_*) = e_*^{\min} \in Z$. If $e_* \neq e_*^{\max}$, then by Remark 2, there exists N' such that for any $n > N'$, $\theta_{\mathbf{B}}(e_*)_n = e_n$. If $N' \leq N$, then $\theta_{\mathbf{B}}(e_*) \in Z$. If $N' > N$, then $\theta_{\mathbf{B}}(e_*) \in \bigcap_{n>N'} Z_n$. Since $Z_{n+1} \subset Z_n$, $\theta_{\mathbf{B}}(e_*) \in Z$.

The existence of a periodic point follows Claim 2 and $\#Z < \infty$. □

LEMMA 4. *Let* $\mathbf{B}$ *be an ordered Bratteli diagram of constant rank and* $(\sigma_n)_{n\in\mathbf{N}}$ *be the natural substitution system of* $\mathbf{B}$. *Then*

$$v_{l-1}^{-1} \circ s(\overrightarrow{P_{[l,m]}(v_m(i))}) = \sigma_l \sigma_{l+1} \cdots \sigma_m(i)\,.$$

PROOF. Fix $l \in \mathbf{N}$. We use induction on m ($\geq l$). It is clear for $m = l$. Suppose the claim holds for m. Here we have the following partition

$$P_{[l,m+1]}(v_{m+1}(i)) = \bigcup_{e_{m+1}\in r^{-1}(v_{m+1}(i))} \{e_{[l,m]}e_{m+1} \mid e_{[l,m]} \in P_{[l,m]}(s(e_{m+1}))\}\,.$$

Let $e_{[l,m+1]}, f_{[l,m+1]} \in P_{[l,m+1]}(v_{m+1}(i))$. If $e_{m+1} < f_{m+1}$, or $e_{m+1} = f_{m+1}$ and $e_{[l,m]} < f_{[l,m]}$, then $e_{[l,m+1]} < f_{[l,m+1]}$. Hence when

$$\overrightarrow{r^{-1}(v_{m+1}(i))} = e_{m+1}^1 e_{m+1}^2 \cdots e_{m+1}^k \quad (\text{where } k = \#r^{-1}(v_{m+1}(i)))\,,$$

we see that

$$\begin{aligned} v_{l-1}^{-1} \circ s(\overrightarrow{P_{[l,m+1]}(v_{m+1}(i))}) &= v_{l-1}^{-1} \circ s(\overrightarrow{P_{[l,m]}(s(e_{m+1}^1))}) \cdots v_{l-1}^{-1} \circ s(\overrightarrow{P_{[l,m]}(s(e_{m+1}^k))}) \\ &= \sigma_l \cdots \sigma_m(v_m^{-1} \circ s(e_{m+1}^1)) \ldots \sigma_l \cdots \sigma_m(v_m^{-1} \circ s(e_{m+1}^k)) \\ &= \sigma_l \cdots \sigma_{m+1}(i)\,. \end{aligned}$$

□

By the definition of $\theta_{\mathbf{B}}$, we have the following.

OBSERVATION 2. *Suppose that* $\mathbf{B}$ *is a properly ordered Bratteli diagram. Let* $e_* \in X_{\mathbf{B}}$ *and* $P = P_{[1,l]}(r(e_l))$. *If* $e_{[1,l]} = \min P$, *then*

$$\theta_{\mathbf{B}}^0(e_*)_{[1,l]}\theta_{\mathbf{B}}^1(e_*)_{[1,l]} \cdots \theta_{\mathbf{B}}^{\#P-1}(e_*)_{[1,l]} = \overrightarrow{P}\,.$$

PROOF OF THEOREM 1. Let $e_* \in X_{\mathbf{B}}$ and $P = P_{[1,l]}(r(e_l))$. If $e_{[1,l]} = \min P$, then by Observation 2 and Lemma 4,

$$J_{\mathbf{B}}(e_*)_0 J_{\mathbf{B}}(e_*)_1 \cdots J_{\mathbf{B}}(e_*)_{\#P-1} = \sigma_1\sigma_2 \cdots \sigma_l(v_l^{-1} \circ r(e_l))\,.$$

