

## Invariants of varieties and singularities inspired by Kähler-Einstein problems

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**Abstract:** We extend the framework of K-stability [Tia], [Don1] to a more general algebro-geometric setting, such as partial desingularisations of fixed singularities, (not necessarily flat) families over higher dimensional base and birational geometry of surfaces.

We also observe that “concavity” of the volume function implies decrease of the (generalised) Donaldson–Futaki invariants along the Minimal Model Program, in our generalised settings. Several related results on the connection with the MMP theory, some of which are new even in the original setting of families over curves, are also proved.

**Key words:** Intersection number; Minimal Model Program; K-stability.

**1. Introduction.** Working on canonical (Kähler) metrics via the use of *numerical invariants* has its origin in the seminal and pioneering paper of Futaki [Fut]. Shortly after its introduction, Mabuchi [Mab] introduced important functionals over the space of Kähler potentials which are connected to the Futaki invariant. Those two fundamental and pioneering works, in turn, after a decade later, reveal their true faces as shadows of a version of GIT stability — K-stability introduced by Tian and Donaldson [Tia], [Don1]. It is also compatible with Yau’s original insightful expectation [Yau] that some “stability” should be equivalent to the existence of canonical metrics.

This paper aims at clarifying and generalising those invariants in a more general algebro-geometric setting than families over a curve, such as (partial) desingularisations of singularities, families over higherdimensional base or classical (absolute) birational geometry of surfaces. We also add results which are new even in the original setting of families over curves. The author also hopes that this would serve as a supplementary introduction for algebraic geometers to the subject.

**2. Generalised setting.** Throughout the paper, we work over an algebraically closed field of characteristic 0. However, the arguments which do not depend on the resolution of singularities or

the Minimal Model Program (MMP), work also over positive characteristics fields.

We fix a normal  $\mathbf{Q}$ -Gorenstein projective variety  $B$  as a base. We also fix a base point  $p \in B$  and a projective morphism  $\pi^o: (\mathcal{X}^o, \mathcal{L}^o) \rightarrow B \setminus \{p\}$  where  $\mathcal{L}^o$  is  $\pi^o$ -ample and  $\mathcal{X}^o$  is a normal  $\mathbf{Q}$ -Gorenstein projective variety of dimension  $n$ . Much of our theory extends naturally also to non-normal varieties which are reduced, equidimensional algebraic schemes,  $\mathbf{Q}$ -Gorenstein, Gorenstein in codimension 1, and satisfying Serre’s condition  $S_2$ .

We consider all the completions of  $(\mathcal{X}^o, \mathcal{L}^o)$  to  $(\mathcal{X}, \mathcal{L})$  over  $B$  i.e. projective morphisms  $\pi: (\mathcal{X}, \mathcal{L}) \rightarrow B$  with  $\pi$ -nef  $\mathcal{L}$  such that  $\pi^{-1}(B \setminus \{p\}) = (\mathcal{X}^o, \mathcal{L}^o)$ . In other words, we consider birational modifications along  $\pi^{-1}(p)$ .

Original setting after Mumford, Futaki, Tian, Donaldson is that  $(\mathcal{X}^o, \mathcal{L}^o) = (X, L) \times (B \setminus \{p\})$  with  $B = \mathbf{P}^1$  (or  $\mathbf{A}^1$  originally),  $(X, L)$  is a polarized variety, and  $(\mathcal{X}, \mathcal{L})$  is a “test configuration” [Don1]. Hence, our main point of the extension is that we allow all kinds of projective morphism, for example,  $\pi$  can be non-flat or even birational.

To our general  $\pi: (\mathcal{X}, \mathcal{L}) \rightarrow B$ , we assign the two invariants  $DF$  and  $\mathcal{V}$  whose Kähler analogues appeared in [Fut, Tia, Don1] and [Aub, Mab].

**2.1. Normalised volume functional.** The first invariant we introduce is the following  $\mathcal{V}$ . It will become clear only later why this simple but somehow modified version of volume function is important for us.

