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Whitney's Umbrellas in Stable Perturbations of a Map Germ $(\mathbf{R}^n, 0) \to (\mathbf{R}^{2n-1}, 0)$

Mariko OHSUMI

Nihon University

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Abstract. Let $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ be a C^∞ map-germ. We are interested in whether the number modulo 2 of stable singular points of codimension n that appear near the origin in a generic perturbation of f is a topological invariant. In this paper we concentrate on investigating the problem when p is $2n-1$, where stable singular points of codimension n are only Whitney's umbrellas, and give a positive answer to the problem.

1. Introduction

1.1. Main theorem. In this paper we show that the number modulo 2 of Whitney's umbrellas that appear in stable perturbations of a generic C^∞ map-germ $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^{2n-1}, 0)$ is a topological invariant.

A C^∞ map-germ $f : (\mathbf{R}^n, p) \to (\mathbf{R}^{2n-1}, q)$ is called Whitney's umbrella if it is $\mathcal{A}$-equivalent to the map-germ from $(\mathbf{R}^n, 0)$ to $(\mathbf{R}^{2n-1}, 0)$ defined by

$$(x_1, \dots, x_n) \to (x_1, \dots, x_{n-1}, x_n^2, x_1x_n, \dots, x_{n-1}x_n)\,.$$

Here two C^∞ map-germs $f : (M_1, p_1) \to (N_1, q_1)$ and $g : (M_2, p_2) \to (N_2, q_2)$ are said to be $\mathcal{A}$-equivalent if there exist C^∞ diffeomorphism-germs $h : (M_1, p_1) \to (M_2, p_2)$ and $k : (N_1, q_1) \to (N_2, q_2)$ such that $k \circ f = g \circ h$.

Let $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^{2n-1}, 0)$ be a generic C^∞ map-germ and let $\bar{f} : U \to \mathbf{R}^{2n-1}$ be a C^∞ representive of f, U being a small open neighborhood of the origin 0 in $\mathbf{R}^n$. By Whitney's theorem([29], [31]), $\bar{f}$ can be approximated by a stable mapping $\tilde{f} : U \to \mathbf{R}^{2n-1}$ whose singularities are only Whitney's umbrellas. We call such $\tilde{f} : U \to \mathbf{R}^{2n-1}$ a stable perturbation of $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^{2n-1}, 0)$.

We are interested in the number of Whitney's umbrellas of $\tilde{f}$.

Let $\mathcal{E}_n$ be the ring of C^∞ function-germs of $(\mathbf{R}^n, 0)$ into $\mathbf{R}$. Let $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^{2n-1}, 0)$ be a C^∞ map-germ. Let $\mathcal{I}(\Sigma^1(f))$ be the ideal in $\mathcal{E}_n$ generated by $n \times n$ minor determinants of the jacobian matrix of f.

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MAIN THEOREM. *Let* $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^{2n-1}, 0)$ *be a generic* C^∞ *map-germ such that* $\dim_{\mathbf{R}} \mathcal{E}_n/\mathcal{I}(\Sigma^1(f)) < +\infty$. *The number of Whitney's umbrellas that appear in a stable perturbation of* f *is equal to* $\dim_{\mathbf{R}} \mathcal{E}_n/\mathcal{I}(\Sigma^1(f))$ (modulo 2) *and it is a topological invariant of* f.

Here we call a map-germ "generic map-germ" in a strong sense. See Definition 2.4 in §2 for the precise definition.

REMARK 1.1. The statement that the number of Whitney's umbrellas that appear in the stable perturbation is equal to $\dim_{\mathbf{R}} \mathcal{E}_n/\mathcal{I}(\Sigma^1(f))$ (mod 2) is a consequence of [4], [5], [19], [21], [22]. Our assertion in the above theorem is that it is a topological invariant of f.

1.2. History of the problem. The problem of counting isolated singular points in stable perturbations of a degenerated map-germ is old and new.

The case of complex holomorphic functions is rather classical. Let $f : (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ be a holomorphic function-germ which defines an isolated singularity at 0. It is well known that Milnor number $\mu(f)$ of f is the number of critical points of a Morse function near f and it is a topological invariant of f (J. W. Milnor [17]).

In the real case also, it is known that for a C^∞ map-germ $f : (\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$ with $\mu(f) < +\infty$, $\mu(f)$ modulo 2 is a topological invariant of f (C. T. C. Wall [27]).

The problem in the case of map-germs was investigated first by Fukuda and Ishikawa [3]. Let $f : (\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$ be a generic C^∞ map-germ, let U be a sufficiently small neighborhood of the origin and let $\bar{f} : U (\subset \mathbf{R}^2) \to \mathbf{R}^2$ be a representive mapping of f. Then we may suppose that $\bar{f}$ has no degenerate singular points except for the origin. By Whitney's theorem [32], $\bar{f}$ can be approximated by a C^∞ stable mapping $\tilde{f} : U \to \mathbf{R}^2$. The degenerate singularity of $\bar{f}$ at the origin of $\mathbf{R}^2$ bifurcates into stable singular points of $\tilde{f}$. Again by Whitney's theorem [32], the singular points of $\tilde{f}$ are $\mathcal{A}$-equivalent to one of the following two map-germs from $(\mathbf{R}^2, 0)$ to $(\mathbf{R}^2, 0)$:

$$\begin{aligned} &(1) \quad (x, y) \mapsto (x, y^2), \quad \text{fold} \\ &(2) \quad (x, y) \mapsto (x, y^3 + xy), \quad \text{cusp}. \end{aligned}$$

Suppose that $f : (\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$ is generic and U is a sufficiently small neighborhood of the origin so that $\bar{f} : U \to \mathbf{R}^2$ has only fold singular points off the origin. The cusp singular points of $\tilde{f}$ are isolated. Let f_1 and f_2 denote the component function germs of f:

$$f = (f_1, f_2) : (\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0).$$

Let $Jf = J(f_1, f_2)$ denote the Jacobian determinant of f:

$$Jf(x) = \det\left(\frac{\partial f_i}{\partial x_j}(x)\right)_{1 \le i, j \le 2}.$$

Set

$$J_1 f = J(Jf, f_2) = \det\begin{pmatrix} \frac{\partial Jf}{\partial x_1} & \frac{\partial Jf}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix},$$

$$J_2 f = J(f_1, Jf) = \det\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial Jf}{\partial x_1} & \frac{\partial Jf}{\partial x_2} \end{pmatrix}.$$

Throughout this paper we use the following notations.

$\langle a, b, \dots \rangle$; the ideal generated by $a, b, \dots$.

THEOREM 1.2 (Fukuda and Ishikawa, [3]). *Let $f : (\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$ be a C^∞ map-germ such that* $\dim_{\mathbf{R}} \mathcal{E}_2/\langle Jf, J_1 f, J_2 f\rangle < +\infty$. *Then the following holds for any stable perturbation $\tilde{f} : U(\subset \mathbf{R}^2) \to \mathbf{R}^2$ of f.*

(1) *The number of cusps of $\tilde{f}$ that appear near the origin is less than or equal to* $\dim_{\mathbf{R}} \mathcal{E}_2/\langle Jf, J_1 f, J_2 f\rangle$.