Let $n \geq 2$ and $Q = P_{[1,n-1]}(r(e_{n-1}^{\min}))$. Notice that $e_{[1,n-1]}^{\min} = \min Q$. By Lemma 2 and 4, there exists N such that

$$v_{n-1}^{-1} \circ s(e_n^{\min}) = v_{n-1}^{-1} \circ s(\min P_{[n,N]}(v_N(i))) = (\sigma_n \sigma_{n+1} \cdots \sigma_N(i))_1 \quad (i \in \mathcal{A}) .$$

Hence

$$J_{\mathbf{B}}(e_*^{\min})_0 J_{\mathbf{B}}(e_*^{\min})_1 \cdots J_{\mathbf{B}}(e_*^{\min})_{\#Q-1} = \sigma_1 \sigma_2 \cdots \sigma_{n-1}((\sigma_n \sigma_{n+1} \cdots \sigma_N(i))_1) .$$

By Lemma 3, $\#Q \to \infty$. So $(\sigma_n)_{n \in \mathbf{N}}$ generates $J_{\mathbf{B}}(e_*^{\min})_0 J_{\mathbf{B}}(e_*^{\min})_1 \cdots$.

Similarly, we can see that $(\overleftarrow{\sigma_n})_{n \in \mathbf{N}}$ generates $J_{\mathbf{B}}(e_*^{\min})_{-1} J_{\mathbf{B}}(e_*^{\min})_{-2} \cdots$. □

6.2. Proof of Main Theorem. By Proposition 5, we have a properly ordered Bratteli diagram $\mathbf{B} = \mathbf{B}(\{\mathcal{P}_n^X\})$ and an HPS-adic presentation of (X, T): $(X_{\mathbf{B}}, \theta_{\mathbf{B}})$. But $\mathbf{B}$ is not of constant rank ($c_0 = 1$, $c_n = d + 1$ $(n \geq 1)$). Here, define a properly ordered Bratteli diagram $\mathbf{D}$ of constant rank by

$$\begin{aligned}
&V_n = \{w_n(0), w_n(1), \dots, w_n(d)\} , \\
&E_n = \{(w_{n-1}(j), s_n(x), w_n(i)) \mid (x, i) \in D(\xi_n, g_n)_j\} , \\
&(w_{n-1}(j), \rho, w_n(i)) \leq (w_{n-1}(j'), \rho', w_n(i')) \text{ if } i = i' \text{ and } \rho \leq \rho' .
\end{aligned}$$

Moreover define $\Psi : \mathbf{D} \to \mathbf{B}$ by

$$\begin{aligned}
&\Psi((w_0(j), \rho, w_1(i))) = (v_0(0), \rho, v_1(i)) , \\
&\Psi((w_{n-1}(j), \rho, w_n(i))) = (v_{n-1}(j), \rho, v_n(i)) \quad (n \geq 2) .
\end{aligned}$$

Then by Lemma 1 (1), Ψ induces a conjugacy Ψ: $(X_{\mathbf{D}}, \theta_{\mathbf{D}}) \to (X_{\mathbf{B}}, \theta_{\mathbf{B}})$ with $\Psi(e_*^{\min}) = e_*^{\min}$.

PROPOSITION 6. $J_{\mathbf{D}}(e_*^{\min}) = J_X(\inf I_\alpha)$.

PROOF. By Proposition 5, $\phi \circ \Psi(e_*^{\min}) = \inf I_\alpha$. Note

$$\Psi(\{e_* \in X_{\mathbf{D}} \mid s(e_*) = j\}) = \{e_* \in X_{\mathbf{B}} \mid e_1 = (v_0(0), s_1(x), v_1(i)),\ (x, i) \in D(\xi_1, g_1)_j\} .$$

By Proposition 5,

$$\phi \circ \Psi(\{e_* \in X_{\mathbf{D}} \mid s(e_*) = j\}) \subset z_0(j) .$$

This completes the proof. □

So we will study the natural substitution system of $\mathbf{D}$.