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**Definition 2.1.**  $\mathcal{V}(\mathcal{X}, \mathcal{L}) := \frac{(\mathcal{L}^n)}{(\mathcal{L}^o|_F)^{\dim(F)} \frac{n-1}{\dim(F)}}$ ,

where  $F$  is the generic fiber of  $\pi^0$ , which people often denote by  $\mathcal{X}_\eta$ . If  $\dim(F) = 0$ , i.e.  $\pi^0$  is generically finite, then we interpret the denominator of the above as 1. The meaning of the somewhat complicated denominator as a normalizing factor, which indeed only depends on the original  $\mathcal{L}^o$ , will be clear in the proof of Theorem 4.1.

We call the above the *normalized volume functional*. For the case with  $\dim(B) = 1$ , this may be able to be seen as a functional of non-Archimedean smooth semipositive metrics from the perspective of [BFJ1], which is concave (cf. next proposition 2.2 (iii)). Prof. S. Boucksom kindly pointed out to the author that the  $\dim(B) = 1$  case of the above essentially coincides with the non-Archimedean analogue of the *Aubin–Mabuchi functional* (also called “Monge-Ampère energy”) they discussed in [BFJ2]. Please note it is different from the so-called K-energy of [Mab].

**Proposition 2.2** (Basic properties of  $\mathcal{V}$ ). *Regarding the above normalised volume functional  $\mathcal{V}$ , we have the following basic properties:*

(i) *If  $f: \mathcal{X}' \rightarrow \mathcal{X}$  is a birational morphism, then we have*

$$\mathcal{V}(\mathcal{X}', f^* \mathcal{L}) = \mathcal{V}(\mathcal{X}, \mathcal{L}).$$

(ii) *If  $\dim(F) > 0$ , then the functional is homogeneous of degree 1 i.e., for any  $a \in \mathbf{Z}_{>0}$ ,*

$$\mathcal{V}(\mathcal{X}, \mathcal{L}^{\otimes a}) = a \cdot \mathcal{V}(\mathcal{X}, \mathcal{L}).$$

(iii) *For any Cartier divisor  $E$  supported on a fiber,*

$$\frac{\partial^2}{\partial E^2} \mathcal{V}(\mathcal{X}, \mathcal{L}) \leq 0,$$

*that is, this functional  $\mathcal{V}$  is concave along the space of divisors supported on  $\pi^{-1}(p)$ .*

(iii) could be regarded as algebraic version of the convexity of (differential geometric) Aubin–Mabuchi functional.

*Proof.* (i), (ii) are straightforward to see.

We prove that (iii) essentially follows from the Hodge index theorem. Also, here is the place we use the assumption that birational modifications are all along fibres over finite points of  $B$ . Let us take an arbitrary ample divisor  $H$  on  $B$ .

Consider variation of  $\mathcal{L}$  to  $\mathcal{L}(tD)$  where  $D$  is a Cartier divisor supported on  $\pi^{-1}(p)$ . What we need to prove is

$$\frac{\partial^2}{\partial D^2} (\mathcal{L}^n) = n(n-1)(\mathcal{L}^{n-2}.D^2) \leq 0.$$

The above follows from the Hodge index theorem since

$$(\mathcal{L}^{n-2}.D.\pi^*H) = 0,$$

because  $\pi(\text{Supp}(D))$  is zero-dimensional. An important note is that our (iii) above is an analogue of the concavity of the Aubin–Mabuchi functional (cf., e.g. [BBGZ]).  $\square$

We note that via (ii) of the above proposition, for  $\dim(B) = 1$  case, we can regard  $\mathcal{V}$  as a functional over a space of all  $\mathbf{R}$ -line bundles up to pull back (“infinite dimensional nef cone”). In other words, the space consists of smooth non-Archimedean semipositive metrics on the analytification  $\mathcal{L}^{an}$  of  $\mathcal{X}^{an}$  from the viewpoint of [BFJ1] (cf. also [KT]). I would like to thank Prof. S. Boucksom for teaching me about the non-Archimedean metrics they use.

**2.2. Generalising Futaki’s invariants further.** Now we introduce our second invariant — an extension of the Futaki invariants, generalising the Donaldson’s extension of the Futaki invariant [Don1] further.