(2) *The number of cusps of $\tilde{f}$ that appear near the origin is equal to* $\dim_{\mathbf{R}} \mathcal{E}_2/\langle Jf, J_1 f, J_2 f\rangle$ modulo 2.

(3) *The number* modulo 2 *of cusps of $\tilde{f}$ is a topological invariant of f.*

In the complex case, Gaffney and Mond [8] showed that Theorem 1.2 holds more precisely. Let $f : (\mathbf{C}^2, 0) \to (\mathbf{C}^2, 0)$ be a holomorphic map-germ, let U be a sufficiently small neighborhood of the origin in $\mathbf{C}^2$ and let $\bar{f} : U(\subset \mathbf{C}^2) \to \mathbf{C}^2$ be a representive mapping of f. Then Whitney's theorem [32] also hold in the complex case, and $\bar{f}$ can be approximated by a stable holomorphic mapping $\tilde{f} : U \to \mathbf{C}^2$ that has only fold and cusp type singular points.

THEOREM 1.3 (Gaffney and Mond, [8]). *Let $f : (\mathbf{C}^2, 0) \to (\mathbf{C}^2, 0)$ be an analytic map-germ such that* $\dim_{\mathbf{C}} \mathcal{O}_2/\langle Jf, J_1 f, J_2 f\rangle < +\infty$. *Then the following holds for any stable perturbation $\tilde{f} : U(\subset \mathbf{C}^2) \to \mathbf{C}^2$ of f.*

(1) *The number of cusps of $\tilde{f}$ that appear near the origin is equal to* $\dim_{\mathbf{C}} \mathcal{O}_2/\langle Jf, J_1 f, J_2 f\rangle$.

(2) *The number of cusps of $\tilde{f}$ that appear near the origin is a topological invariant of f.*

REMARK 1.4. The conditions that $\dim_{\mathbf{R}} \mathcal{E}_2/\langle Jf, J_1 f, J_2 f\rangle < +\infty$ and that $\dim_{\mathbf{C}} \mathcal{O}_2/\langle Jf, J_1 f, J_2 f\rangle < +\infty$ are generic conditions in a strong sense. That is, the set of map-germs which do not fulfill this condition is of ∞-codimension in the set of all map germs.

Apart from the problem of topological invariance, the study on the number of 0-dimensional singular points in generic perturbations of a degenerate map-germ is recently widely developed.

For a k-tuple of integers $I = (i_1, i_2, \dots, i_k)$ with $i_1 \geq i_2 \geq \dots \geq i_k \geq 0$, there is a submanifold Σ^I of $J^l(\mathbf{C}^n, \mathbf{C}^p)(l \geq k)$ called Thom-Boadman singularity set with symbol I. We will not give the definition of Σ^I, see [1] and [20] for the definition. If codim Σ^I in $J^l(\mathbf{C}^n, \mathbf{C}^p) = n$, then for a generic mapping $\tilde{f} : \mathbf{C}^n \to \mathbf{C}^p$ singular points of f with type Σ^I appear isolatedly.

D. Mond [19] investigated the number of Σ^1 type singular points, that is, Whitney's umbrellas, for a holomorphic map-germ $(\mathbf{C}^2, 0) \to (\mathbf{C}^3, 0)$.

A generalization of Theorems 1.2, 1.3 and [19] on the number of codim n Thom-Boardman singular points was first done by J. Nuño Ballesteros and M. Saia [21], then was followed by T. Fukui, J. Nuño Ballesteros and M. Saia [4], J. Nuño Ballesteros and M. Saia [22], T. Fukui, J. Nuño Ballesteros and M. Saia [5], T. Fukui and J. Weyman [6].

For a holomorphic map-germ $f : (\mathbf{C}^n, 0) \to (\mathbf{C}^p, 0)$, we suppose that $j^l f(0) \in \overline{\Sigma^I}$ in $J^l(\mathbf{C}^n, \mathbf{C}^p)$. Let $\mathcal{I}(\Sigma^I)$ denote the defining ideal of the set-germ $\overline{\Sigma^I}$ in $(J^l(\mathbf{C}^n, \mathbf{C}^p), j^l f(0))$:

$$\mathcal{I}(\Sigma^I) = \{\alpha \in \mathcal{O}_{j^l(\mathbf{C}^n,\mathbf{C}^p), j^l f(0)} |\, \alpha|_{\overline{\Sigma^I}} = 0\} \subset \mathcal{O}_{J^l(\mathbf{C}^n,\mathbf{C}^p), j^l f(0)}$$

and we define an ideal $\mathcal{I}(\Sigma^I(f))$ in $\mathcal{O}_n$ by

$$\mathcal{I}(\Sigma^I(f)) = (j^l f)^*(\mathcal{I}(\Sigma^I)) \, .$$

For example, the Thom-Boardman singularity of cusp singularity $(\mathbf{C}^2, 0) \to (\mathbf{C}^2, 0)$ is $\Sigma^{1,1,0}$ and we have

$$\overline{\Sigma^{1,1,0}} = \overline{\Sigma^{1,1}} \, .$$

And for a holomorphic map-germ $f : (\mathbf{C}^2, 0) \to (\mathbf{C}^2, 0)$, the ideal $\mathcal{I}(\Sigma^{1,1}(f))$ is the ideal $\langle Jf, J_1 f, J_2 f\rangle$ appeared in Theorems 1.2 and 1.3.

The Thom-Boardman singularity of Whitney's umbrella $(\mathbf{C}^n, 0) \to (\mathbf{C}^{2n-1}, 0)$ is $\Sigma^{1,0}$ and we have

$$\overline{\Sigma^{1,0}} = \overline{\Sigma^1} \, .$$

And for a holomorphic map-germ $f : (\mathbf{C}^n, 0) \to (\mathbf{C}^p, 0)$ with $n \leq p$ and for

$$\Sigma^{i_1} = \{j^l g(q) \in J^l(\mathbf{C}^n, \mathbf{C}^p) |\, \operatorname{corank} Jg(q) = i_1\} \, ,$$

$\mathcal{I}(\Sigma^{i_1}(f))$ is the ideal generated by $(n - i_1 + 1) \times (n - i_1 + 1)$ minor determinants of the jacobian matrix of f and for a map-germ $f : (\mathbf{C}^n, 0) \to (\mathbf{C}^{2n-1}, 0)$, $\mathcal{I}(\Sigma^1(f))$ is the ideal generated by $n \times n$ minor determinants of the jacobian matrix of f, which appeared in our main theorem.

THEOREM 1.5 (T. Fukui, J. Nuño Ballesteros and M. Saia, [5], [22]). *Let $f : (\mathbf{C}^n, 0) \to (\mathbf{C}^p, 0)$ be a holomorphic map-germ such that $\dim_{\mathbf{C}} \mathcal{O}_n/\mathcal{I}(\Sigma^I(f)) < +\infty$. Then the following properties hold for any generic perturbation $\tilde{f} : U(\subset \mathbf{C}^n) \to \mathbf{C}^p$ of f.*

(1) *The number of singular points of type Σ^I of $\tilde{f}$ is equal to or less than*

$\dim_{\mathbf{C}} \mathcal{O}_n/\mathcal{I}(\Sigma^I(f))$.