LEMMA 5. $i \leq g_n(d) \Longleftrightarrow \xi_{n+1}(i) = -1$.

PROOF. By (♯) in Section 2, notice that

$$(C_n^s)^{-1}(d) = \begin{cases} \alpha & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even.} \end{cases}$$

Therefore

$$i \leq g_n(d) \Longleftrightarrow x^s_{(n,\infty)} \circ (C^s_n)^{-1}(i) \leq_{\mathrm{lex}} (-1)b_{n+2}(-1)b_{n+4}\cdots \Longleftrightarrow \xi_{n+1}(i) = -1$$

□

PROPOSITION 7. *Let*

$$\tau_n(i) = \begin{cases} \lceil b_n, i\rceil^s_n & \text{if } x^s_{n+1} \circ (C^s_n)^{-1}(i) = -1 \\ \lceil -1, i\rceil^s_n \lceil 0, i\rceil^s_n \cdots \lceil b_n - 1, i\rceil^s_n & \text{otherwise.} \end{cases}$$

Then the natural substitution system of **D** *is* $(\tau_1, \overleftarrow{\tau_2}, \tau_3, \overleftarrow{\tau_4}, \ldots)$.

PROOF. By Lemma 1,

$$r^{-1}(w_n(i)) = \begin{cases} \left\{(w_{n-1}(\lceil a_n, i\rceil^s_n), s_n(b_n), w_n(i))\right\} & \text{if } i \leq g_n(d) \\ \left\{(w_{n-1}(\lceil x, i\rceil^s_n), s_n(x), w_n(i)) \mid -1 \leq x < b_n\right\} & \text{otherwise.} \end{cases}$$

By Lemma 5, $i \leq g_n(d)$ is equivalent to $x^s_{n+1} \circ (C^s_n)^{-1}(i) = -1$.

Let $i \leq g_n(d)$. Then

$$\sigma_n(i) = w^{-1}_{n-1} \circ s(w_{n-1}(\lceil b_n, i\rceil^s_n), s_n(b_n), w_n(i)) = \lceil b_n, i\rceil^s_n = \tau_n(i) = \overleftarrow{\tau_n}(i)\,.$$

Let $i > g_n(d)$ and n be odd. Since s_n is increasing on $\{-1, 0, \ldots, b_n - 1\}$, we see that

$$\begin{aligned}\overrightarrow{r^{-1}(w_n(i))} = (w_{n-1}(\lceil -1, i\rceil^s_n), s_n(-1), w_n(i))(w_{n-1}(\lceil 0, i\rceil^s_n), s_n(0), w_n(i)) \\ \cdots (w_{n-1}(\lceil b_n - 1, i\rceil^s_n), s_n(b_n - 1), w_n(i))\,.\end{aligned}$$

Therefore

$$w^{-1}_{n-1}\left(\overrightarrow{r^{-1}(w_n(i))}\right) = \lceil -1, i\rceil^s_n \lceil 0, i\rceil^s_n \cdots \lceil b_n - 1, i\rceil^s_n = \tau_n(i)\,.$$

Let $i > g_n(d)$ and n be even. Since s_n is decreasing on $\{-1, 0, \ldots, b_n - 1\}$, similarly, we can see that $w^{-1}_{n-1}\left(\overrightarrow{r^{-1}(w_n(i))}\right) = \overleftarrow{\tau_n}(i)$. □

Combining Proposition 2 (1), 6, 7 and Theorem 1, we get the following.

PROPOSITION 8. *The sequence* $(\tau_1, \overleftarrow{\tau_2}, \tau_3, \overleftarrow{\tau_4}, \cdots)$ *generates* $J(\alpha)$.

