On holomorphic curves extremal for the truncated defect relation and some applications

By Nobushige Toda*)

Professor Emeritus, Nagoya Institute of Technology

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Abstract: We consider extremal holomorphic curves for the truncated defect relation when the number of vectors whose truncated defects are equal to 1 is large. Some applications to another defect are given.

Key words: Holomorphic curve; truncated defect relation; extremal.

1. Introduction. Let $f = [f_1, \dots, f_{n+1}]$ be a holomorphic curve from $\boldsymbol{C}$ into the n-dimensional complex projective space $P^n(\boldsymbol{C})$ with a reduced representation

$$(f_1, \dots, f_{n+1}) : \boldsymbol{C} \to \boldsymbol{C}^{n+1} - \{\boldsymbol{0}\},$$

where n is a positive integer. We use the notations:

$$||f(z)|| = (|f_1(z)|^2 + \cdots + |f_{n+1}(z)|^2)^{1/2};$$

$$T(r, f) = \frac{1}{2\pi}\int_0^{2\pi} \log ||f(re^{i\theta})||d\theta - \log ||f(0)||.$$

We suppose throughout the paper that f is transcendental: $\lim_{r\to\infty} T(r, f)/\log r = \infty$ and that f is linearly non-degenerate over $\boldsymbol{C}$; namely, $f_1, \dots, f_{n+1}$ are linearly independent over $\boldsymbol{C}$.

It is well-known that f is linearly non-degenerate over $\boldsymbol{C}$ if and only if the Wronskian $W = W(f_1, \dots, f_{n+1})$ of $f_1, \dots, f_{n+1}$ is not identically equal to zero.

For a vector $\boldsymbol{a} = (a_1, \dots, a_{n+1}) \in \boldsymbol{C}^{n+1} - \{\boldsymbol{0}\}$, we put

$$||\boldsymbol{a}|| = (|a_1|^2 + \cdots + |a_{n+1}|^2)^{1/2};$$

$$(\boldsymbol{a}, f) = a_1 f_1 + \cdots + a_{n+1} f_{n+1};$$

$$(\boldsymbol{a}, f(z)) = a_1 f_1(z) + \cdots + a_{n+1} f_{n+1}(z);$$

$$N(r, \boldsymbol{a}, f) = N(r, 1/(\boldsymbol{a}, f))$$

as in [6, Introduction]. We call the quantity

$$\delta(\boldsymbol{a}, f) = 1 - \limsup_{r\to\infty} N(r, \boldsymbol{a}, f)/T(r, f)$$

the deficiency (or defect) of $\boldsymbol{a}$ with respect to f. We have that $0 \le \delta(\boldsymbol{a}, f) \le 1$.

Further, let $\nu(c)$ be the order of zero of $(\boldsymbol{a}, f(z))$ at $z = c$ and for a positive integer k, let

$$n_k(r, \boldsymbol{a}, f) = \sum_{|c|\le r} \min\{\nu(c), k\};$$

$$N_k(r, \boldsymbol{a}, f) = \int_0^r \frac{n_k(t, \boldsymbol{a}, f) - n_k(0, \boldsymbol{a}, f)}{t}dt + n_k(0, \boldsymbol{a}, f)\log r \quad (r > 0).$$

We put

$$\delta_k(\boldsymbol{a}, f) = 1 - \limsup_{r\to\infty} N_k(r, \boldsymbol{a}, f)/T(r, f).$$

It is easy to see that

$$(1) \qquad 0 \le \delta(\boldsymbol{a}, f) \le \delta_k(\boldsymbol{a}, f) \le 1.$$

We denote by $S(r, f)$ any quantity satisfying $S(r, f) = o\{T(r, f)\}$ as $r \to +\infty$, possibly outside a set of r of finite linear measure and by $\boldsymbol{e}_1, \dots, \boldsymbol{e}_{n+1}$ the standard basis of $\boldsymbol{C}^{n+1}$.

Let X be a subset of $\boldsymbol{C}^{n+1} - \{\boldsymbol{0}\}$ in N-subgeneral position; that is to say, $\#X \ge N + 1$ and any $N + 1$ elements of X generate $\boldsymbol{C}^{n+1}$, where N is an integer satisfying $N \ge n$. We say that X is in general position when X is in $n-$ subgeneral position.