(2) *The number of singular points of type* Σ^I *of* $\tilde{f}$ *is equal to* $\dim_{\mathbf{C}} \mathcal{O}_n/\mathcal{I}(\Sigma^I(f))$ *if and only if the Zariski closure of* Σ^I *is Cohen-Macaulay at a point* $j^k f(0) \in \Sigma^I$.

(3) *When the length is equal to* 1, *the Zariski closure of* Σ^I *is always Cohen-Macaulay at* $j^1 f(0)$.

REMARK 1.6. T. Fukui and J. Weyman [6, 7] investigate when the Zariski closure of Σ^I is Cohen-Macaulay and proved that the defining ideals of the Zariski closure of some $\Sigma^{i,j}$, for example $\Sigma^{2,1}((n, p) = (3, 2))$, $\Sigma^{3,1}((n, p) = (4, 2))$, are Cohen-Macaulay.

In the real case, for a C^∞ map-germ $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$, the defining ideal $\mathcal{I}(\Sigma^I(f))$ can be defined in the same way as in the complex case. From Theorem 1.5, we have

THEOREM 1.7. *Let* Σ^I *have codimension* n. *Let* $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ *be a* C^∞ *map-germ such that* $\dim_{\mathbf{R}} \mathcal{E}_n/\mathcal{I}(\Sigma^I(f)) < +\infty$. *Let* U *be an open neighborhood of the origin* 0 *in* $\mathbf{R}^n$ *and let* $\tilde{f} : U(\subset \mathbf{R}^n) \to \mathbf{R}^p$ *be a generic perturbation of* f. *Then the number of singular points of type* Σ^I *that appear in* $\tilde{f}$ *is equal to* $\dim_{\mathbf{R}} \mathcal{E}_n/\mathcal{I}(\Sigma^I(f))$ modulo 2.

As seen in the above, the numbers of singular points that appear in generic perturbations of map-germs are well investigated. However, strangely enough, the topological invariance of these numbers is not considered after [3], [8]. Thus, the following natural problem arises.

PROBLEM. *Let* Σ^I *be a Thom-Boardman singularity with codimension* n.

(1) *Is the number of singular points of type* Σ^I *that appear in a generic perturbation* $\tilde{f} : U(\subset \mathbf{C}^n) \to \mathbf{C}^p$ *of a holomorphic map-germ* $f : (\mathbf{C}^n, 0) \to (\mathbf{C}^p, 0)$ *a topological invariant of* f *?*

(2) *Is the number* (modulo 2) *of singular points of type* Σ^I *that appear in a generic perturbation* $\tilde{f} : U(\subset \mathbf{R}^n) \to \mathbf{R}^p$ *of a* C^∞ *map-germ* $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ *a topological invariant of* f *?*

In this paper, we answer this problem for Whitney's umbrellas in the real case.

2. A generic property of map-germs

We recall Fukuda's theorem [2] on generic properties of C^∞ map-germs. Let $C^\infty(\mathbf{R}^n, \mathbf{R}^p; 0, 0)$ denote the set of all C^∞ map-germ from $(\mathbf{R}^n, 0)$ to $(\mathbf{R}^p, 0)$. Let $\pi_r : C^\infty(\mathbf{R}^n, \mathbf{R}^p; 0, 0) \to J^r(\mathbf{R}^n, \mathbf{R}^p)$ be the canonical projection defined by $\pi_r(f) = j^r f(0)$. A subset Σ of $C^\infty(\mathbf{R}^n, \mathbf{R}^p; 0, 0)$ is said to be ∞-codimensional in $C^\infty(\mathbf{R}^n, \mathbf{R}^p; 0, 0)$, if for any positive integer k, there exist a positive integer r and a semi-algebraic subset Σ_k in $J^r(\mathbf{R}^n, \mathbf{R}^p)$ with codimension $\geq k$ such that $\Sigma \subset \pi_r^{-1}(\Sigma_k)$.

Since $\dim_{\mathbf{R}} \mathcal{E}_n/\mathcal{I}(\Sigma^1(f)) < +\infty$ if and only if $\mathcal{I}(\Sigma^1(f)) \supset \langle x_1, \ldots, x_m \rangle^k$ for some k, we have

LEMMA 2.1. *The set*

$$\Sigma^* = \{f \in C^\infty(\mathbf{R}^n, \mathbf{R}^p; 0, 0) \mid \dim_{\mathbf{R}} \mathcal{E}_n/\mathcal{I}(\Sigma^1(f)) = +\infty\}$$

is an ∞*-codimensional subset of* $C^\infty(\mathbf{R}^n, \mathbf{R}^p; 0, 0)$.

THEOREM 2.2 (Fukuda [2], Theorem 1). *Let* X *be a semi-algebraic submanifold of the multi-jet space* ${}_mJ^k(\mathbf{R}^n, \mathbf{R}^p)$. *Then there exists an* ∞*-codimensional subset* Σ_∞ *of* $C^\infty(\mathbf{R}^n, \mathbf{R}^p; 0, 0)$ *such that any* $f \in C^\infty(\mathbf{R}^n, \mathbf{R}^p; 0, 0) - \Sigma_\infty$ *has a* C^∞ *representative* $\bar{f} : U \to \mathbf{R}^p$ *that satisfies the following two properties*;

(1) *for any* m*-tuple* $S = \{x_1, \dots, x_m\}$ *of distinct points of* $U - \{0\}$, *the multi jet extension* ${}_mj^k\bar{f} : U^{(m)} \to {}_mJ^k(\mathbf{R}^n, \mathbf{R}^p)$ *is transversal to* X *at* $(x_1, \dots, x_m)$,

(2) *if* $\operatorname{codim} X \geq mn$, *then*

$$ {}_mj^k\bar{f}((U - \{0\})^{(m)}) \cap X = \emptyset\,.$$

As an easy Corollary of Theorem 2.2, we have

COROLLARY 2.3. *There exists an* ∞*-codimensional subset* Σ_∞ *of* $C^\infty(\mathbf{R}^n, \mathbf{R}^{2n-1}; 0, 0)$ *such that any* $f \in C^\infty(\mathbf{R}^n, \mathbf{R}^{2n-1}; 0, 0) - \Sigma_\infty$ *has a* C^∞ *representive* $\bar{f} : U \to \mathbf{R}^{2n-1}$ *that satisfies the following properties.*

(1) $\bar{f}$ *has no singular points except for the origin,*

(2) *if* $x_1, x_2, \dots, x_m$ *are distinct points in* $U - \{0\}$ *such that* $\bar{f}(x_1) = \bar{f}(x_2) = \cdots = \bar{f}(x_m)$, *then the images of the germs of* $\bar{f}$ *at* $x_1, x_2, \dots, x_m$ *meet transversally at* $y = \bar{f}(x_1) = \bar{f}(x_2) = \cdots = \bar{f}(x_m)$,

(3) *as a consequence of* (1) *and* (2), $\bar{f} : U - \{0\} \to \mathbf{R}^{2n-1}$ *is* $\mathcal{A}$*-stable.*

DEFINITION 2.4. A map-germ $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^{2n-1}, 0)$ is said to be *generic* if $\dim_{\mathbf{R}} \mathcal{E}_n/\mathcal{I}(\Sigma^1(f)) < +\infty$ and f has a representative $\bar{f} : U \to \mathbf{R}^{2n-1}$ that satisfies conditions (1), (2) and (3) in Corollary 2.3. Such a representative $\bar{f} : U \to \mathbf{R}^{2n-1}$ is called a *proper representive* of f.