To complete the proof of Main Theorem, we observe the relation between $(\tau_1, \overleftarrow{\tau_2}, \cdots)$ and $(\sigma_1, \overleftarrow{\sigma_2}, \cdots)$. Indeed, it suffices to show that

$$\sigma_1\overleftarrow{\sigma_2}\sigma_3\overleftarrow{\sigma_4}\cdots = \tau_1\overleftarrow{\tau_2}\tau_3\overleftarrow{\tau_4}\cdots\,.$$

To show this, we prepare some definitions. Let

$$\tau'_n(i) = \begin{cases} \lceil a_n, i'\rceil_n & \text{if } x^s_{n+1} \circ (C^s_n)^{-1}(i) = -1 \\ \lceil 0, i'\rceil_n \lceil 1, i'\rceil_n \cdots \lceil a_n - 1, i'\rceil_n & \text{otherwise} \end{cases}$$

where $i' = C_n \circ (C_n^s)^{-1}(i)$. Moreover define i_n, i_n^s, Δ by

$$M_n = \{x_{(n,\infty)} \mid x_* \in M_\alpha\}\,,\ i_n : M_n \to \mathcal{A},\ i_n(x_{(n,\infty)}) = \#\{\lambda \in \Lambda_1 \mid x_{(n,\infty)}(\lambda) <_{\text{lex}} x_{(n,\infty)}\};$$

$$M_n^s = \{y_{(n,\infty)} \mid y_* \in M_\alpha^s\}\,,\ i_n^s : M_n^s \to \mathcal{A},\ i_n^s(y_{(n,\infty)}) = \#\{\lambda \in \Lambda_1 \mid x^s_{(n,\infty)}(\lambda) <_{\text{lex}} y_{(n,\infty)}\};$$

$$\Delta : \bigcup_{n=1}^{\infty} M_n^s \to \bigcup_{n=1}^{\infty} M_n,\ \Delta(x_{(n,\infty)}) = (x_{n+1})_+(x_{n+2})_+ \cdots .$$

We have the following:

- For each $i \in \mathcal{A}$, there exist $x_{(n,\infty)} \in M_n$, $y_{(n,\infty)} \in M_n^s$ such that $i_n(x_{(n,\infty)}) = i_n^s(y_{(n,\infty)}) = i$.
- $x_{(n,\infty)} = \Delta \circ x^s_{(n,\infty)}$.
- For $y_{(n,\infty)}, y'_{(n,\infty)} \in M_n^s$, if $y_{n+1} = y'_{n+1} = -1$, or $y_{n+1}, y'_{n+1} \geq 0$, then

$$y_{(n,\infty)} <_{\text{lex}} y'_{(n,\infty)} \Longleftrightarrow \Delta(y_{(n,\infty)}) <_{\text{lex}} \Delta(y'_{(n,\infty)})\,.$$

The following formula gives characterizations of σ_n, τ_n and τ'_n.

FORMULA 1. *The following holds.*

$$(1)\quad \sigma_n(i_n(x_{(n,\infty)})) = \begin{cases} i_{n-1}(0x_{(n,\infty)}) \cdots i_{n-1}(a_n x_{(n,\infty)}) & \text{if } x_{n+1} = 0 \\ i_{n-1}(0x_{(n,\infty)}) \cdots i_{n-1}((a_n - 1)x_{(n,\infty)}) & \text{otherwise.} \end{cases}$$

$$(2)\quad \tau_n(i_n^s(y_{(n,\infty)})) = \begin{cases} i^s_{n-1}(b_n y_{(n,\infty)}) & \text{if } y_{n+1} = -1 \\ i^s_{n-1}((-1)y_{(n,\infty)}) \cdots i^s_{n-1}((b_n - 1)y_{(n,\infty)}) & \text{otherwise.} \end{cases}$$

$$(3)\quad \tau'_n(i_n^s(y_{(n,\infty)})) = \begin{cases} i_{n-1}(a_n \Delta(y_{(n,\infty)})) & \text{if } y_{n+1} = -1 \\ i_{n-1}(0\Delta(y_{(n,\infty)})) \cdots i_{n-1}((a_n - 1)\Delta(y_{(n,\infty)})) & \text{otherwise.} \end{cases}$$