Cartan ([1], $N = n$) and Nochka ([4], $N > n$) gave the following theorem:

Theorem A (truncated defect relation)**.** For any q elements $\boldsymbol{a}_j$ $(j = 1, \dots, q)$ of X,

$$\sum_{j=1}^{q} \delta_n(\boldsymbol{a}_j, f) \le 2N - n + 1,$$

where $2N - n + 1 \le q \le \infty$ (see [3]).

We are interested in the holomorphic curve f extremal for the truncated defect relation:

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*) Present address: Chiyoda 3-16-15-302, Naka-ku, Nagoya, Aichi 460-0012.

$$(2) \qquad \sum_{j=1}^{q} \delta_n(\boldsymbol{a}_j, f) = 2N - n + 1.$$

We gave several results in [5]. The purpose of this paper is to give some results on $\delta_n(\boldsymbol{a}, f)$ when (2) holds and $\#\{\boldsymbol{a} \in X \mid \delta_n(\boldsymbol{a}, f) = 1\}$ is large. Some applications to another defect are also given.

2. Preliminaries and lemmas. Let $f = [f_1, \dots, f_{n+1}]$ and X etc. be as in Section 1 and q be an integer satisfying $N + 1 < q < \infty$. For a non-empty subset P of X, we denote by $V(P)$ the vector space spanned by the elements of P and by $d(P)$ the dimension of $V(P)$.

Lemma 2.1 (see [3, (2.4.3), p. 68])**.** *If* $\#P \leq N + 1$, *then* $\#P - d(P) \leq N - n$.

We put for $\nu = 1, \dots, n+1$

$$X_\nu(0) = \{\boldsymbol{a} = (a_1, a_2, \dots, a_{n+1}) \in X \mid a_\nu = 0\}.$$

Then, $0 \leq \#X_\nu(0) \leq N$ as X is in N-subgeneral position. By Lemma 2.1, we have the inequality

$$(3) \qquad \#X_\nu(0) - d(X_\nu(0)) \leq N - n.$$

Let $X_\nu^1(0)$ be a subset of $X_\nu(0)$ satisfying

(i) $\#X_\nu^1(0) = d(X_\nu(0))$;

(ii) All elements of $X_\nu^1(0)$ are linearly independent,

and we put $X_\nu^0(0) = X_\nu(0) - X_\nu^1(0)$. Then, from (3) we have the inequality $\#X_\nu^0(0) \leq N - n$.

Lemma 2.2. *For any q vectors $\boldsymbol{a}_1, \dots, \boldsymbol{a}_q$ in $X - X_\nu^0(0)$, we have the following inequality for any ν $(1 \leq \nu \leq n+1)$:*

$$(q - N - 1)T(r, f) \leq \sum_{j=1}^{q} N_n(r, \boldsymbol{a}_j, f) + (N - n) \sum_{j=1; j \neq \nu}^{n+1} N_n(r, \boldsymbol{e}_j, f) + S(r, f).$$

Proof. As the proof proceeds in the same way for any ν, we prove this lemma for $\nu = n + 1$. For simplicity we put

$$W_1(f_1, \dots, f_{n+1}) = W(f_1, \dots, f_{n+1})/(f_1 \cdots f_{n+1}).$$

We put $(\boldsymbol{a}_j, f) = F_j$ $(1 \leq j \leq q)$ and for any $z (\neq 0)$ arbitrarily fixed, let

$$|F_{j_1}(z)| \leq |F_{j_2}(z)| \leq \cdots \leq |F_{j_q}(z)|,$$

where $1 \leq j_1, \dots, j_q \leq q$ and $j_1, \dots, j_q$ are distinct. Then, there is a positive constant K such that

$$||f(z)|| \leq K |F_{j_\nu}(z)| \quad (\nu = N+1, \dots, q)$$
$$|F_{j_\nu}(z)| \leq K ||f(z)|| \quad (\nu = 1, \dots, q).$$

(From now on we denote by K a constant, which may be different from each other when it appears.)