LEMMA 2.5. *Let* $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^{2n-1}, 0)$ *be a generic* C^∞ *map-germ and let* $\bar{f} : U \to \mathbf{R}^{2n-1}$ *be a proper representative of* f, U *being a sufficiently small neighborhood of the origin* $0 \in \mathbf{R}^n$. *Let* U' *be an open neighborhood of* 0 *such that*

$$0 \in U' \subset \overline{U}' \subset U\,.$$

Let $\tilde{f} : U \to \mathbf{R}^{2n-1}$ *be a stable perturbation of* $\bar{f}$ *sufficiently close to* $\bar{f}$. *Then the restricted mapping* $\bar{f}|_{U-\overline{U}'}$ *and* $\tilde{f}|_{U-\overline{U}'}$ *are* $\mathcal{A}$*-equivalent.*

PROOF. By Corollary 2.3 (3), we have that $\bar{f}|_{U-\{0\}} : U - \{0\} \to \mathbf{R}^{2n-1}$ is $\mathcal{A}$-stable. Thus the restricted mapping $\bar{f}|_{U-\overline{U}'}$ is $\mathcal{A}$-stable. Since $\tilde{f} : U \to \mathbf{R}^{2n-1}$ approximates $\bar{f} : U \to \mathbf{R}^{2n-1}$ sufficiently closely with respect to the Whitney topology of $C^\infty(U, \mathbf{R}^{2n-1}; 0, 0)$, $\tilde{f}|_{U-\overline{U}'}$ is also sufficiently close to $\bar{f}|_{U-\overline{U}'}$ with respect to the Whitney topology of $C^\infty(U -$

$\overline{U}'$, $\mathbf{R}^{2n-1}$; 0, 0). Since $\bar{f}|_{U-\overline{U}'}$ is $\mathcal{A}$-stable, $\tilde{f}|_{U-\overline{U}'}$ and $\bar{f}|_{U-\overline{U}'}$ are $\mathcal{A}$-equivalent from the definition of stability. □

REMARK 2.1. Even when $\tilde{f} : U \to \mathbf{R}^{2n-1}$ approximates $\bar{f} : U \to \mathbf{R}^{2n-1}$ sufficiently closely with respect to the Whitney topology of $C^\infty(U, \mathbf{R}^{2n-1}; 0, 0)$, it is not necessarily that $\tilde{f}|_{U-\{0\}} : U - \{0\} \to \mathbf{R}^{2n-1}$ approximates $\bar{f}|_{U-\{0\}} : U - \{0\} \to \mathbf{R}^{2n-1}$ sufficiently closely with respect to the Whitney topology of $C^\infty(U - \{0\}, \mathbf{R}^{2n-1}; 0, 0)$. Therefore even if $\bar{f}|_{U-\{0\}}$ is $\mathcal{A}$-stable, we can not claim that $\bar{f}|_{U-\{0\}}$ and $\tilde{f}|_{U-\{0\}}$ are $\mathcal{A}$-equivalent.

3. Double points of a mapping

The key of the proof of our main theorem is an observation of double points of a mapping.

DEFINITION 3.1. A *double point* of a mapping $f : X \to Y$ is a point x for which there exists a different point y from x such that $f(x) = f(y)$. We denote by $D(f)$ the set of double points of f.

EXAMPLE 3.2. The double point set of Whitney's umbrella $f : \mathbf{R}^n \to \mathbf{R}^{2n-1}$,

$$f(x_1, \dots, x_n) = (x_1, \dots, x_{n-1}, x_n^2, x_1x_n, \dots, x_{n-1}x_n),$$

is given by

$$D(f) = \{(0, \dots, 0, x_n) | \, x_n \neq 0\}.$$

The singular point set $\{(0, \dots, 0)\}$ of Whitney's umbrella f is coincident with $\overline{D(f)} - D(f)$, where $\overline{D(f)}$ is the topological closure of $D(f)$. See Figure 1.

From Corollary 2.3, we have

LEMMA 3.3. *For a proper representative $\bar{f} : U(\subset \mathbf{R}^n) \to \mathbf{R}^{2n-1}$ of a generic map-germ $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^{2n-1}, 0)$, $D(\bar{f})$ is a smooth curve and consists of a finite number of connected components.*

DEFINITION 3.4. Let $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^{2n-1}, 0)$ be a generic C^∞ map-germ and let $\bar{f} : U \to \mathbf{R}^{2n-1}$ be a proper representative of f. Then, $D(\bar{f})$ consists of an even number of

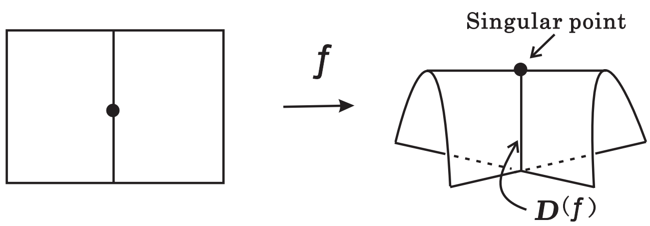

FIGURE 1. Whitney's umbrella

connected smooth curves which we call *half branches of* $\overline{D(\bar{f})}$. For every half branch γ of $\overline{D(\bar{f})}$, there exists a distinct half branch γ^* of $\overline{D(\bar{f})}$ such that for every point x of γ, there exists a point y of γ^* with $f(x) = f(y)$, which we call the *partner branch of* γ. We call the union of a half branch, its partner branch and the origin, $\gamma \cup \gamma^* \cup \{0\}$, a *branch of* $\overline{D(\bar{f})}$.

LEMMA 3.5. *Let f, g : $(\mathbf{R}^n, 0) \to (\mathbf{R}^{2n-1}, 0)$ be generic C^∞ map-germs and let $\bar{f} : U \to \mathbf{R}^{2n-1}$ and $\bar{g} : V \to \mathbf{R}^{2n-1}$ be their proper representive mappings respectively. If $\bar{f}$ and $\bar{g}$ are topological equivalent, that is, if there exist homeomorphisms $h_1 : U \to V$ and $h_2 : \mathbf{R}^{2n-1} \to \mathbf{R}^{2n-1}$ such that $h_2 \circ \bar{f} = \bar{g} \circ h_1$, then $\overline{D(\bar{f})}$ and $\overline{D(\bar{g})}$ are homeomorphic and the number of branches of $\overline{D(\bar{f})}$ and the number of branches of $\overline{D(\bar{g})}$ coincide.*

4. Proof of the main theorem

From Lemma 3.5, to prove the main theorem, it suffices to prove.