The point is, by the intersection number formula [Wan] and [Od2], that the Donaldson–Futaki invariant [Don1] is the “derivative” along the direction of the (relative) canonical divisor, which is exactly encoding the infinitesimal behaviour along the Minimal Model Program with scaling [BCHM] (or its analytic counterpart, i.e. unnormalised Kähler-Ricci flow cf., e.g., [CL], [ST]).

**Proposition 2.3** ([Wan], [Od2]). *If  $(\mathcal{X}, \mathcal{L})$  is a test configuration of a polarised projective variety  $(X, L)$ , we have*

$$\frac{\partial}{\partial K_{\mathcal{X}/B}} \mathcal{V}(\mathcal{L}) = \frac{DF(\mathcal{X}, \mathcal{L})}{(L^{n-1})^2},$$

*where the  $DF(\mathcal{X}, \mathcal{L})$  is the Donaldson–Futaki invariant [Don1]. In general, if  $B$  is a curve, then the above equation holds once we replace the Donaldson–Futaki number by  $\text{deg}(\lambda_{CM}(\mathcal{X}, \mathcal{L}))$ , where the  $\lambda_{CM}$  is the CM line bundle introduced by Paul-Tian [PT] (also see [FR]).*

Given the above, we define our further generalisation of the Donaldson–Futaki invariants similarly as follows:

**Definition 2.4.** For our extended framework of the previous section, we define our (generalised) Donaldson–Futaki invariant as

$$DF(\mathcal{X}, \mathcal{L}) := \left( \frac{\partial}{\partial K_{\mathcal{X}/B}} \mathcal{V}(\mathcal{L}) \right) \cdot (\mathcal{L}^o|_F)^{\frac{2(n-1)}{dim(F)}}.$$

More explicitly, we can write  $DF(\mathcal{X}, \mathcal{L})$  as follows:

$$n(\mathcal{L}^{n-1} \cdot K_{\mathcal{X}/B})(\mathcal{L}|_F)^{\frac{n-1}{dim(F)}} - (n-1)(\mathcal{L}^n)(\mathcal{L}|_F)^{dim(F)-1} \cdot K_F(\mathcal{L}|_F)^{\frac{dim(B)-1}{dim(F)}}.$$

Note that the last term  $(\mathcal{L}|_F)^{\frac{dim(F)}{dim(F)}}$  did not appear in the original setting of  $dim(B) = 1$  since the exponent was 0 in that case.

The following basic property says that Donaldson–Futaki invariant is a functional of the space of polarisations (up to pull backs).

**Proposition 2.5.** *Consider the pull back of  $\mathcal{L}$  on  $\mathcal{X}$  by a birational morphism  $f: \mathcal{X}' \rightarrow \mathcal{X}$ . Then we have*

$$DF(\mathcal{X}', f^* \mathcal{L}) = DF(\mathcal{X}, \mathcal{L}).$$

The proof follows straightforwardly from the above description of our (generalised) Donaldson–Futaki invariants via intersection numbers. After this proposition, we often write the above generalisation simply as  $DF(\mathcal{L})$  and call it the DF invariant or DF (of the polarisations) from now on in this paper.

Note that the above (2.5) would be an inequality  $\leq$  in general if  $\mathcal{X}$  is non-normal, whose difference in turn reflects the presence of conductor of normalisation. This extends the old result of Ross–Thomas [RT, (5.1), (5.2)] and again also matches to the non-Archimedean framework of [BFJ1].

We can extend our Futaki invariant further to log setting by using Shokurov’s “b-divisors” [Sho], which are roughly speaking infinite linear combination of divisors above  $\mathcal{X}$ . Indeed,  $\{K_{\mathcal{X}/B}\}_{\mathcal{X}}$  also forms a Weil b-divisor and it is enough to replace it by  $\{K_{\mathcal{X}/B}\}_{\mathcal{X}} + \mathcal{D}$  where  $\mathcal{D}$  is some other Weil b-divisor. Accordingly, all the contents of our theory extend in a straightforward manner but we omit them. The extension includes the usual log K-stability [Don2], [OS].

