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A Characterization of a Coaction Reduced to That of a Closed Subgroup

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Abstract. It is shown that, for any coaction α of a locally compact group K on a properly infinite von Neumann algebra A and a closed subgroup H of K, α is cocycle conjugate to a coaction which comes from a coaction of H if and only if the dual action $\hat{\alpha}$ is induced by an action of H. We also include applications of the result concerning almost periodic coactions and the ranges of 1-cocycles on measured equivalence relations.

1. Introduction

Let K be a locally compact group and A be a von Neumann algebra. The coactions of K on A are special unital normal $*$-isomorphisms from A into $W^*(K) \otimes A$, where $W^*(K)$ is the von Neumann algebra which is generated by the left regular representation of K. It is known that group coactions are defined as a dual notion of group actions. Indeed, if a group K is abelian, then all coactions of K can be considered as actions of the dual group $\hat{K}$, and vice versa.

For each closed subgroup H of K, there exists a natural inclusion map I from $W^*(H)$ to $W^*(K)$. So, if the range of a coaction α of K on A is contained in $I(W^*(H)) \otimes A$, then α comes from a coaction of H, i.e., $(I^{-1} \otimes \mathrm{id}) \circ \alpha$ is a coaction of H on A. In the recent work of the author and T. Yamanouchi, they treated the exchangeability of coactions which fix Cartan subalgebras ([1]). Namely, for each coaction α of K which fixes a Cartan subalgebra and closed subgroup H of K, they gave a necessary and sufficient condition that α is cocycle conjugate to a coaction which fixes the common Cartan subalgebra and comes from a coaction of H. They proved that, a coaction α of K on A satisfies such properties if and only if the dual action $\hat{\alpha}$ of K on the crossed product $\hat{K}\,{}_{\alpha}\!\ltimes A$ is induced from an action of H ([1, Theorem 7.2]). But, in their proof, they use the fact that α fixes a Cartan subalgebra and comes from a 1-cocycle on a discrete measured equivalence relation. Hence their arguments can not be directly applied to general situations.

So it is natural to ask that their theorem is valid for general coactions. Our aim of this paper is to give an affirmative answer to this question for all coactions on properly infinite von Neumann algebras (Theorem 3.5). By using our arguments, we succeed in characterizing

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the ranges of 1-cocycles on (not necessarily discrete) measured equivalence relations and the flow of weights in terms of corresponding coactions (Corollary 4.1 and Corollary 4.2).

Moreover, we also apply our arguments to almost periodic coactions. We will prove that, if a coaction α is almost periodic, then the dual action $\hat{\alpha}$ is induced by an action of a discrete group (Theorem 3.2). We note that the assumption of our theorem contains the case that α is an almost periodic group action. So our theorem is a generalization of the theory in [3] which describes the connection between discrete decompositions and continuous decompositions of type III von Neumann algebras (Corollary 3.3).

The idea to prove our main theorem is to develop the arguments in [3] to general coactions. We will show that, if a coaction α of K comes from a coaction α^0 of H, then the system $\{\hat{K}\ {}_{\alpha}\!\ltimes A, \hat{\alpha}\}$ is induced by $\{\hat{H}\ {}_{\alpha^0}\!\ltimes A, \widehat{\alpha^0}\}$ (Proposition 3.1). To prove the converse, we use the Takesaki duality theorem. We will prove that, if an action κ of K is induced by an action of H, then the dual coaction $\hat{\kappa}$ is cocycle conjugate to a coaction which comes from a coaction of H (Proposition 3.4).

The organization of this paper is as follows: Section 2 is devoted to summarizing the basic facts about coactions and induced actions. In Section 3, we will give a proof of our main theorem. We also give applications to 1-cocycles in Section 4. In Appendix, we mention the construction of coactions from 1-cocycles on the flows of weights.

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2. Preparation

In this section, we recall the basic facts about coactions and induced actions. Further details about these matters are found in [1], [2], [3], [4], [7], [8], [9] and [10].

We assume that all von Neumann algebras in this paper have separable preduals.

For a Hilbert space $\mathcal{H}$ with a inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$, we denote by $B(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. For a faithful normal semifinite weight φ on a von Neumann algebra A, set $\mathfrak{n}_\varphi := \{x \in A : \varphi(x^*x) < \infty\}$ and $\mathfrak{m}_\varphi := \mathfrak{n}_\varphi^* \mathfrak{n}_\varphi$. We denote the Hilbert space obtained from φ by the GNS-construction by $\mathcal{H}_\varphi$, and the natural injection from $\mathfrak{n}_\varphi$ to $\mathcal{H}_\varphi$ by Λ_φ.

2.1. Group coactions on von Neumann algebras. Let K be a (second countable) locally compact group with the left Haar measure μ and the module δ_K. We denote the left (resp. right) regular representation of K on $L^2(K, \mu)$ by λ_K (resp. ρ_K) and the von Neumann algebra generated by $\{\lambda_K(k)\}_{k \in K}$ (resp. $\{\rho_K(k)\}_{k \in K}$) by $W^*(K)$ (resp. $W_r^*(K)$).

We recall that the coproduct Δ_K of $W^*(K)$ is a unital normal $*$-isomorphism from $W^*(K)$ into $W^*(K) \otimes W^*(K)$ which is defined by the following:

$$\Delta_K(x) := W_K(1 \otimes x)W_K^* \quad (x \in W^*(K)),$$

where W_K is a unitary on $L^2(K \times K, \mu \times \mu)$ defined by

$$(W_K\xi)(k_1, k_2) := \xi(k_2^{-1}k_1, k_2) \quad (\xi \in L^2(K \times K),\ k_1, k_2 \in K)\,.$$

Suppose that A is a von Neumann algebra. A unital normal $*$-isomorphism α from A into $W^*(K) \otimes A$ is called a coaction of K on A if α satisfies the following:

$$(\mathrm{id} \otimes \alpha) \circ \alpha = (\Delta_K \otimes \mathrm{id}) \circ \alpha\,.$$

We note that, by using the Fourier transformation, if K is abelian, then coactions of K coincide with actions of the dual group $\hat{K}$.

DEFINITION 2.1. Let α be a coaction of a locally compact group K on a von Neumann algebra A.

(1) α is called faithful if $\{(\mathrm{id} \otimes \omega)\alpha(a) : a \in A,\ \omega \in A_*\}''$ is equal to $W^*(K)$, where A_* in general stands for the predual of A.

(2) The fixed-point algebra A^α is the von Neumann subalgebra of A defined by

$$A^\alpha := \{a \in A : \alpha(a) = 1 \otimes a\}\,.$$

(3) Suppose that a is in A and V is a closed subset of K. The spectrum $\mathrm{Sp}(\alpha)$, the spectrum $\mathrm{Sp}_\alpha(a)$ of a, the spectral subspace $A^\alpha(V)$ and the discrete spectrum $\mathrm{Sp}_d(\alpha)$ are defined by the following:

$$\begin{aligned}
\mathrm{Sp}(\alpha) &= \bigcap\{\mathrm{Ker}(\phi) : \phi \in A(K), \alpha_\phi = 0\}\,,\\
\mathrm{Sp}_\alpha(a) &= \bigcap\{\mathrm{Ker}(\phi) : \phi \in A(K), \alpha_\phi(a) = 0\}\,,\\
A^\alpha(V) &= \{a \in A : \mathrm{Sp}_\alpha(a) \subseteq V\}\,,\\
\mathrm{Sp}_d(\alpha) &= \{k \in K : A^\alpha(\{k\}) \neq \{0\}\}\,,
\end{aligned}$$

where $A(K) = (W^*(K))_*$ is the Fourier algebra of K and α_ϕ is determined by the following:

$$\omega(\alpha_\phi(a)) = (\phi \otimes \omega)\alpha(a) \quad (\phi \in A(K),\ a \in A,\ \omega \in A_*)\,.$$

(4) α is called almost periodic if (i) α is faithful; (ii) The spectral subspaces $\{A^\alpha(\{k\})\}_{k\in K}$ generates a σ-strongly$*$ dense subspace of A; (iii) There exists a faithful normal state φ on A which satisfies the equation $(\mathrm{id} \otimes \varphi)\alpha(a) = \varphi(a)$ for each $a \in A$. We call φ an α-invariant state on A.