PROOF. (1) Let $x_{(n,\infty)} \in M_n$. Since $0a_{n+2}0a_{n+4} \cdots \in x_{(n,\infty)}(\Lambda_1)$, we have

$$x_{n+1} = 0 \Longleftrightarrow x_{n+1} C_n^{-1} i_n(x_{(n,\infty)}) = 0\,.$$

Assume that there exist $\lambda \in \Lambda_1$ and $0 \leq c \leq a_n$ such that

$$c\, x_{(n,\infty)} \leq_{\text{lex}} x_{(n-1,\infty)}(\lambda) <_{\text{lex}} c\, x_{(n,\infty)} C_n^{-1}(i_n(x_{(n,\infty)}))\,.$$

Then we can see

$$x_{(n,\infty)} \leq_{\text{lex}} x_{(n,\infty)}(\lambda) <_{\text{lex}} x_{(n,\infty)} C_n^{-1}(i_n(x_{(n,\infty)}))\,.$$

This contradicts the definition of i_n and C_n. Therefore we have

$$\lceil c, i_n(x_{(n,\infty)}) \rceil_n = i_{n-1}(c\, x_{(n,\infty)})\,.$$

This completes the proof of (1). Similarly we can show (2).

(3) Let $y_{(n,\infty)} \in M_n^s$. Since $(-1)b_{n+2}(-1)b_{n+4}\cdots \in x^s_{(n,\infty)}(\Lambda_1)$, we have

$$y_{n+1} = -1 \Longleftrightarrow x^s_{n+1}(C_n^s)^{-1}(i_n^s(y_{(n,\infty)})) = -1\,.$$

First, consider the case of $y_{n+1} = -1$. Assume that there exists $\lambda \in \Lambda_1$ such that

$$a_n\ \Delta(y_{(n,\infty)}) \leq_{\text{lex}} x_{(n-1,\infty)}(\lambda) <_{\text{lex}} a_n\ x_{(n,\infty)}(C_n^s)^{-1}(i_n^s(y_{(n,\infty)}))\,.$$

Then we can see

$$y_{(n,\infty)} \leq_{\text{lex}} x^s_{(n,\infty)}(\lambda) <_{\text{lex}} x^s_{(n,\infty)}(C_n^s)^{-1}(i_n^s(y_{(n,\infty)}))\,.$$

This contradicts the definitions of i_n^s and C_n^s. Therefore we have

$$\lceil a_n, C_n(C_n^s)^{-1}(i_n^s(y_{(n,\infty)}))\rceil_n = i_{n-1}(a_n\ \Delta(y_{(n,\infty)}))\,.$$

Next, consider the case of $y_{n+1} \geq 0$. Assume there exist $\lambda \in \Lambda_1$ and $0 \leq c < a_n$ such that

$$c\ \Delta(y_{(n,\infty)}) \leq_{\text{lex}} x_{(n-1,\infty)}(\lambda) <_{\text{lex}} c\ x_{(n,\infty)}(C_n^s)^{-1}(i_n^s(y_{(n,\infty)}))\,.$$

Then we can see

$$y_{(n,\infty)} \leq_{\text{lex}} x^s_{(n,\infty)}(\lambda) <_{\text{lex}} x^s_{(n,\infty)}(C_n^s)^{-1}(i_n^s(y_{(n,\infty)}))\,.$$

This contradicts the definitions of i_n^s and C_n^s. Therefore we have

$$\lceil c, C_n(C_n^s)^{-1}(i_n^s(y_{(n,\infty)}))\rceil_n = i_{n-1}(c\ \Delta(y_{(n,\infty)}))\,. \qquad \square$$