**3. Extending K-semistability.** We extend the idea of K-stability [Don1] to our generalised framework. More precisely, we extend K-semistability and study its properties as follows:

**Definition 3.1.** We follow the notation of the previous section. We say that  $\pi: (\mathcal{X}, \mathcal{L}) \rightarrow B$  is *generically K-semistable* if the set of all the Donaldson–Futaki invariants of all possible birational transforms along  $\pi$ -preimages of finite points are bounded below.

Note that if  $\pi$  is a test configuration, the above implies the K-semistability of general fibers. However, in general, our definition is not the same as K-semistability of the generic fiber as we will see soon in Proposition 3.2. The above is a little analogous to the fact that original K-semistability for [Tia] and [Don1] corresponds to lower boundedness of Mabuchi functional. It also indicates that the (Donaldson–)Futaki invariant itself shares some features of the Mabuchi functional (K-energy) [Mab], as the author learnt from S. Boucksom.

**Proposition 3.2.** *If  $\mathcal{X}^o \rightarrow B \setminus \{p\}$  is an isomorphism, so that any completion of  $\pi^o$  is birational, then it is generically K-semistable if and only if  $B$  has only canonical singularities. Moreover, all the non-trivial Donaldson–Futaki invariants are positive if and only if  $B$  has only terminal singularities.*

*Proof of Proposition 3.2.* Suppose  $B$  has non-canonical singularity at  $p \in B$ . Then we take a relative canonical model  $\mathcal{X}^{can}$  over  $B$  by [BCHM]. By the negativity lemma [KM, 3.39], it follows that  $K_{\mathcal{X}^{can}/B}$  is anti-effective (and non-zero). We take  $\mathcal{L}^{can} := K_{\mathcal{X}^{can}/B}$  which is  $\pi$ -ample from our construction. Thus, the Donaldson–Futaki invariant  $DF(\mathcal{X}^{can}, \mathcal{L}^{can})$ , which is  $((\mathcal{L}^{can})^{n-1} \cdot K_{\mathcal{X}^{can}/B})$  up to a positive constant multiple, is negative. Thus we end the proof of the first half of the assertions.

If  $B$  is strictly canonical at  $p$ , i.e. canonical but not terminal, take a terminalisation  $\mathcal{X}^{min}$  of  $B$  again using [BCHM]. Then since we know  $K_{\mathcal{X}^{min}/B} = 0$ , for an arbitrary  $\pi$ -nef line bundle  $\mathcal{L}$  on  $\mathcal{X}^{min}$ , the corresponding Donaldson–Futaki invariants vanish.  $\square$

For the general fibration case, we also see that

**Proposition 3.3.** *If  $K_F = a\mathcal{L}|_F$  with  $a \geq 0$ , and  $\mathcal{X} \setminus \pi^{-1}(p)$  has only canonical singularities with  $dim(B) = 1$ , then  $(\mathcal{X}, \mathcal{L})$  is generically K-semistable.*

*Proof.* We give a case by case proof depending on the Fujita–Kawamata type semipositivity (cf., e.g., [Kol], [Fuj]).

If  $a = 0$ , this is a Calabi–Yau fibration. From the semipositivity theorem, we can take  $K_{\mathcal{X}/B}$  as an effective vertical divisor. Therefore, the corresponding Donaldson–Futaki invariant  $(\mathcal{L}^{n-1} \cdot K_{\mathcal{X}/B})$ , multiplied by some positive constant, is non-negative.

If  $a > 0$ , here we only prove the case when  $K_{\mathcal{X}/B}$  is relatively ample and  $\mathcal{X}$  is canonical, i.e. the relative canonical model over  $B$  and leave the rest to Proposition 4.2. In that case, the corresponding Donaldson–Futaki invariant is simply  $(K_{\mathcal{X}/B}^n)$  up to a positive constant multiple. Then it is the leading coefficient of  $\text{deg}_B(\det(\pi_*\mathcal{O}_{\mathcal{X}}(mK_{\mathcal{X}/B})))$ , thanks to the Grothendieck–Riemann–Roch theorem. The semipositivity theorem [Kol], [Fuj] implies that the above quantity is all non-negative for sufficiently divisible  $m \in \mathbf{Z}_{>0}$ , so the assertion holds for the relative canonical model case. For general case, from Proposition 4.2 shows that birational modifications of this relative canonical model have bigger or equal Donaldson–Futaki invariants so that the assertion holds.  $\square$