By [2, Proposition 3.3], if α is almost periodic, then $\mathrm{Sp}_d(\alpha)$ is countable and there exists a mutually orthogonal family of projections $\{P_k\}_{k\in\mathrm{Sp}_d(\alpha)}$ with sum 1 such that the canonical implementation V_K of α is determined by the following:

$$V_K := \sum_{k\in\mathrm{Sp}_d(\alpha)} \lambda_K(k) \otimes P_k\,.$$

Namely, the equation $\alpha(a) = V_K(1 \otimes a)V_K^*$ holds for each $a \in A$.

(5) The crossed product $\hat{K}\ {}_{\alpha}\!\ltimes A$ is a von Neumann algebra which is generated by $\alpha(A)$ and $L^{\infty}(K)\otimes \mathbf{C}$. It is known that $\hat{K}\ {}_{\alpha}\!\ltimes A$ is equal to the fixed-point algebra of the stabilization $\tilde{\alpha} := \mathrm{Ad}(W_K^{*}\otimes 1)\circ(\sigma\otimes 1)\circ(\mathrm{id}\otimes\alpha)$ on $B(L^2(K))\otimes A$, where σ in general stands for the flip.

(6) The map $\hat{\alpha}_k := \mathrm{Ad}(\rho_K(k)\otimes 1)|_{\hat{K}\ {}_{\alpha}\ltimes A}$ is an action of K on $\hat{K}\ {}_{\alpha}\!\ltimes A$ and called the dual action of α. It is known that the fixed-point algebra of $\hat{\alpha}$ on $\hat{K}\ {}_{\alpha}\!\ltimes A$ is equal to $\alpha(A)$.

(7) A unitary R in $W^{*}(K)\otimes A$ is called an α-1-cocycle if R satisfies the following equation:

$$(\Delta_K\otimes\mathrm{id})(R) = (1\otimes R)(\mathrm{id}\otimes\alpha)(R)\,.$$

For each α-1-cocycle R, the map $\mathrm{Ad}(R)\circ\alpha$ is also a coaction of K. We denote it by ${}_{R}\alpha$. We call a coaction α' of K on A is cocycle conjugate to α if there exist an α-1-cocycle R and an automorphism π on A such that α' is equal to $(\mathrm{id}\otimes\pi)\circ{}_{R}\alpha\circ\pi^{-1}$. We note that the system $\{\hat{K}\ {}_{\alpha}\!\ltimes A, \hat{\alpha}\}$ is determined by only the cocycle conjugacy class of α.

Let H be a closed subgroup of K. By [7, Lemma 3.1], there exists a unique normal $*$-isomorphism I from $W^{*}(H)$ into $W^{*}(K)$ which satisfies the equation $I(\lambda_H(h)) = \lambda_K(h)$ for each $h\in H$. It follows that the equation $\Delta_K\circ I = (I\otimes I)\circ\Delta_H$ holds. So, for each coaction α^0 of H on A, $(I\otimes\mathrm{id})\circ\alpha^0$ is a coaction of K on A.

We further suppose that β is an action of K on a von Neumann algebra M. The crossed product $K\ {}_{\beta}\!\ltimes M$ is a von Neumann algebra which is generated by $W^{*}(K)\otimes\mathbf{C}$ and $\pi_\beta(M)$, where $\pi_\beta(X)(k) := \beta_{k^{-1}}(X)$ for each $X\in M$ and $k\in K$. Set $\tilde{\beta}_k := \mathrm{Ad}(\rho_K(k))\otimes\beta_k$ for each $k\in K$. It is known that $K\ {}_{\beta}\!\ltimes M$ is equal to the fixed-point algebra of the action $\tilde{\beta}$ of K on $B(L^2(K))\otimes M$. Moreover, the map $\hat{\beta}(X) := \mathrm{Ad}(W_K\otimes 1)(1\otimes X)$ for $X\in K\ {}_{\beta}\!\ltimes M$ is a coaction of K on $K\ {}_{\beta}\!\ltimes M$. We call $\hat{\beta}$ the dual coaction of β. By the Takesaki duality theorem, the bidual coaction $\hat{\hat{\alpha}}$ on $K\ {}_{\hat{\alpha}}\!\ltimes \hat{K}\ {}_{\alpha}\!\ltimes A$ is conjugate to $\tilde{\alpha}$ on $B(L^2(K))\otimes A$. By using the arguments as in the proof [9, Theorem V.2.4], if A is properly infinite, then there exists an α-1-cocycle R such that $A^{{}_{R}\alpha}$ is also properly infinite. It follows that, for each coaction α on a properly infinite von Neumann algebra A, $\hat{\hat{\alpha}}$ is cocycle conjugate to α.

For each action β of K on a von Neumann algebra M, the map T_β defined by $T_\beta(x) := \int_K \beta_k(x)d\mu(k)$ is an operator valued weight from M to M^{β}. Set $\mathfrak{n}_{T_\beta} := \{x\in M : T_\beta(x^{*}x)\in M^{\beta}\}$ and $\mathfrak{m}_{T_\beta} := \mathfrak{n}_{T_\beta}^{*}\mathfrak{n}_{T_\beta}$. It is known that $T_{\hat{\alpha}}$ is a semifinite operator valued weight from $\hat{K}\ {}_{\alpha}\!\rtimes A$ to $\alpha(A)$, i.e., $\mathfrak{m}_{T_{\hat{\alpha}}}$ is dense in $\hat{K}\ {}_{\alpha}\!\rtimes A$.

2.2. Inclusions of group–subgroup with the induced actions on von Neumann algebras. Let P be a von Neumann algebra and $H\subseteq K$ be an inclusion of (second countable) locally compact group–subgroup. For each action β of H on P, we define (mutually commute) actions γ of H and κ of K on $L^\infty(K) \otimes P$ by the following:

$$\gamma_h(X)(l) := \beta_h(X(lh)), \quad \kappa_k(X(l)) := X(k^{-1}l) \quad (X \in L^\infty(K) \otimes P,\ h \in H,\ k, l \in K).$$

Set $Q := (L^\infty(K) \otimes P)^\gamma$. The restriction of κ to Q is called the induced action by β. We denote the pair $\{Q, \kappa\}$ by $\mathrm{Ind}_H^K\{P, \beta\}$.