By Formula 1, $\tau_1 = \tau_1'$. Moreover $\tau_n'\overleftarrow{\tau_{n+1}} = \sigma_n\overleftarrow{\tau'_{n+1}}$ (equivalently, $\overleftarrow{\tau_n'}\tau_{n+1} = \overleftarrow{\sigma_n}\tau'_{n+1}$). Indeed, if $y_{n+2} = -1$,

$$\begin{aligned}
\tau_n'\overleftarrow{\tau_{n+1}}(i^s_{n+1}(y_{(n+1,\infty)}))) &= \tau_n'(i_n^s(b_{n+1}\ y_{(n+1,\infty)}))\\
&= i_{n-1}(0\Delta(b_{n+1}\ y_{(n+1,\infty)}))\cdots i_{n-1}((a_n-1)\Delta(b_{n+1}\ y_{(n+1,\infty)}))\\
&= i_{n-1}(0a_{n+1}\Delta(y_{(n+1,\infty)}))\cdots i_{n-1}((a_n-1)a_{n+1}\Delta(y_{(n+1,\infty)}))\\
&= \sigma_n(i_n(a_{n+1}\Delta(y_{(n+1,\infty)}))) = \sigma_n\overleftarrow{\tau'_{n+1}}(i^s_{n+1}(y_{(n+1,\infty)})))\,.
\end{aligned}$$

If $y_{n+2} \neq -1$,

$$\begin{aligned}
&\tau_n'\overleftarrow{\tau_{n+1}}(i^s_{n+1}(y_{(n+1,\infty)}))) = \tau_n'(i_n^s((b_{n+1}-1)\ y_{(n+1,\infty)})\cdots i_n^s((-1)\ y_{(n+1,\infty)}))\\
&= \underbrace{i_{n-1}(0\Delta((b_{n+1}-1)\ y_{(n+1,\infty)}))\cdots i_{n-1}((a_n-1)\Delta((b_{n+1}-1)\ y_{(n+1,\infty)}))}\\
&\quad \cdots\\
&\quad \underbrace{i_{n-1}(0\Delta(0\ y_{(n+1,\infty)}))\cdots i_{n-1}((a_n-1)\Delta(0\ y_{(n+1,\infty)}))}\ i_{n-1}(a_n\Delta((-1)\ y_{(n+1,\infty)}))\\
&= \underbrace{i_{n-1}(0(a_{n+1}-1)\Delta(y_{(n+1,\infty)}))\cdots i_{n-1}((a_n-1)(a_{n+1}-1)\Delta(y_{(n+1,\infty)}))}
\end{aligned}$$

...

$$\underbrace{i_{n-1}(00\Delta(y_{(n+1,\infty)}))\cdots i_{n-1}((a_n-1)0\Delta(y_{(n+1,\infty)}))\ i_{n-1}(a_n0\Delta(y_{(n+1,\infty)}))}$$

$$= \sigma_n(i_n((a_{n+1}-1)\Delta(y_{(n+1,\infty)}))\cdots i_n(0\Delta(y_{(n+1,\infty)}))) = \sigma_n \overleftarrow{\tau'_{n+1}}(i^s_{n+1}(y_{(n+1,\infty)}))) \,.$$

Therefore we have

$$\tau_1\overleftarrow{\tau_2}\tau_3\overleftarrow{\tau_4}\cdots = \tau'_1\overleftarrow{\tau_2}\tau_3\overleftarrow{\tau_4}\cdots = \sigma_1\overleftarrow{\tau'_2}\tau_3\overleftarrow{\tau_4}\cdots = \sigma_1\overleftarrow{\sigma_2}\tau'_3\overleftarrow{\tau_4}\cdots = \cdots = \sigma_1\overleftarrow{\sigma_2}\sigma_3\overleftarrow{\sigma_4}\cdots \,.$$

This completes the proof of Main Theorem.

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