Motivated by the above, the next section 4, and the relation of CM line bundles with the Weil-Petersson metrics, we naturally conjecture as follows:

**Conjecture 3.4.** If  $\dim(B) = 1$ , the generic K-semistability of  $(\mathcal{X}, \mathcal{L})$  is equivalent to that generic fiber  $(F, \mathcal{L}|_F)$  of  $\pi$  is K-semistable in the original sense [Don1].

A similar question was asked before by X. Wang during my visit to Hong Kong in spring of 2010.

**4. To minimise the DF by the MMP.**

**4.1. Decrease of the DF along MMP.**

Recall that [Od1] observes, very vaguely speaking, that birationally “small” models in the MMP theory have “small” Donaldson–Futaki invariants. A while after [Od1], the *continuous decrease* of the Donaldson–Futaki invariants along the MMP was first proved in [LX] for families of Fano varieties over curves. As a Kähler analogue, this should corresponds to decrease of K-energy along the normalised Kähler-Ricci flow.

We generalise the phenomenon to our much extended framework as follows:

**Theorem 4.1.** Suppose  $K_F = a\mathcal{L}|_F$  and let us consider the  $K_{\mathcal{X}/B}$ -MMP with “scaling  $\mathcal{L}$ ” (precisely speaking, the scaling divisor is  $\frac{1}{l}\mathcal{D}$  where  $\mathcal{D}$  corresponds to a general section of  $(\mathcal{L} \otimes \pi^*\mathcal{M})^{\otimes l}$  with sufficiently ample  $\mathcal{M}$  on  $B$  and  $l \gg 0$ ).

Along that MMP with scaling, which we see as the linear and continuous change of polarisations  $\mathcal{L}_t$  to the relative canonical divisor, the Donaldson–Futaki invariants  $DF(\mathcal{L}_t)_{t \geq 0}$  monotonously decrease when  $t$  increases.

*Proof.* We have  $\mathcal{L}_t = tE + \mathcal{L}$  where  $E := K_{\mathcal{X}/B} - a\mathcal{L}$ . Thus what we need to show is in the direction of  $E$ , the Donaldson–Futaki invariant decreases i.e.,  $\frac{\partial}{\partial E}(DF(\mathcal{L})) < 0$ . It follows from the properties of Mabuchi functional  $\mathcal{V}$  (2.2). Indeed, after some simple calculation using (2.2(ii)), this derivative can be rewritten via  $\mathcal{V}$  as  $\frac{\partial^2}{\partial E^2}\mathcal{V}$  which is negative by the convexity of  $\mathcal{V}$  (2.2(iii)).  $\square$

It is known that Kähler-Ricci flow is compatible with (K-)MMP with ample scaling ([CL], [ST] etc.). This compatibility can be also seen when we transfer the Kähler-Ricci flow into the non-Archimedean setting. It would be interesting to see the above phenomenon from a differential geometric Ricci flow point of view.

**4.2. Minimisation —semistable case—.**

In our algebraic setting, the observations below show that *minimisation (critical points) of our generalised Donaldson–Futaki invariants* give “canonical limits” of “semistable” objects, which we see as an analogue of the fact that critical points of the K-energy [Mab] are those with constant scalar curvature (e.g. Kähler-Einstein metrics).

These phenomena are motivated by [Od1] and [LX]. Indeed the method of [Od1] to get small (in that case, negative) Futaki invariants, taking some “canonical model” in the MMP theory, can be interpreted as a phenomenon that “canonical limits (made by the MMP) give small Futaki invariants”. Furthermore, [LX] later proved the decrease of Futaki invariants of Fano varieties case directly, which we generalise (4.1).