We will assume in what follows that H is a closed subgroup of K. By [4, Section 2.6], there exists a rho-function ρ on K associated to a quasi-invariant measure on K/H. For the (left) projection π_K from K to K/H, choose a Borel cross section $\theta : K/H \to K$. It is easy to check that the unitary $V : L^2(K) \to L^2(K/H) \otimes L^2(H)$ defined by $(V\xi)(p, h) := \xi(\theta(p)h)\rho(\theta(p)h)^{-1/2}$ yields an isomorphism from $B(L^2(K))$ onto $B(L^2(K/H)) \otimes B(L^2(H))$. Moreover, since $\pi_K(k\theta(p))$ is equal to kp, we have that $\theta(kp)^{-1}k\theta(p)$ belongs to H for all $p \in K/H$ and $k \in K$. So the map $\chi : K \times K/H \to H$ defined by $\chi(k, p) := \theta(kp)^{-1}k\theta(p)$ is a Borel 1-cocycle. It is known that the system $\mathrm{Ind}_H^K\{P, \beta\}$ is conjugate to $\{L^\infty(K/H) \otimes P, \delta\}$, where the action δ of K is defined by the following (see [1, Theorem A.1]):

$$\delta_k(X)(p) := \beta^{-1}_{\chi(k^{-1},p)}(X(k^{-1}p)) \quad (X \in L^\infty(K/H) \otimes P,\ p \in K/H,\ k \in K).$$

Conversely, suppose that κ is an action of K on a von Neumann algebra Q. If there exists a K-equivariant embedding of $L^\infty(K/H)$ into the center $Z(Q)$ of Q, then there exists an action β of H on a von Neumann algebra P such that the pair $\{Q, \kappa\}$ is conjugate to $\mathrm{Ind}_H^K\{P, \beta\}$ (Imprimitivity Theorem).

3. Exchangeability of coactions by their dual actions

In this section, by showing a connection between the ranges of coactions and their dual actions, we will give a proof of our main theorem. Firstly, we will show the following:

PROPOSITION 3.1. *Suppose that α is a coaction of a locally compact group K on a von Neumann algebra A. If there exists a closed subgroup H of K such that the subalgebra $\{(\mathrm{id} \otimes \omega)\alpha(a) : a \in A,\ \omega \in A_*\}''$ is contained in $I(W^*(H))$, the range of the $*$-isomorphism I from $W^*(H)$ into $W^*(K)$ defined by $I(\lambda_H(h)) = \lambda_K(h)$ $(h \in H)$. Then the dual action $\hat{\alpha}$ of K on $\hat{K}\,{}_\alpha\!\ltimes A$ is induced by some dual action β of H on a von Neumann algebra P.*

PROOF. By the assumptions, $\alpha^0 := (I^{-1} \otimes \mathrm{id}) \circ \alpha$ is a coaction of H on A. Set $P := \hat{H}\,{}_{\alpha^0}\!\ltimes A$, $\beta := \widehat{\alpha^0}$ and $\{Q, \kappa\} := \mathrm{Ind}_H^K\{P, \beta\}$. We will show that there exists a $*$-isomorphism Φ from $\hat{K}\,{}_\alpha\!\ltimes A$ onto Q which satisfies the equation $(\mathrm{id} \otimes \Phi) \circ \hat{\alpha} = \kappa \circ \Phi$. For this, we first define a map $\bar{\alpha}$ on $\hat{K}\,{}_\alpha\!\ltimes A$ by the following:

$$\bar{\alpha} := (\sigma \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \alpha^0) = (I^{-1} \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\sigma \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \alpha).$$

We claim that $\bar{\alpha}$ is a coaction of H on $\hat{K}\,{}_\alpha\!\ltimes A$.

Indeed, a direct computation shows that the following equations hold on $\hat{K}\,{}_{\alpha}\ltimes A$:

$$\begin{aligned}\bar{\alpha} &= (I^{-1}\otimes \mathrm{id}\otimes \mathrm{id})\circ(\mathrm{Ad}(W_K)\otimes \mathrm{id})\circ\tilde{\alpha}\,,\\ (\mathrm{id}\otimes\bar{\alpha})\circ\bar{\alpha} &= (\Delta_H\otimes \mathrm{id}\otimes \mathrm{id})\circ\bar{\alpha}\,.\end{aligned}$$

So, for each $z\in\hat{K}\,{}_{\alpha}\ltimes A$, we have

$$\bar{\alpha}(z) = (I^{-1}\otimes \mathrm{id}\otimes \mathrm{id})(\mathrm{Ad}(W_K)\otimes \mathrm{id})(1\otimes z)\,.$$

It follows that

$$\begin{aligned}(\mathrm{id}\otimes\tilde{\alpha})\bar{\alpha}(z) &= (\mathrm{id}\otimes \mathrm{Ad}(W_K^*)\otimes \mathrm{id})(\mathrm{id}\otimes I\otimes \mathrm{id}\otimes \mathrm{id})(\mathrm{id}\otimes\bar{\alpha})\bar{\alpha}(z)\\
&= (\mathrm{id}\otimes \mathrm{Ad}(W_K^*)\otimes \mathrm{id})(\mathrm{id}\otimes I\otimes \mathrm{id}\otimes \mathrm{id})(\Delta_H\otimes \mathrm{id}\otimes \mathrm{id})\bar{\alpha}(z)\\
&= (\mathrm{id}\otimes \mathrm{Ad}(W_K^*)\otimes \mathrm{id})((\mathrm{id}\otimes I)\Delta_H\otimes \mathrm{id}\otimes \mathrm{id})\\
&\qquad\qquad (I^{-1}\otimes \mathrm{id}\otimes \mathrm{id})(\mathrm{Ad}(W_K)\otimes \mathrm{id})(1\otimes z)\\
&= (\mathrm{id}\otimes \mathrm{Ad}(W_K^*)\otimes \mathrm{id})((I^{-1}\otimes \mathrm{id})\Delta_K\otimes \mathrm{id}\otimes \mathrm{id})(\mathrm{Ad}(W_K)\otimes \mathrm{id})(1\otimes z)\\
&= (I^{-1}\otimes \mathrm{Ad}(W_K^*)\otimes \mathrm{id})(\Delta_K\otimes \mathrm{id}\otimes \mathrm{id})(\mathrm{Ad}(W_K)\otimes \mathrm{id})(1\otimes z)\\
&= (I^{-1}\otimes \mathrm{Ad}(W_K^*)\otimes \mathrm{id})(\mathrm{id}\otimes\Delta_K\otimes \mathrm{id})(\mathrm{Ad}(W_K)\otimes \mathrm{id})(1\otimes z)\\
&= (I^{-1}\otimes \mathrm{Ad}(W_K^*)\otimes \mathrm{id})(\mathrm{id}\otimes \mathrm{Ad}(W_K)\otimes \mathrm{id})\\
&\qquad\qquad (\sigma\otimes \mathrm{id}\otimes \mathrm{id})(\mathrm{id}\otimes \mathrm{Ad}(W_K)\otimes \mathrm{id})(1\otimes 1\otimes z)\\
&= (I^{-1}\otimes \mathrm{id}\otimes \mathrm{id}\otimes \mathrm{id})(\sigma\otimes \mathrm{id}\otimes \mathrm{id})(\mathrm{id}\otimes \mathrm{Ad}(W_K)\otimes \mathrm{id})(1\otimes 1\otimes z)\\
&= (\sigma\otimes \mathrm{id}\otimes \mathrm{id})(\mathrm{id}\otimes I^{-1}\otimes \mathrm{id}\otimes \mathrm{id})(\mathrm{id}\otimes \mathrm{Ad}(W_K)\otimes \mathrm{id})(1\otimes 1\otimes z)\\
&= (\sigma\otimes \mathrm{id}\otimes \mathrm{id})(1\otimes\bar{\alpha}(z))\,.\end{aligned}$$

So we conclude that $\bar{\alpha}(\hat{K}\,{}_{\alpha}\ltimes A)$ is contained in $W^*(H)\otimes\hat{K}\,{}_{\alpha}\ltimes A$, and $\bar{\alpha}$ is a coaction of H on $\hat{K}\,{}_{\alpha}\ltimes A$.