THEOREM 4.1. *Let f : $(\mathbf{R}^n, 0) \to (\mathbf{R}^{2n-1}, 0)$ be a generic C^∞ map-germ and let $\bar{f} : U \to \mathbf{R}^{2n-1}$ be a proper representive mapping of f, U being a sufficiently small neighborhood of the origin $0 \in \mathbf{R}^n$. Then the number of Whitney's umbrella of a stable perturbation $\tilde{f} : U \to \mathbf{R}^{2n-1}$ of f is equal to the number of branches of $\overline{D(\bar{f})}$* modulo 2

PROOF. The closure $\overline{D(\tilde{f})}$ of the set of double point of $\tilde{f} : U \to \mathbf{R}^{2n-1}$ is the union of $D(\tilde{f})$ and the singular point set of $\tilde{f}$:

$$\overline{D(\tilde{f})} = D(\tilde{f}) \cup \{\text{the singularities of } \tilde{f}\} = D(\tilde{f}) \cup \{\text{Whitney's umbrella of } \tilde{f}\}\,.$$

$\overline{D(\tilde{f})}$ consists of connected smooth curves any two of which have no common points. On the other hand branches of $\overline{D(\bar{f})}$ are not closed curves and they have the origin as a unique common point. See Figure 2.

Let U' be an open neighborhood of 0 such that

$$0 \in U' \subset \overline{U'} \subset U$$

and that $\tilde{f}|_{U-\overline{U'}}$ and $\bar{f}|_{U-\overline{U'}}$ are $\mathcal{A}$-equivalent. The existence of such a neighborhood U' is guaranteed by Lemma 2.5. Now consider connected components of $\overline{D(\tilde{f})}$. There may be connected components of $\overline{D(\tilde{f})}$ that are closed curves. Taking U' wide if necessary, we may suppose that all the closed curve components of $\overline{D(\tilde{f})}$ are contained in U' and that the number of connected components of $\overline{D(\tilde{f})}$ not contained in U' is equal to the number of branches of $\overline{D(\bar{f})}$. See Figure 3.

Let $\tilde{C}$ be a connected component of $\overline{D(\tilde{f})}$. There are four cases to consider.

Case (1); where $\tilde{C} \cap (U - U') = \emptyset$ and there exists another connected component $\tilde{C}'$ of $\overline{D(\tilde{f})}$ such that for every point $x \in \tilde{C}$ there exists a point $x' \in \tilde{C}'$ with $\tilde{f}(x) = \tilde{f}(x')$. See Figure 4.

In this case $\tilde{C}$ and $\tilde{C}'$ contain no singular points of $\tilde{f}$, hence they contain no Whitney's umbrellas.

Case (2); where $\tilde{C} \cap (U - U') \neq \emptyset$ and there exists another connected component $\tilde{C}'$ of $\overline{D(\tilde{f})}$ such that for every point of x in $\tilde{C}$ there exists a point $x' \in \tilde{C}'$ with $\tilde{f}(x) = \tilde{f}(x')$. See Figure 5.

In this case $\tilde{C}$ and $\tilde{C}'$ contain no singular points of $\tilde{f}$, hence they contain no Whitney's umbrellas.

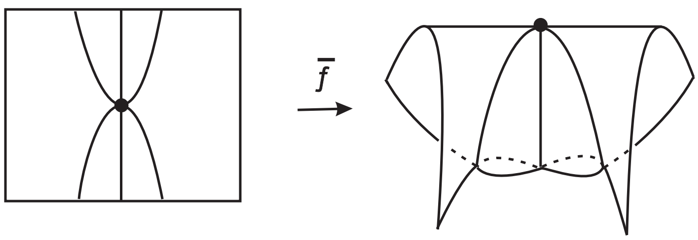

FIGURE 2.

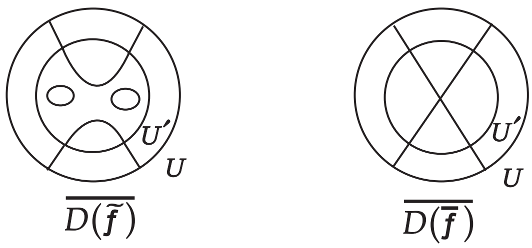

FIGURE 3.

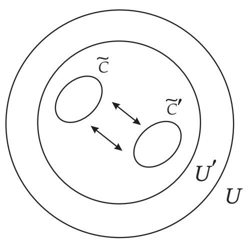

FIGURE 4.

Case (3); where $\tilde{C} \cap (U - U') \neq \emptyset$ and there exist distinct two points x and y in $\tilde{C} \cap (U - \overline{U'})$ such that $\tilde{f}(x) = \tilde{f}(y)$. See Figure 6.

In this case, there is a unique Whitney's umbrella in the middle of x and y in $\tilde{C}$, as seen as follows. Since $\tilde{f}$ meets transversally at x and y, there exist neighborhood $V(x)$ of x and $V(y)$ of y such that $\tilde{f}|_{V(x)}$ and $\tilde{f}|_{V(y)}$ meet transversally along $\tilde{C} \cap V(x)$ and $\tilde{C} \cap V(y)$. Extending such neighborhoods $V(x)$ toward y and $V(y)$ toward x respectively as widely as possible, we have a unique Whitney's umbrella in the middle of x and y in $\tilde{C}$.

Case (4); where $\tilde{C} \cap (U - U') = \emptyset$ and there exist two distinct points $x_0 \in \tilde{C}$ and $y_0 \in \tilde{C}$ such that $\tilde{f}(x_0) = \tilde{f}(y_0)$. See Figure 7.

In this case, since $\tilde{C}$ is a smooth connected curve contained in $\overline{U'}$ and hence $\tilde{C}$ is a closed connected curve, $\tilde{C} - \{x_0, y_0\}$ can be devided into two connected components $\tilde{C}_1, \tilde{C}_2$:

$$\tilde{C} - \{x_0, y_0\} = \tilde{C}_1 \cup \tilde{C}_2 \,.$$

For any point $x_1 \in \tilde{C}_1$ sufficiently close to x_0, there exists a point $y_1 \in \tilde{C}$ sufficiently close to y_0 such that $\tilde{f}(y_1) = \tilde{f}(x_1)$. The point y_1 corresponding to x_1 belongs to either $\tilde{C}_1$ or $\tilde{C}_2$.

(4-1); the case where $y_1 \in \tilde{C}_1$.

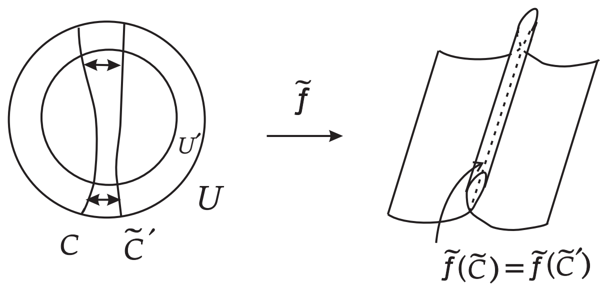

FIGURE 5.

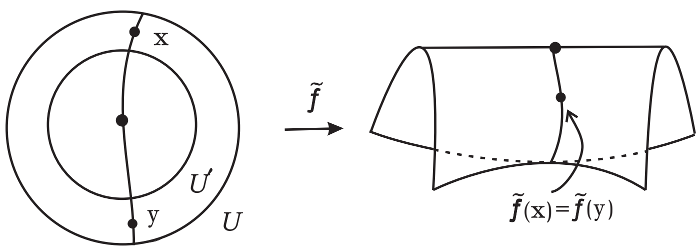

FIGURE 6.