**Proposition 4.2.** If  $(\mathcal{X}, \mathcal{L})$  minimises the Donaldson–Futaki invariants among other (birational) completions of  $(\mathcal{X}^o, \mathcal{L}^o) \rightarrow B^o$ , it satisfies the following basic properties.

- (i)  $\mathcal{X}$  has only canonical singularities (we assume  $\mathcal{X}$  is  $\mathbf{Q}$ -Gorenstein) if  $(\mathcal{L}^{\dim(F)}.K_F) \leq 0$  (e.g. if  $-K_F$  is nef).
- (ii) If  $K_F = a\mathcal{L}|_F$ , then for an open neighborhood  $U$  of  $p$  in  $B$ ,  $K_{\mathcal{X}/B}|_{\pi^{-1}(U)} = a\mathcal{L}|_{\pi^{-1}(U)}$ .

*Proof.* (i): Suppose the contrary and take the relative canonical model of  $\mathcal{X}$  as  $f: \mathcal{X}^{can} \rightarrow \mathcal{X}$  by [BCHM] again. Putting  $E := -K_{\mathcal{X}^{can}/B}$ , we know that  $E$  is effective and non-zero by the negativity lemma again [KM, 3.39]. Then, we have

$$\begin{aligned} & \frac{1}{n(n-1)} \frac{d}{dt} \Big|_{t=0} DF(\mathcal{X}^{can}, f^*\mathcal{L} - tE) \\ &= -(f^*\mathcal{L}^{n-2}.E.f^*K_{\mathcal{X}/B} - E)(\mathcal{L}|_F^{\dim(F)})^{\frac{n-1}{\dim(F)}} \end{aligned}$$

$$+ (f^* \mathcal{L}^{n-1} \cdot E)(\mathcal{L}|_F)^{\dim(F)} \frac{\dim(B)-1}{\dim(F)} (\mathcal{L}|_F)^{\dim(F)-1} \cdot K_F).$$

The first term is negative from the Hodge index theorem and the second is also non-positive due to our assumption. Thus we get a contradiction.

(ii): It simply follows from Theorem 4.1.  $\square$

Concerning the case over a curve, are basically similar to the author’s older works. The essential difference with the above general case is that we allow base change and consider “normalised Futaki” invariants involving the degree of base change.

Our modified setting is as follows: We fix  $(\mathcal{X}^o, \mathcal{L}^o) \rightarrow B^o$  ( $:= B \setminus \{p\}$  where  $B$  is a smooth curve) as before, and consider all finite morphisms  $\phi: \tilde{B} \rightarrow B$  and the normal  $\mathbf{Q}$ -Gorenstein completions of  $(\mathcal{X}^o, \mathcal{L}^o) \times_B \tilde{B} \rightarrow \tilde{B} \setminus \{f^{-1}(p)\}$  to  $(\tilde{\mathcal{X}}, \tilde{\mathcal{L}}) \rightarrow \tilde{B}$ , where we assume  $\mathcal{X}$  is normal and  $\mathbf{Q}$ -Gorenstein. For this model, we set

$$nDF(\tilde{\mathcal{X}}, \tilde{\mathcal{L}}) := \frac{DF(\tilde{\mathcal{X}}, \tilde{\mathcal{L}})}{\deg(\phi)},$$

as in [LX]. We call  $(\tilde{\mathcal{X}}, \tilde{\mathcal{L}})$  is *nDF-minimising* if  $nDF(\tilde{\mathcal{X}}, \tilde{\mathcal{L}}) \leq nDF(\tilde{\mathcal{X}}', \tilde{\mathcal{L}}')$  for all other possible  $(\tilde{\mathcal{X}}', \tilde{\mathcal{L}}')$  (we allow all base changes  $\phi$ ).