It follows that $\bar{\alpha}(\hat{K}\,{}_{\alpha}\ltimes A)$ is equal to the fixed-point algebra of the dual action $\hat{\bar{\alpha}}_h := \mathrm{Ad}(\rho_H(h)\otimes 1\otimes 1)$ on $\hat{H}\,{}_{\bar{\alpha}}\ltimes(\hat{K}\,{}_{\alpha}\ltimes A)$. Now, we define a unitary U from $L^2(K\times H)$ to $L^2(H\times K)$ by the following:

$$(U\xi)(h,k) := \delta_K(k^{-1}h)^{1/2}\xi(k^{-1}h,h)\quad(\xi\in L^2(K\times H),\ k\in K,\ h\in H)\,.$$

It is easy to check that, for each $k\in K$ and $h\in H$, the following equations hold:

$$\begin{aligned}\mathrm{Ad}(U)(1\otimes\lambda_H(h)) &= \lambda_H(h)\otimes\lambda_K(h) = (I^{-1}\otimes \mathrm{id})\Delta_K(\lambda_K(h))\,,\\ \mathrm{Ad}(U)(\rho_K(h)\otimes\rho_H(h)) &= \rho_H(h)\otimes 1\,,\\ \mathrm{Ad}(U)(\lambda_K(k)\otimes 1) &= 1\otimes\rho_K(k)\,.\end{aligned}$$

So, for each $k \in K$, $h \in H$ and $a \in A$, we have

$$\mathrm{Ad}(U)(L^\infty(K) \otimes L^\infty(H)) = L^\infty(H) \otimes L^\infty(K)\,,$$

$$\mathrm{Ad}(U \otimes 1)(1 \otimes \alpha^0(a)) = \bar{\alpha}(\alpha(a))\,,$$

$$\mathrm{Ad}(U \otimes 1)\,\mathrm{Ad}(\rho_K(h) \otimes \rho_H(h) \otimes 1) = \mathrm{Ad}(\rho_H(h) \otimes 1 \otimes 1)\,,$$

$$\mathrm{Ad}(U \otimes 1)\,\mathrm{Ad}(\lambda_K(k) \otimes 1 \otimes 1) = \mathrm{Ad}(1 \otimes \rho_K(k) \otimes 1)\,.$$

Hence the map $\Phi := \mathrm{Ad}(U^* \otimes 1) \circ \bar{\alpha}$ is an isometry from $\hat{K}\,{}_\alpha\!\ltimes A$ onto Q which satisfies the desired condition.

Therefore we complete the proof. □

We note that the above arguments can be applied to almost periodic coactions. Namely, we get the following:

THEOREM 3.2. *Let α be a coaction of a locally compact group K on a von Neumann algebra A. If α is almost periodic with the discrete spectrum* $\mathrm{Sp}_d(\alpha)$, *then there exists a coaction α^0 of Γ on A such that the dual action $\hat{\alpha}$ is induced by $\widehat{\alpha^0}$, where Γ is a subgroup of K generated by* $\mathrm{Sp}_d(\alpha)$.

PROOF. Suppose that φ is a faithful normal α-invariant state with the GNS-Hilbert space $\mathcal{H}_\varphi$. By [2, Proposition 3.3], there exists a mutually orthogonal family of projections $\{P_k\}_{k\in\mathrm{Sp}_d(\alpha)}$ on $\mathcal{H}_\varphi$ with sum 1 such that $V_K := \sum_{k\in\mathrm{Sp}_d(\alpha)} \lambda_K(k) \otimes P_K$ is the canonical implementation of α associated to φ. Set $V_\Gamma := \sum_{k\in\mathrm{Sp}_d(\alpha)} \lambda_\Gamma(k) \otimes P_k$. It is easy to check that V_Γ is a unitary on $L^2(\Gamma) \otimes \mathcal{H}_\varphi$, and $\alpha^0 : a \mapsto V_\Gamma(1 \otimes a)V_\Gamma^*$ is a coaction of Γ on A. Set $V_{\Gamma 13} := (\sigma \otimes \mathrm{id})(1_{L^2(K)} \otimes V_\Gamma)$. We have the following equations:

$$V_{\Gamma 13}(1_{L^2(\Gamma)} \otimes V_K) = \sum_{k\in\mathrm{Sp}_d(\alpha)} \lambda_\Gamma(k) \otimes \lambda_K(k) \otimes P_k = (1_{L^2(\Gamma)} \otimes V_K)V_{\Gamma 13}\,,$$

$$V_{\Gamma 13}(1_{L^2(\Gamma)} \otimes V_\Gamma) = (\Delta_\Gamma \otimes \mathrm{id}) \sum_{k\in\mathrm{Sp}_d(\alpha)} \lambda_\Gamma(k) \otimes P_k = (\Delta_\Gamma \otimes \mathrm{id})(V_\Gamma)\,.$$

Hence the range of the map $\bar{\alpha} := (\sigma \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \alpha^0)$ on $\hat{K}\,{}_\alpha\!\ltimes A$ is contained in $W^*(\Gamma) \otimes \hat{K}\,{}_\alpha\!\ltimes A$, and $\bar{\alpha}$ is a coaction of Γ on $\hat{K}\,{}_\alpha\!\ltimes A$ with the canonical implementation $V_{\Gamma 13}$. Now we define a unitary U from $L^2(K \times \Gamma)$ to $L^2(\Gamma \times K)$ by the following:

$$(U\xi)(\gamma, k) := \delta_K(k^{-1}\gamma)^{1/2}\xi(k^{-1}\gamma, \gamma) \quad (\xi \in L^2(K \times \Gamma),\ k \in K,\ \gamma \in \Gamma)\,.$$

By using the same arguments as in the proof of Proposition 3.1, the map $\mathrm{Ad}(U^* \otimes 1) \circ \bar{\alpha}$ is a $*$-isomorphism from $\{\hat{K}\,{}_\alpha\!\ltimes A, \hat{\alpha}\}$ to $\mathrm{Ind}_\Gamma^K\{\hat{\Gamma}\,{}_{\alpha^0}\!\ltimes A, \widehat{\alpha^0}\}$.

Therefore we get our conclusion. □

COROLLARY 3.3 ([3, Corollary 3.4]). *Suppose that M is a type* III *von Neumann algebra with a faithful normal semifinite almost periodic weight φ such that the point spectrum* *of the modular operator* Δ_φ *generates the discrete subgroup* Γ. *Let* $\{N_1, \Gamma, \beta_1\}$ *giving rise to* $M \cong \Gamma_{\beta_1} \ltimes N_1$ *be the discrete decomposition of* M *associated to* φ, *and* $\{N_2, \mathbf{R}, \beta_2\}$ *giving rise to* $M \cong \mathbf{R}_{\beta_2} \ltimes N_2$ *be the continuous decomposition of* M. *Then the action* β_2 *of* $\mathbf{R}$ *on* N_2 *is induced up to* $\mathbf{R}$ *from the action* β_1 *of* Γ *on* N_1.

We will next show that the converse of Proposition 3.1 holds when the von Neumann algebra A is properly infinite. For this, we will show the following:

PROPOSITION 3.4. *Let* K *be a locally compact group and* H *be a closed subgroup of* K. *If an action* κ *of* K *is induced from an action of* H, *then the dual coaction* $\hat{\kappa}$ *is cocycle conjugate to a coaction which comes from a coaction of* H.