In this case, with the same reason as in the Case (3), there exists a unique Whitney's umbrella in $\tilde{C}_1$. In this case, also for any point $x_2 \in \tilde{C}_2$ sufficiently close to x_0, there exists a point $y_2 \in \tilde{C}_2$ such that $\tilde{f}(y_2) = \tilde{f}(x_2)$. Then, again with the same reason, there exists a unique Whitney's umbrella in $\tilde{C}_2$. Thus, we have exactly two Whitney's umbrellas on $\tilde{C}$. See Figure 8.

(4-2); the case where $y_1 \in \tilde{C}_2$.

In this case, for every point $x \in \tilde{C}_1$, there exists a point $y \in \tilde{C}_2$ such that $\tilde{f}(y) = \tilde{f}(x)$ and there is no Whitney's umbrellas on $\tilde{C}$. See Figure 9.

Now, the connected components $\tilde{C}$ of $\overline{D(\tilde{f})}$ of Case (1), (2) do not contribute to the number of Whitney's umbrellas of $\tilde{f}$. Since connected components of $\overline{D(\tilde{f})}$ of Case (4) contain either two Whitney's umbrellas or none each, they also do not contribute to the number modulo 2 of Whitney's umbrellas of $\tilde{f}$. Thus we have

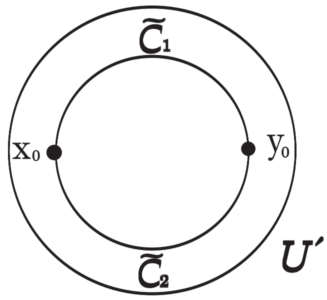

FIGURE 7.

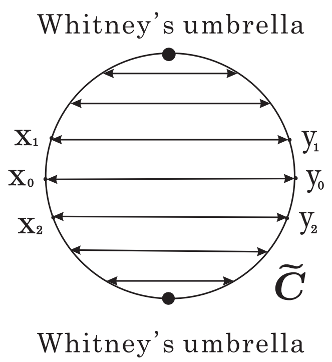

FIGURE 8.

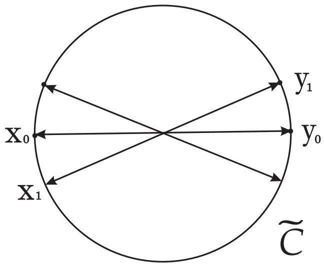

FIGURE 9.

the number of Whitney's umbrella

$\equiv$ the number of the connected components of $\overline{D(\tilde{f})}$ of Case (3)

(modulo 2) .

On the other hand connected components of $\overline{D(\tilde{f})}$ of Cases (1) and (4) are contained in $\overline{U'}$ and connected components of $\overline{D(\tilde{f})}$ of Cases (2) and (3) are not contained in $\overline{U'}$. The number of connected components of $\overline{D(\tilde{f})}$ of Case (2) is even, since for each connected component $\tilde{C}$ of Case (2), the corresponding component $\tilde{C}'$ is also of Case (2). Thus

the number of Whitney's umbrella

$\equiv$ the number of connected components of $\overline{D(\tilde{f})}$ of Case (3) (modulo 2)

$\equiv$ the number of connected components of $\overline{D(\tilde{f})}$ of Cases (2) and (3) (modulo 2)

$=$ the number of connected components of $\overline{D(\tilde{f})}$ not contained in U'

$=$ the number of branches of $\overline{D(\bar{f})}$.

This completes the proof of Theorem 4.1 and hence of the main theorem. $\square$

5. Some examples

In this section, we observe $\mathcal{A}$-simple map-germs $(\mathbf{R}^2, 0) \to (\mathbf{R}^3, 0)$ classified by D. Mond [18], and see that Theorem 4.1 holds for them.

THEOREM 5.1 (D. Mond [18]). *Each of the germs in the following list is $\mathcal{A}$-simple, and every $\mathcal{A}$-simple germ of a map from a 2-manifold to a 3-manifold is equivalent to one of the germs on the list.*

Germ	*$\mathcal{A}$-codimension*	*Name*
$f(x, y) = (x, y)$	0	Immersion
$f(x, y) = (x, y^2, xy)$	2	Cross-cap (S_0)
$f(x, y) = (x, y^2, y^3 \pm x^{k+1}y)$, $k \geq 1$	$k+2$	$S_k^\pm$
$f(x, y) = (x, y^2, x^2y \pm y^{2k+1})$, $k \geq 2$	$k+2$	$B_k^\pm$
$f(x, y) = (x, y^2, xy^3 \pm x^k y)$, $k \geq 3$	$k+2$	$C_k^\pm$
$f(x, y) = (x, y^2, x^3y + y^5)$	6	F_4
$f(x, y) = (x, xy + y^{3k-1}, y^3)$, $k \geq 2$	$k+2$	H_k

EXAMPLE 5.2. We consider the normal form $S_k^\pm : f_k^\pm : (\mathbf{R}^2, 0) \to (\mathbf{R}^3, 0)$ given by

$$f_k^\pm(x, y) = (x, y^2, y^3 \pm x^{k+1}y), \quad k \geq 1 .$$

Since $\mathcal{I}(\Sigma^1(f_k^\pm))$ is the ideal generated by 2×2 minors of the jacobian matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 2y \\ \pm(k+1)x^k y & 3y^2 \pm x^{k+1} \end{pmatrix}$$

of f, we have

$$\mathcal{I}(\Sigma^1(f_k^\pm)) = \langle y, x^{k+1} \rangle$$

and we have

$$\dim_{\mathbf{R}} \mathcal{E}_2/\mathcal{I}(\Sigma^1(f_k^\pm)) = k + 1 .$$

On the other hand,

$$D(f_k^+) = \{(x, y) |\, x^{k+1} + y^2 = 0, y \neq 0\} ,$$

$$D(f_k^-) = \{(x, y) |\, x^{k+1} - y^2 = 0, y \neq 0\} .$$

Thus we see that

$$\text{the number of branches of } \overline{D(f_k^+)} = \begin{cases} 1, & \text{if } k+1 \equiv 1 \mod 2 \\ 0, & \text{if } k+1 \equiv 0 \mod 2 \end{cases}$$

$$\text{the number of branches of } \overline{D(f_k^-)} = \begin{cases} 1, & \text{if } k+1 \equiv 1 \mod 2 \\ 2. & \text{if } k+1 \equiv 0 \mod 2 \end{cases}$$

Hence we have

$$\text{the number of branches of } \overline{D(f_k^\pm)} \equiv \dim_{\mathbf{R}} \mathcal{E}_2/\mathcal{I}(\Sigma^1(f_k{}^\pm)) \pmod 2$$

as Theorem 4.1 asserts.