**Theorem 4.3** ( $\dim(B) = 1$  case). *In the above setting, suppose  $(\mathcal{X}, \mathcal{L})$  is nDF-minimising. Then it holds that*

- (i) *any fibre is reduced and semi-log-canonical.*
- (ii) *if  $F$  is a klt  $\mathbf{Q}$ -Fano variety, then all fibers are klt  $\mathbf{Q}$ -Fano varieties.*
- (iii) *if  $K_F = a\mathcal{L}|_F$  with  $a \geq 0$ , the normalised DF invariant of  $(\mathcal{X}', \mathcal{L}') \rightarrow B'$  is minimum of all the models if and only if any fibre  $G$  is reduced slc with  $K_G = a\mathcal{L}|_G$ .*

*Proof.* (i): If the fiber over  $p \in B$  is not reduced, then we take base change of  $B$  ramifying at  $p$  with sufficiently divisible ramifying degree, and take its normalization. Then it is nontrivial along that preimage of non-reduced component so that the (normalized) DF decreases (recall our remark after Proposition 2.5, and also [RT, 5.1, 5.2]).

So we can and do suppose  $\mathcal{X}_0$  is reduced. The relative lc model of  $(\mathcal{X}, \mathcal{X}_0)$  over  $\mathcal{X}$  exists as [OX] shows, which we denote as  $f: \mathcal{X}^{lc} \rightarrow \mathcal{X}$ .

Let us consider the polarised varieties of  $(\mathcal{X}^{lc}, f^* \mathcal{L}(-tE))$  where  $E := K_{\mathcal{X}^{lc}} - f^* K_{\mathcal{X}} + Exc(f)$  and  $Exc(f)$  denotes the total reduced exceptional divisor. Then the corresponding Donaldson–Futaki invariants decrease when  $t$  increases by an argument very similar to that in [Od1, section 3].

(ii): Supposing  $\mathcal{X}_0$  is *not* klt, then we take the relative log canonical model of  $(\mathcal{X}, (1 - \epsilon)\mathcal{X}_0)$  by [BCHM]. The rest follows from [Od1, section 6].

(iii): The “If” direction was proved earlier in [Od3] (also [WX] for the  $a > 0$  case). The converse holds as well, since if  $\mathcal{L} - aK_{\mathcal{X}/B}$  is not 0, then by (4.1), we have  $DF(\mathcal{L} + \epsilon E) < DF(\mathcal{L})$ .  $\square$

For the Calabi–Yau case, a non-rigorous comment is that the reduced slc CY fibers of  $p \in B$  or its preimages by base changes form *infinite ordered set* which we expect to “converges to tropical varieties” which is homeomorphic to the dual complexes, as in [KS]. Regarding the Fano case (ii) we note that, by applying [Kal], it easily follows that the set of all the completed  $\mathbf{Q}$ -Fano fibrations of relative Picard rank 1 form a *tree* (as a graph) via Sarkisov links [Kal]. That is, there is no loop of Sarkisov links among  $\mathbf{Q}$ -Fano-Mori fibrations. We thank Prof. A.-S. Kaloghiros for answering our question regarding this.

**Optimal destabilization.** Finally we propose a problem regarding “maximal destabilisation”. We still keep the notation in section 2.

**Problem 4.4.** For fixed  $\pi^o$ , formulate the “norms”  $\|\mathcal{L}\|$  of polarisations  $\mathcal{L}$  and show the existence of  $(\mathcal{X}, \mathcal{L})$  (“maximally destabilising”) which minimises the DF divided by the norm  $DF(\mathcal{L})/\|\mathcal{L}\|$  among all birational models  $\pi: (\mathcal{X}, \mathcal{L}) \rightarrow B$ .

We expect that relative canonical model  $\mathcal{X} := B^{can} \rightarrow B$  (cf. [BCHM], also recall the proof of (3.2)) will be the maximally destabilising model of non-(semi-)lc singularities in the case where  $\mathcal{X}^o \simeq B^o$  (so that  $\pi$  are birational). We also note that then the corresponding DF invariant is  $(K_{\mathcal{X}/B})^n$  and its log version also exists. Indeed, take the relative log canonical model of a non-log-canonical base  $B$  by [OX] as  $\mathcal{X} := B^{lc} \rightarrow B$ . Then the corresponding log DF invariant is  $(K_{B^{lc}/B} + E)^n$  with reduced total exceptional divisor  $E$  on  $B^{lc}$ , which coincides with the “local volume” of [BdFF], [Ful] (for the proof of the coincidence, please see [Zha]).

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