PROOF. Suppose that β is an action of H on a von Neumann algebra P. Set $\{Q, \kappa\} := \mathrm{Ind}_H^K\{P, \beta\}$, where

$$Q := (L^\infty(K) \otimes P)^\gamma, \ \gamma_h(X)(l) := \beta_h(X(lh)) \quad (X \in L^\infty(K) \otimes P, \ h \in H, \ l \in K).$$

By using the duality, the action $\mathrm{id} \otimes \gamma$ of H on $K_\kappa \ltimes (L^\infty(K) \otimes P)$ is conjugate to γ' on $B(L^2(K)) \otimes P$, where $\gamma'_h := \mathrm{Ad}(\rho_K(h)) \otimes \beta_h$ for $h \in H$. Moreover, since $K_\kappa \ltimes (L^\infty(K) \otimes P)$ is equal to $(B(L^2(K)) \otimes L^\infty(K) \otimes P)^{\tilde{\kappa}}$ and $\tilde{\kappa}$ commutes with $\mathrm{id} \otimes \gamma$, we have

$$\begin{aligned}\big(K_\kappa \ltimes (L^\infty(K) \otimes P)\big)^{\mathrm{id}\otimes\gamma} &= \big((B(L^2(K)) \otimes L^\infty(K) \otimes P)^{\tilde{\kappa}}\big)^{\mathrm{id}\otimes\gamma} \\ &= \big((B(L^2(K)) \otimes L^\infty(K) \otimes P)^{\mathrm{id}\otimes\gamma}\big)^{\tilde{\kappa}} \\ &= (B(L^2(K)) \otimes Q)^{\tilde{\kappa}} = K_\kappa \ltimes Q.\end{aligned}$$

So the dual coaction $\hat{\kappa}$ on $K_\kappa \ltimes Q$ is conjugate to $\kappa' \otimes \mathrm{id}$ on $(B(L^2(K)) \otimes P)^{\gamma'}$, where κ' is defined by $\kappa'(X) = \mathrm{Ad}(W_K)(1 \otimes X)$ for each $X \in B(L^2(K))$. Now, we define two unitaries $V : L^2(K) \to L^2(K/H) \otimes L^2(H)$ and $W_{K,H}$ on $L^2(K) \otimes L^2(K)$ by the following:

$$(V\xi)(p, h) := \xi(\theta(p)h)\rho(\theta(p)h)^{-1/2} \quad (\xi \in L^2(K), \ p \in K/H, \ h \in H),$$

$$(W_{K,H}\zeta)(k_1, k_2) := \zeta(\theta(\pi_K(k_2))k_1, k_2) \quad (\zeta \in L^2(K \times K), \ k_1, k_2 \in K),$$

where θ is a Borel cross section for the left projection $\pi_K : K \to K/H$ and ρ is the rho-function on K associated to a quasi-invariant measure on K/H. Set $\Psi := \mathrm{Ad}(V \otimes 1)$ and $\kappa_{K,H} := (\mathrm{id} \otimes \Psi) \circ (\kappa' \otimes \mathrm{id}) \circ \Psi^{-1}$. It is easy to check that the map Ψ is an isomorphism from $B(L^2(K)) \otimes P$ to $B(L^2(K/H)) \otimes B(L^2(H)) \otimes P$ which satisfies the equation $\Psi \circ \gamma'_h = (\mathrm{id} \otimes \tilde{\beta}_h) \circ \Psi$ for each $h \in H$. It follows that $(B(L^2(K)) \otimes P)^{\gamma'}$ is isomorphic to $B(L^2(K/H)) \otimes (B(L^2(H)) \otimes P)^{\tilde{\beta}} = B(L^2(K/H)) \otimes H_\beta \ltimes P$. So $\kappa_{K,H}$ is a coaction of K on $B(L^2(K/H)) \otimes H_\beta \ltimes P$. On the other hand, a direct computation shows that $W_{K,H}$ is a κ'-1-cocycle which satisfies the following equation for each $\xi \in L^2(K \times K)$ and $k, k_1, k_2 \in K$:

$$\begin{aligned}(\mathrm{Ad}(W_{K,H})\kappa'(\lambda_K(k))\xi)(k_1, k_2) &= ((\lambda_K(k) \otimes \lambda_K(k))W_{K,H}^*\xi)(\theta(\pi_K(k_2))k_1, k_2) \\ &= \xi(\theta(\pi_K(k^{-1}k_2))^{-1}k^{-1}\theta(\pi_K(k_2))k_1, k^{-1}k_2).\end{aligned}$$

Since $W_{K,H}\otimes 1$ is in $W^*(K)\otimes(B(L^2(K)\otimes P)^{\gamma'}$ and $\theta(\pi_K(k^{-1}k_2))^{-1}k^{-1}\theta(\pi_K(k_2))$ is in H for each $k, k_2 \in K$, we conclude that $(\mathrm{id}\otimes\Psi)(W_{K,H}\otimes 1)$ is a $\kappa_{K,H}$-1-cocycle such that the range of a coaction ${}_{(\mathrm{id}\otimes\Psi)(W_{K,H}\otimes 1)}\kappa_{K,H}$ is contained in $I(W^*(H))\otimes B(L^2(K/H))\otimes H\ {}_{\beta}\ltimes P$. So our claim has been proven. □

By the above propositions, we get our main theorem.

THEOREM 3.5. *Let α be a coaction of a locally compact group K on a properly infinite von Neumann algebra A, and H be a closed subgroup of K with the $*$-isomorphism I : $W^*(H)\ni\lambda_H(h)\mapsto\lambda_K(h)\in W^*(K)$. Then the following are equivalent*:

(1) *There exists an α-1-cocycle R such that the subalgebra $\{(\mathrm{id}\otimes\omega)_R\alpha(a) : a\in A,\ \omega\in A_*\}''$ is contained in $I(W^*(H))$.*

(2) *The dual action $\hat\alpha$ of K on $\hat K\ {}_{\alpha}\ltimes A$ is induced by some action of H.*

(3) *There exists an injective $*$-homomorphism Θ from $L^\infty(K/H)$ to the center of $\hat K\ {}_{\alpha}\ltimes A$ such that $\Theta\circ\ell_k=\hat\alpha_k\circ\Theta$ for each $k\in K$, where ℓ_k comes from the left translation by k on K/H.*

Moreover, if one of the above conditions occurs, then there exists a coaction α' of H on A such that the system $\{\hat K\ {}_{\alpha}\ltimes A, \hat\alpha\}$ is induced by the system $\{\hat H\ {}_{\alpha'}\ltimes A, \widehat{\alpha'}\}$.

PROOF. By the above propositions and the Takesaki duality theorem, the conditions (1) and (2) are equivalent. The equivalence of (2) and (3) comes from the Imprimitivity Theorem in [10]. The last assertion follows from Proposition 3.1. □

4. Applications for the coactions which come from 1-cocycles

We conclude this paper with two applications associated to the ranges of 1-cocycles.

We first consider 1-cocycles on the measured (not necessarily discrete) equivalence relations. Suppose that $\mathcal{R}$ is a measured equivalence relation on a base space X with a Haar system $\{\lambda^x\}_{x\in X}$. We denote by ν a σ-finite measure on $\mathcal{R}$ obtained by $\{\lambda^x\}_{x\in X}$. Set $\delta := d\nu/d\nu^{-1}$. For each 2-cocycle σ on $\mathcal{R}$, we denote by $W^*(\mathcal{R},\sigma)$ the left von Neumann algebra of a left Hilbert algebra $\mathfrak{A}_I$ which is defined by the following:

$$\mathfrak{A}_I := \{\xi\in L^2(\mathcal{R},\nu) : \xi \text{ is } \delta\text{-bounded and } \|\xi\|_I<\infty\},$$

$$(f*g)(\gamma) := \int f(\gamma_1)g(\gamma_1^{-1}\gamma)\sigma(\gamma_1,\gamma_1^{-1}\gamma)d\lambda^{r(\gamma)}(\gamma_1),$$

$$f^\sharp(\gamma) := \sigma(\gamma,\gamma^{-1})\delta(\gamma)^{-1}\overline{f(\gamma^{-1})}.$$

It is known that $W^*(\mathcal{R},\sigma)$ has a special abelian subalgebra $W^*(X)$ which is isomorphic to $L^\infty(X,\mu)$ and called the diagonal subalgebra of $W^*(\mathcal{R},\sigma)$. We note that, if $\mathcal{R}$ is a discrete measured equivalence relation, then $W^*(X)$ is a Cartan subalgebra of $W^*(\mathcal{R},\sigma)$. For the details about these matters, refer to [5].