For any integer l with $0 \leq l \leq k+1$ and with $l \equiv k+1 \pmod 2$, we have a stable perturbation of f

$$\tilde{f}^{\pm}_{k,l} : U(\subset \mathbf{R}^2) \to \mathbf{R}^3$$

such that the number of Whitney's umbrellas of $\tilde{f}_{\pm l}$ is exactly l, constructed as follows. Let $\varepsilon_1, \dots, \varepsilon_l$ be sufficiently small distinct real numbers. Set $m = \dfrac{k+1-l}{2}$ and let $\delta_1, \dots, \delta_m$ be small positive numbers. Then,

$$\tilde{f}^{\pm}_{k,l}(x, y) = (x, y^2, y^3 \pm y(x-\varepsilon_1)\cdots(x-\varepsilon_l)(x^2+\delta_1)\cdots(x^2+\delta_m))$$

is a stable perturbation of f. Whitney's umbrellas of $\tilde{f}^{\pm}_{k,l}$ are the points $(\varepsilon_1, 0), \dots, (\varepsilon_l, 0)$. Thus the number of Whitney's umbrellas of $\tilde{f}^{\pm}_{k,l}$ is exactly l. See Figure 10, 11, 12, 13, 14, 15.

EXAMPLE 5.3. Now we consider the normal form $B^{\pm}_k : f^{\pm}_k : (\mathbf{R}^2, 0) \to (\mathbf{R}^3, 0)$ given by

$$f^{\pm}_k(x, y) = (x, y^2, x^2 y \pm y^{2k+1}), \quad k \geq 2.$$

Since $\mathcal{I}(\Sigma^1(f^{\pm}_k)) = \langle y, x^2 \rangle$, we have

$$\dim_{\mathbf{R}} \mathcal{E}_2/\mathcal{I}(\Sigma^1(f^{\pm}_k)) = 2.$$

On the other hand,

$$D(f^{+}_k) = \{(x, y) \mid x^2 + y^{2k} = 0,\ y \neq 0\} = \emptyset,$$
$$D(f^{-}_k) = \{(x, y) \mid x^2 - y^{2k} = 0,\ y \neq 0\}.$$

Thus we see that

$$\text{the number of branches of } \overline{D(\overline{f^{+}_k})} = 0$$

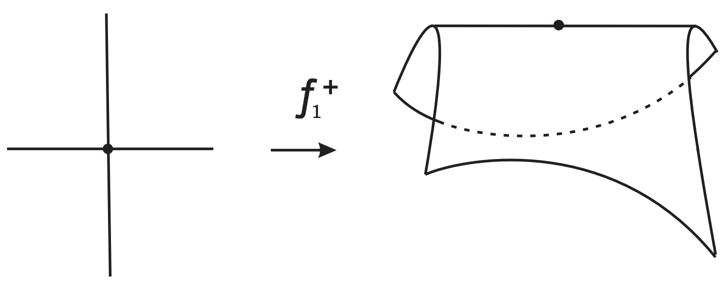

FIGURE 10. $S^{+}_1 : f^{+}_1(x, y) = (x, y^2, y^3 + x^2 y)$, $\dim_{\mathbf{R}} \mathcal{E}_2/\mathcal{I}(\Sigma^1(f^{+}_1)) = 2$.

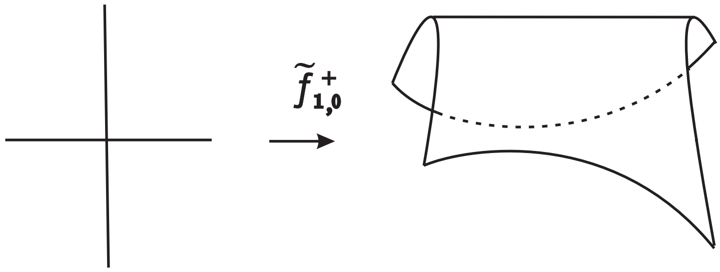

FIGURE 11. $\tilde{f}^{+}_{1,0}(x, y) = (x, y^2, y^3 + y(x^2 + \delta_1)),\ D(\tilde{f}^{+}_{1,0}) = \emptyset$.

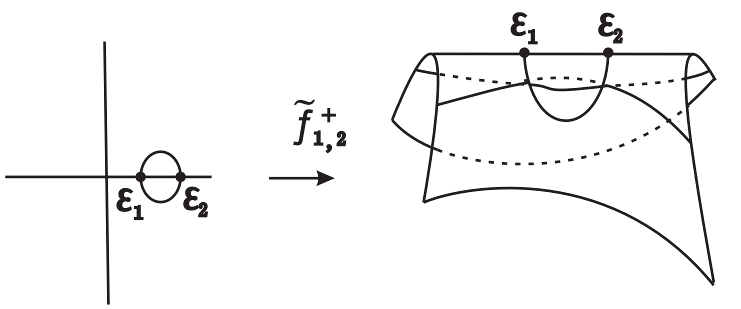

FIGURE 12. $\tilde{f}^{+}_{1,2}(x, y) = (x, y^2, y^3 + y(x - \varepsilon_1)(x - \varepsilon_2)), D(\tilde{f}^{+}_{1,2}) = \{(x, y) \in \mathbf{R}^2 | \, y^2 + (x - \varepsilon_1)(x - \varepsilon_2) = 0, y \neq 0\}$.

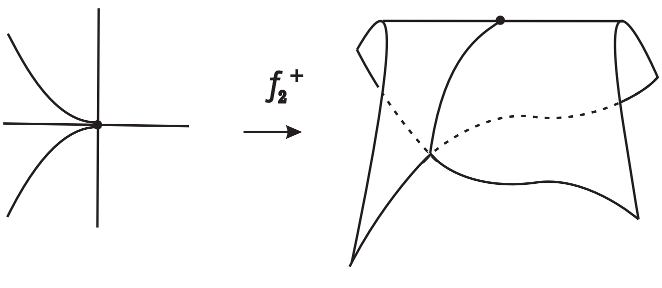

FIGURE 13. $S_2{}^{+} : f_2{}^{+}(x, y) = (x, y^2, y^3 + x^3 y),\ \dim_{\mathbf{R}} \mathcal{E}_2/\mathcal{I}(\Sigma^1(f_2{}^{+})) = 3$.

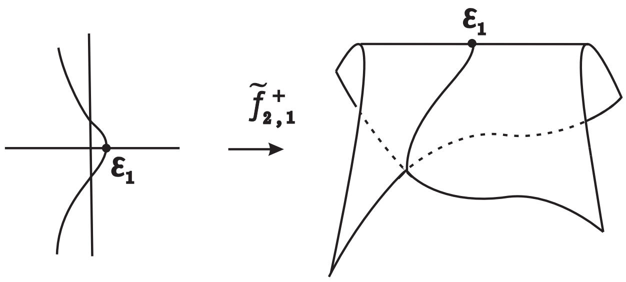

FIGURE 14. $\tilde{f}^+_{2,1}(x,y) = (x, y^2, y^3 + y(x-\varepsilon_1)(x^2+\delta_1))$, $D(\tilde{f}^+_{2,1}) = \{(x,y) \in \mathbf{R}^2 | \, y^2 + (x-\varepsilon_1)(x^2+\delta_1) = 0, \, y \neq 0\}$.