By [1, Theorem 5.8], for each coaction α of a locally compact group K on $W^*(\mathcal{R},\sigma)$, if α fixes each element of $W^*(X)$, then there exists a (Borel) 1-cocycle c on $\mathcal{R}$ of a locally compact group K such that α is equal to α_c, where

$$\alpha_c(a) := U_c(1 \otimes a)U_c^* \quad (a \in W^*(\mathcal{R}, \sigma)),$$

$$(U_c\xi)(k, \gamma) := \xi(c(\gamma)^{-1}k, \gamma) \quad (\xi \in L^2(K \times \mathcal{R}),\ k \in K,\ \gamma \in \mathcal{R}).$$

By [1, Proposition 5.10], we have that a 1-cocycle c is cohomologous to c' if and only if there exists an α_c-1-cocycle R such that $\alpha_{c'}$ is equal to ${}_R\alpha_c$. Moreover, by using the arguments as in the proof of [1, Theorem 6.3], we have that the spectrum $\mathrm{Sp}(\alpha_c)$ is equal to the essential range of c. Namely, the essential range of c is contained in a closed subgroup H if and only if the coaction α_c comes from a coaction of H. So we get the following:

COROLLARY 4.1 (cf. [1, Theorem 7.2]). *Let $\mathcal{R}$ be a measured equivalence relation with a 2-cocycle σ. For a Borel 1-cocycle c from $\mathcal{R}$ to a locally compact group K with a closed subgroup H, the following are equivalent*:

(1) *The 1-cocycle c is cohomologous to a 1-cocycle c' such that the essential range of c' is contained in H.*

(2) *There exist a coaction α^0 of H on $W^*(\mathcal{R}, \sigma)$ and an isomorphism Φ from $\{\hat{K}\ {}_{\alpha_c}\ltimes W^*(\mathcal{R}, \sigma), \widehat{\alpha_c}\}$ to $\mathrm{Ind}_H^K\{\hat{H}\ {}_{\alpha^0}\ltimes W^*(\mathcal{R}, \sigma), \widehat{\alpha^0}\}$ which satisfy $W^*(X) \subseteq W^*(\mathcal{R}, \sigma)^{\alpha^0}$ and $\Phi(\mathbf{C} \otimes W^*(X)) = \mathbf{C} \otimes \mathbf{C} \otimes W^*(X)$.*

PROOF. Put $A := W^*(\mathcal{R}, \sigma)$.

(1)$\Rightarrow$(2): Set $\alpha^0 := (I^{-1} \otimes \mathrm{id}) \circ \alpha_{c'}$. By Proposition 3.1, there exists an isomorphism Φ from $\{\hat{K}\ {}_{\alpha_{c'}}\ltimes A, \widehat{\alpha_{c'}}\}$ to $\mathrm{Ind}_H^K\{\hat{H}\ {}_{\alpha^0}\ltimes A, \widehat{\alpha^0}\}$ which satisfies $\Phi(1 \otimes a) = 1 \otimes 1 \otimes a$ for each $a \in A^{\alpha_{c'}}$. Moreover, by [1, Proposition 5.10], there exists an α_c-1-cocycle R such that $\alpha_{c'}$ is equal to ${}_R\alpha_c$. So the condition (2) follows.

(2)$\Rightarrow$(1): By [1, Theorem 5.8], there exists a Borel 1-cocycle $c' : \mathcal{R} \to K$ such that $(I \otimes \mathrm{id}) \circ \alpha^0$ is equal to α_{c_0}. By the construction, we have that the essential range of c_0 is contained in H. By using the same arguments as in the proof of Proposition 3.1, there exists an isomorphism Ψ from $\{\hat{K}\ {}_{\alpha_{c_0}}\ltimes A, \widehat{\alpha_{c_0}}\}$ to $\mathrm{Ind}_H^K\{\hat{H}\ {}_{\alpha^0}\ltimes A, \widehat{\alpha^0}\}$ such that $\Psi(\mathbf{C} \otimes W^*(X)) = \mathbf{C} \otimes \mathbf{C} \otimes W^*(X)$. Hence the map $\Psi^{-1} \circ \Phi$ is an isomorphism from $\{\hat{K}\ {}_{\alpha_c}\ltimes A, \widehat{\alpha_c}\}$ to $\{\hat{K}\ {}_{\alpha_{c_0}}\ltimes A, \widehat{\alpha_{c_0}}\}$ which satisfies $(\Psi^{-1} \circ \Phi)(\mathbf{C} \otimes W^*(X)) = \mathbf{C} \otimes \mathbf{C} \otimes W^*(X)$. So there exist an α_c-1-cocycle R and a $*$-isomorphism θ on A which satisfy $\alpha_{c_0} = (\mathrm{id} \otimes \theta^{-1}) \circ {}_R\alpha_c \circ \theta$ and $\theta(W^*(X)) = W^*(X)$. Set $\alpha' := {}_R\alpha_c (= (\mathrm{id} \otimes \theta) \circ \alpha_{c_0} \circ \theta^{-1})$. It is easy to check that $A^{\alpha'}$ also contains $W^*(X)$, and there exists a 1-cocycle $c' : \mathcal{R} \to K$ such that α' is equal to $\alpha_{c'}$. By [1, Proposition 5.10], we have that c' is cohomologous to c.

Thus our claim has been proven. □

The second application is concerning 1-cocycles on the (smooth) flows of weights.

Let N be a type III factor with the dominant weight φ and the flow of weights $(X_N, \mathbf{R}, F^N)$.

In [11], T. Yamanouchi showed that, if a coaction α of a locally compact group K on N fixes each element of the centralizer N_φ, then there exists a Borel 1-cocycle c from $\mathbf{R} \times X_N$ to

K such that α is equal to the extended modular coaction β_c^φ. We recall that, for the continuous decomposition $N \cong \mathbf{R}_\theta \ltimes N_\varphi = (\{u(t)\}_{t\in\mathbf{R}} \vee N_\varphi)$, β_c^φ is determined by the following:

$$\beta_c^\varphi(a) = 1 \otimes a \quad (a \in N_\varphi), \quad \beta_c^\varphi(u(t)) = Q_t(1 \otimes u(t)) \quad (t \in \mathbf{R}),$$

where $Q_t \in W^*(K) \otimes L^\infty(X_N)$ is defined by $Q_t(\omega) := \lambda_K(c(-t, \omega))$ for $\omega \in X_N$. For the precise definition of the extended modular coactions, refer to [11, Section 3] (see also Appendix).

It follows that the range of c is contained in a closed subgroup H if and only if the coaction β_c^φ comes from a coaction of H. By using [11, Theorem 4.1] and [11, Proposition 3.3] instead of [1, Theorem 5.8] and [1, Proposition 5.10], we can use the same arguments as in the proof of Corollary 4.1. Namely, we get the following:

COROLLARY 4.2. *Let N be a type* III *factor with a dominant weight φ and c be a* 1*-cocycle from $\mathbf{R} \times X_N$ to a locally compact group K. For any closed subgroup H of K, the following are equivalent*:

(1) *The* 1*-cocycle c is cohomologous to a* 1*-cocycle c' such that the range of c' is contained in H.*

(2) *There exist a coaction α^0 of H on N and an isomorphism Φ from $\{\hat{K}\ {}_{\beta_c^\varphi}\!\ltimes N, \widehat{\beta_c^\varphi}\}$ to* $\mathrm{Ind}_H^K\{\hat{H}\ {}_{\alpha^0}\!\ltimes N, \widehat{\alpha^0}\}$ *which satisfy $N_\varphi \subseteq N^{\alpha^0}$ and $\Phi(\mathbf{C} \otimes N_\varphi) = \mathbf{C} \otimes \mathbf{C} \otimes N_\varphi$.*

Appendix A. A remark on coactions which come from the 1-cocycles on the flow of weights

Let N be a type III factor with a faithful normal semifinite weight φ and the modular automorphism group $\{\sigma_t^\varphi\}_{t\in\mathbf{R}}$. Set $\tilde{N} := \mathbf{R}_{\sigma^\varphi} \ltimes N$. The smooth flow of weights $(X_N, \mathbf{R}, F^N)$ of N is given by the restriction of $\{\widehat{\sigma^\varphi}_t\}_{t\in\mathbf{R}}$ to $Z(\tilde{N}) \cong L^\infty(X_N, \nu_N)$. In what follows we assume that c is in $Z^1(F^N, K)$, a 1-cocycle from $\mathbf{R} \times X_N$ into a locally compact group K. Then we have an action of $\mathbf{R}$ on $K \times X_N$ by the following:

$$t \cdot (k, \omega) := (kc(t, \omega), F_t^N(\omega)) \quad (\omega \in X_N,\ k \in K,\ t \in \mathbf{R}).$$

The action is called the skew product action associated with c and denoted by $(K\ {}_c\!\times X_N, \mathbf{R})$. The 1-cocycle c is called minimal if the skew product action acts ergodically.

In [6], M. Izumi shows that, if K is compact, then for each minimal 1-cocycle c in $Z^1(F^N, K)$, there exists a dual action β of K on a type III factor M such that the fixed-point algebra M^β is equal to N, and the flow of weights of M coincides with the skew product action on $K\ {}_c\!\times X_N$. It follows that there exists a coaction α of K on N such that the flow of weights of the fixed-point algebra N^α is given as the skew product action on $K\ {}_c\!\times X_N$.

On the other hand, in [11], by using a different method, T. Yamanouchi showed that such a coaction can be constructed for each $c \in Z^1(F^N, K)$ with a locally compact (not necessarily compact) group K. The action is called the extended modular coaction and denoted by β_c^φ.

In this appendix, we will modify the arguments in [6], and show that Izumi's arguments can be extended to the case that K is not necessarily compact and c is not necessarily minimal. Moreover, we will show that the coactions obtained by our arguments are cocycle conjugate to the extended modular coactions.

We denote the skew product action of $\mathbf{R}$ on $L^\infty(K\,{}_c\!\times X_N, \mu\times\nu_N)$ by $\{\vartheta_t\}_{t\in\mathbf{R}}$, the canonical left translation of K on $L^\infty(K\,{}_c\!\times X_N, \mu\times\nu_N)$ by $\{\lambda_k\}_{k\in K}$ and the canonical semifinite operator valued weight from $L^\infty(K\,{}_c\!\times X_N)$ onto $L^\infty(X_N)\cong L^\infty(K\,{}_c\!\times X_N)^\lambda$ by T. It is easy to check that the equation $T\circ\vartheta_t=\widehat{\sigma^\varphi}_t\circ T$ holds for each $t\in\mathbf{R}$.

THEOREM A.1 (cf. [6, Lemma 5.8]). *Under the above conditions, fix a $*$-isomorphism $\pi : Z(\tilde{N})\to L^\infty(X_N)$. There exists a unique system $\{M,\beta\}$ which satisfies the following*:

(1) *M is a von Neumann algebra which contains N.*

(2) *β is a dual action of K on M, and the fixed-point algebra M^β is equal to N.*

(3) *$\tilde{M}:=\mathbf{R}_{\sigma^{\varphi\circ T_\beta}}\ltimes M$ is generated by $Z(\tilde{M})$ and $\tilde{N}$.*

(4) *$M\cap N'$ is equal to the fixed-point algebra of $\widehat{\sigma^{\varphi\circ T_\beta}}$ on $Z(\tilde{M})$.*

(5) *The inclusion of the flows of weights*

$$\{Z(\tilde{M})\supseteq Z(\tilde{N}), \widehat{\sigma^{\varphi\circ T_\beta}}|_{Z(\tilde{M})}, T_{\tilde{\beta}}|_{Z(\tilde{M})}, \mathrm{mod}(\beta)\}$$

is conjugate to

$$\{L^\infty(K\,{}_c\!\times X_N)\supseteq L^\infty(X_N), \vartheta_t, T, \lambda\}$$

with a $*$-isomorphism $\rho : Z(\tilde{M})\to L^\infty(K\,{}_c\!\times X_N)$ satisfying $\rho|_{Z(\tilde{N})}=\pi$, where $\tilde{\beta}$ is the canonical extension of β and $\mathrm{mod}(\beta)$ *is the Connes–Takesaki module of β.*

PROOF. We first show the uniqueness of $\{M,\beta\}$. Suppose that both $\{M_1,\beta_1\}$ and $\{M_2,\beta_2\}$ satisfy the desired properties. Then there exists an isomorphism ρ from $Z(\tilde{M}_1)$ onto $Z(\tilde{M}_2)$ such that the equations $\rho\circ\widehat{\sigma^{\varphi\circ T_{\beta_1}}}=\widehat{\sigma^{\varphi\circ T_{\beta_2}}}\circ\rho$, $\rho\circ\mathrm{mod}(\beta_1)=\mathrm{mod}(\beta_2)\circ\rho$ and $\rho\circ T_{\widetilde{\beta_1}}=T_{\widetilde{\beta_2}}\circ\rho$ hold on $Z(\tilde{M}_1)$, and the restriction of ρ to $Z(\tilde{N})$ is the identity map. Fix a faithful normal semifinite weight ψ on $\tilde{N}$. Since $\widetilde{M_i}$ is generated by $Z(\widetilde{M_i})$ and $\tilde{N}$, the subalgebra generated by $\mathfrak{m}_\psi\mathfrak{m}_{T_{\widetilde{\beta_i}}|_{Z(\tilde{M}_i)}}$ is dense in $\widetilde{M_i}$ $(i=1,2)$. So $\psi\circ T_{\widetilde{\beta_1}}$ (resp. $\psi\circ T_{\widetilde{\beta_2}}$) is a faithful normal semifinite weight on $\widetilde{M_1}$ (resp. $\widetilde{M_2}$). Hence we may and do assume that $\tilde{M}_1$ (resp. $\tilde{M}_2$) acts on $\mathcal{H}_{\psi\circ T_{\widetilde{\beta_1}}}$ (resp. $\mathcal{H}_{\psi\circ T_{\widetilde{\beta_2}}}$). For each $a,b\in\mathfrak{n}_{T_{\widetilde{\beta_1}}|_{Z(\tilde{M}_1)}}$ and $x,y\in\mathfrak{n}_\psi$, we have

$$\psi(T_{\widetilde{\beta_2}}(y^*\rho(b^*)\rho(a)x))=\psi(y^*T_{\widetilde{\beta_2}}(\rho(b^*a))x)=\psi(y^*T_{\widetilde{\beta_1}}(b^*a)x)=\psi(T_{\widetilde{\beta_1}}(y^*b^*ax)).$$