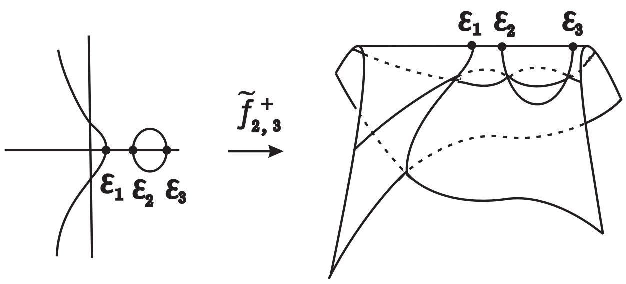

FIGURE 15. $\tilde{f}^+_{2,3}(x,y) = (x, y^2, y^3 + y(x-\varepsilon_1)(x-\varepsilon_2)(x-\varepsilon_3))$, $D(\tilde{f}^+_{2,3}) = \{(x,y) \in \mathbf{R}^2 | \, y^2 + (x-\varepsilon_1)(x-\varepsilon_2)(x-\varepsilon_3) = 0, \, y \neq 0\}$.

$$\text{the number of branches of } \overline{D(\overline{f_k^-})} = 2\,.$$

Hence we have

$$\text{the number of branches of } \overline{D(\overline{f_k^\pm})} \equiv \dim_{\mathbf{R}} \mathcal{E}_2/\mathcal{I}(\Sigma^1(f_k^\pm)) \pmod 2$$

as Theorem 4.1 asserts.

For any integer l with $0 \leq l \leq 2$ and with $l \equiv 2 \pmod 2$, (that is, l is 0 or 2), we have a stable perturbation of f

$$\tilde{f}^\pm_{k,l} : U(\subset \mathbf{R}^2) \to \mathbf{R}^3$$

such that the number of Whitney's umbrellas of $\tilde{f}^\pm_{k,l}$ is exactly l, constructed as follows. Let $\varepsilon_1, \varepsilon_2$ be sufficiently small distinct real numbers. Let δ_1 be small positive number. Then,

$$\tilde{f}^\pm_{k,0}(x,y) = (x, y^2, \pm y^{2k+1} + y(x^2+\delta_1))\,,$$
$$\tilde{f}^\pm_{k,2}(x,y) = (x, y^2, \pm y^{2k+1} + y(x-\varepsilon_1)(x-\varepsilon_2))$$

are stable perturbations of f. $\tilde{f}^{\pm}_{k,0}$ has no Whitney's umbrellas and Whitney's umbrellas of $\tilde{f}^{\pm}_{k,2}$ are the points $(\varepsilon_1, 0)$, $(\varepsilon_2, 0)$. Thus the number of Whitney's umbrellas of $\tilde{f}^{\pm}_{k,0}$ and $\tilde{f}^{\pm}_{k,2}$ is exactly 0 and 2 respectively. See Figure 16, 17, 18.

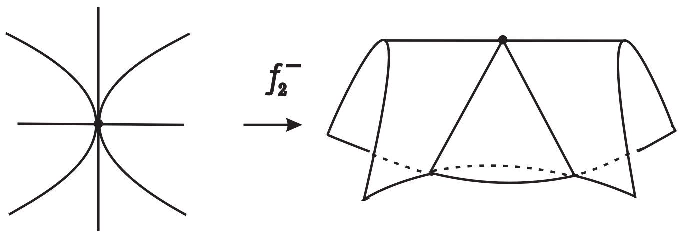

FIGURE 16. $B_2^- : f_2^-(x, y) = (x, y^2, x^2 y - y^5)$, $\dim_{\mathbf{R}} \mathcal{E}_2/\mathcal{I}(\Sigma(f_2^-)) = 2$.

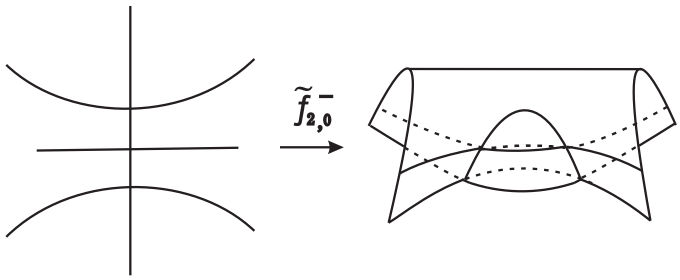

FIGURE 17. $\tilde{f}^-_{2,0}(x, y) = (x, y^2, -y^5 + y(x^2 + \delta_1))$, $D(\tilde{f}^-_{2,0}) = \{(x, y) \in \mathbf{R}^2 | \, y^4 - (x^2 + \delta_1) = 0\}$.

In this way, we have the following table and we see that for Mond's normal forms $\dim_{\mathbf{R}} \mathcal{E}_2/\mathcal{I}(\Sigma^1(f_k^{\pm})) \equiv$ the number of branches of $\overline{D(\overline{f_k^{\pm}})}$ (mod 2) as Theorem 4.1 asserts.

	$\dim_{\mathbf{R}} \mathcal{E}_2/\mathcal{I}(\Sigma^1(f))$	number of branches of $\overline{D(\bar{f})}$
S_k^+	$k+1$	$\begin{cases} 1, & \text{if } k+1 \equiv 1 \mod 2 \\ 0, & \text{if } k+1 \equiv 0 \mod 2 \end{cases}$
S_k^-	$k+1$	$\begin{cases} 1, & \text{if } k+1 \equiv 1 \mod 2 \\ 2, & \text{if } k+1 \equiv 0 \mod 2 \end{cases}$
B_k^+	2	0,
B_k^-	2	2,
C_k^+	k	$\begin{cases} 1, & \text{if } k \equiv 1 \mod 2 \\ 2, & \text{if } k \equiv 0 \mod 2 \end{cases}$
C_k^-	k	$\begin{cases} 3, & \text{if } k \equiv 1 \mod 2 \\ 2, & \text{if } k \equiv 0 \mod 2 \end{cases}$
F_4	3	1,
H_k	2	0.

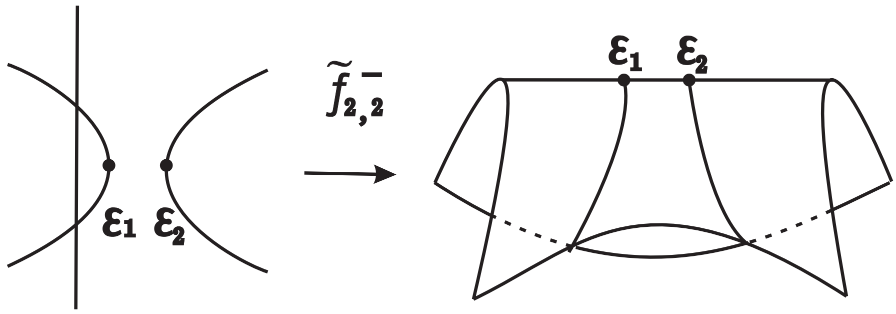

FIGURE 18. $\tilde{f}^{-}_{2,2}(x, y) = (x, y^2, -y^5 + y(x - \varepsilon_1)(x - \varepsilon_2))$, $D(\tilde{f}^{-}_{2,2}) = \{(x, y) \in \mathbf{R}^2 | \, y^4 - (x - \varepsilon_1)(x - \varepsilon_2) = 0, y \neq 0\}$.

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Present Address:
DEPARTMENT OF MATHEMATICS, COLLEGE OF HUMANITIES AND SCIENCES
NIHON UNIVERSITY,
SAKURAJOSUI SETAGAYA-KU, TOKYO, 156–8550 JAPAN.
e-mail: m_ohsumi@math.chs.nihon-u.ac.jp