It follows that the map U defined by

$$U(\Lambda_{\psi\circ T_{\widetilde{\beta_1}}}(ax))=\Lambda_{\psi\circ T_{\widetilde{\beta_2}}}(\rho(a)x)\quad (a\in\mathfrak{n}_{T_{\widetilde{\beta_1}}|_{Z(\tilde{M}_1)}},\ x\in\mathfrak{n}_\psi)$$

is a unitary from $\mathcal{H}_{\psi\circ T_{\widetilde{\beta_1}}}$ to $\mathcal{H}_{\psi\circ T_{\widetilde{\beta_2}}}$. A direct computation shows that $\mathrm{Ad}(U)$ is an automorphism from $\tilde{M}_1$ to $\tilde{M}_2$ which satisfies the equations $\mathrm{Ad}(U)\circ\widehat{\sigma^{\varphi\circ T_{\beta_1}}} = \widehat{\sigma^{\varphi\circ T_{\beta_2}}}\circ\mathrm{Ad}(U)$ and $\mathrm{Ad}(U)\circ\widetilde{\beta}_1 = \widetilde{\beta}_2\circ\mathrm{Ad}(U)$ on $\widetilde{M}_1$. Hence $\mathrm{Ad}(U)$ yields an isomorphism from $\{M_1, \beta_1\}$ to $\{M_2, \beta_2\}$. So such a pair $\{M, \beta\}$ is unique.

We next show the existence of $\{M, \beta\}$. Suppose that $\mathcal{H}$ is the standard Hilbert space of $\tilde{N}$. Set $A := L^\infty(K)\otimes Z(\tilde{N})$. We denote by $\mathcal{H}_1$ the completion of the algebraic tensor product $\mathfrak{n}_T \odot_{Z(\tilde{N})} \mathcal{H}$ by a inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}_1}$ defined by the following:

$$\langle a\odot\xi, b\odot\eta\rangle_{\mathcal{H}_1} := \langle T(b^*a)\xi, \eta\rangle_{\mathcal{H}}, \quad (a, b\in\mathfrak{n}_T,\ \xi,\eta\in\mathcal{H}).$$

The von Neumann algebra generated by $\tilde{N}$ and A in $B(\mathcal{H}_1)$ is denoted by L. By using the same arguments as in the proof of [6, Lemma 5.8], we have

$$L = \int_{X_N}^{\oplus} L^\infty(K)\otimes\tilde{N}(\omega)d\mu_N(\omega),$$

and $L\cap N' = A$. Let $u(t)$ be the canonical implementation of $\widehat{\sigma^\varphi}_t$ on $\mathcal{H}$. We define a 1-parameter unitary group $\{U(t)\}_{t\in\mathbf{R}}$ on $\mathcal{H}_1$ by the following:

$$U(t)\Lambda(a\odot\xi) := \Lambda(\vartheta_t(a)\odot u(t)\xi) \quad (\xi\in\mathcal{H},\ a\in\mathfrak{n}_T).$$

Then the restriction of $\mathrm{Ad}(U(t))$ to L is an automorphism of L. We denote it by Θ_t. It is easy to check that the equation $\Theta_t(ax) = \vartheta_t(a)\widehat{\sigma^\varphi}_t(x)$ holds for each $a\in A$ and $x\in\tilde{N}$. Set $M := L^\Theta$. By the definition, N is contained in M.

We construct the action β of K on M as follows. For each $k\in K$, we define a unitary $V(k)$ by

$$V(k)\Lambda(a\odot\xi) := \Lambda(\lambda_k(a)\odot\xi) \quad (a\in\mathfrak{n}_T,\ \xi\in\mathcal{H}).$$

Then $\{\mathrm{Ad}(V(k))\}_{k\in K}$ is a dual action of K on L. We denote it by β'. By the construction, β' commutes with Θ. So the restriction of β' to $M = L^\Theta$ is an action of K on M. We denote it by β. Since the equations $\beta'_k(ax) = \lambda_k(a)x$ and $T_{\beta'}(bx) = T(b)x$ hold for each $x\in\tilde{N}$, $a\in A$ and $b\in\mathfrak{m}_T$, we have that $L^{\beta'}$ is equal to $\tilde{N}$. Hence we obtain $N = (\tilde{N})^\Theta = (L^{\beta'})^\Theta = (L^\Theta)^\beta = M^\beta$. We note that, by the definition of β', $T_{\beta'}$ satisfies the equation $T_{\beta'}\circ\Theta_t = \widehat{\sigma^\varphi}_t\circ T_{\beta'}$ and the map $T_\beta = T_{\beta'}|_M$ is a semifinite operator valued weight from M to N.

Since L is generated by $A = L^\infty(K)\otimes Z(\tilde{N})$ and $\tilde{N}$, for each faithful normal semifinite trace τ on $\tilde{N}$ satisfying the equation $\tau\circ\widehat{\sigma^\varphi}_t = e^{-t}\tau$, the map $\tau_1 := \tau\circ T_{\beta'}$ is a faithful normal semifinite trace on L which satisfies the equation

$$\tau_1\circ\Theta_t = \tau\circ T_{\beta'}\circ\Theta_t = \tau\circ\widehat{\sigma^\varphi}_t\circ T_{\beta'} = e^{-t}\tau\circ T_{\beta'} = e^{-t}\tau_1.$$

So, by using the same argument as in the proof of [6, Lemma 5.8] again, we conclude that $\{L, \Theta, \beta'\}$ coincides with $\{\tilde{M}, \widehat{\sigma^{\varphi\circ T_\beta}}, \tilde{\beta}\}$. Hence the flow of weights of M is equal to the skew product action $(K\,{}_c\!\times X_N, \mathbf{R})$. Moreover, by using the duality, $\mathrm{id}\otimes\tilde{\beta}$ is conjugate to the action $\mathrm{id}\otimes\beta$ of K on $B(L^2(\mathbf{R}))\otimes M$. Since N is properly infinite, we can replace $\{N\subseteq M, \beta\}$ by $\{B(L^2(\mathbf{R}))\otimes N \subseteq B(L^2(\mathbf{R}))\otimes M, \mathrm{id}\otimes\beta\}$. So we may assume that β is dual.

Finally, since $\tilde{M}\cap N'$ is equal to $Z(\tilde{M})$, we have $M\cap N' = (\tilde{M}\cap N')^{\widehat{\sigma^{\varphi\circ T_\beta}}} = Z(\tilde{M})^{\widehat{\sigma^{\varphi\circ T_\beta}}}$. Therefore we complete the proof. □

COROLLARY A.2. *Under the above setting, there exists a coaction α of K on N such that the flow of weights of N^α is equal to the skew product action $(K\,{}_c\!\times X_N, \mathbf{R})$. Moreover, the coaction α is cocycle conjugate to the extended modular coaction β_c^φ.*

PROOF. By Theorem A.1, there exists a coaction α of K on N such that the pair $\{M,\beta\} := \{\hat{K}\,{}_\alpha\!\ltimes N, \hat{\alpha}\}$ satisfies the conditions of the above theorem. Since the flow of weight of N^α coincides with that of M, α satisfies the desired condition. Moreover, since such an action β is unique, α and β_c^φ are cocycle conjugate. So we get our conclusion. □

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