As X is in N-subgeneral position, there are $n + 1$ linearly independent functions in $\{F_{j_1}, \dots, F_{j_{N+1}}\}$. Let $\{G_1, \dots, G_{n+1}\}$ be linearly independent functions in $\{F_{j_1}, \dots, F_{j_{N+1}}\}$ such that $\{G_1, \dots, G_{n+1}\} \supset \{F_{j_1}, \dots, F_{j_{N+1}}\} \cap \{F_j \mid \boldsymbol{a}_j \in X_{n+1}^1(0)\}$ and put

$$\{G_{n+2}, \dots, G_{N+1}\} = \{F_{j_1}, \dots, F_{j_{N+1}}\} - \{G_1, \dots, G_{n+1}\}.$$

Then, $\{G_{n+2}, \dots, G_{N+1}\} \cap \{F_j \mid \boldsymbol{a}_j \in X_{n+1}(0)\} = \phi$ and we have the equality

$$\begin{aligned} &\frac{F_{j_{N+2}}(z) \cdots F_{j_q}(z)}{W_1(G_1, \dots, G_{n+1}) \Pi_{k=1}^{N-n} W_1(f_1, \dots, f_n, G_{n+1+k})} \\ &= \frac{\Pi_{j=1}^{q} F_j(z) (\Pi_{j=1}^{n} f_j(z))^{N-n}}{W(G_1, \dots, G_{n+1}) \Pi_{k=1}^{N-n} W(f_1, \dots, f_n, G_{n+1+k})} \\ &= K \frac{\Pi_{j=1}^{q} F_j(z) (\Pi_{j=1}^{n} f_j(z))^{N-n}}{W(f_1, \dots, f_{n+1})^{N+1-n}} \equiv H(z) \end{aligned}$$

since $W(G_1, \dots, G_{n+1}) = c_0 W(f_1, \dots, f_{n+1})$ and $W(f_1, \dots, f_n, G_{n+1+k}) = c_k W(f_1, \dots, f_{n+1})$ for $k = 1, \dots, N - n$. $(c_k \neq 0 \ (0 \leq k \leq N - n))$.

From this equality we obtain the inequality which holds for any $z \neq 0$:

$$\begin{aligned} (q - N - 1) \log ||f(z)|| &\leq \log |H(z)| \\ &+ \sum_{(\nu_1, \dots, \nu_{n+1})} \log^+ |W_1(F_{\nu_1}, \dots, F_{\nu_{n+1}})(z)| \\ &+ \sum_{\{F_j | \boldsymbol{a}_j \notin X_{n+1}(0)\}} \log^+ |W_1(f_1, \dots, f_n, F_j)(z)| \\ &+ \log^+ |K|, \end{aligned}$$

where the summation $\sum_{(\nu_1, \dots, \nu_{n+1})}$ is taken over all systems $\{F_{\nu_1}, \dots, F_{\nu_{n+1}}\}$ of $n+1$ functions which are linearly independent and taken from $\{F_1, \dots, F_q\}$. By integrating both sides of this inequality with respect to θ $(z = re^{i\theta})$, we obtain this lemma as in [1]. Here, we used the facts that

$$\frac{1}{2\pi} \int_0^{2\pi} \log |H(re^{i\theta})| d\theta \leq \sum_{j=1}^{q} N_n(r, \boldsymbol{a}_j, f) + (N - n) \sum_{j=1}^{n} N_n(r, \boldsymbol{e}_j, f) + O(1)$$

and

$$\frac{1}{2\pi} \int_0^{2\pi} \log^+ |W_1(F_{\nu_1}, \dots, F_{\nu_{n+1}})(re^{i\theta})| d\theta = S(r, f)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \log^+ |W_1(f_1,\dots,f_n,F_j)(re^{i\theta})|d\theta.$$

□

Corollary 2.1. *For $1 \le \nu \le n+1$*

$$\sum_{\boldsymbol{a}\in X-X_\nu^0(0)} \delta_n(\boldsymbol{a},f) + (N-n)\sum_{j=1;j\ne\nu}^{n+1} \delta_n(\boldsymbol{e}_j,f) \le N+1+(N-n)n.$$

Proof. From Lemma 2.2 we easily obtain this corollary by a usual manner to obtain the defect relation. □

Lemma 2.3. *Suppose that $\delta_n(\boldsymbol{e}_j,f)=1$ $(1\le j\le n+1,\ j\ne\nu)$ for some ν $(1\le\nu\le n+1)$. Let*

$$X_\nu^0(0)=\{\boldsymbol{c}_1^\nu,\dots,\boldsymbol{c}_{p(\nu)}^\nu\}\ (0\le p(\nu)\le N-n).$$

Then, $\sum_{\boldsymbol{a}\in X}\delta_n(\boldsymbol{a},f)\le N+1+\sum_{j=1}^{p(\nu)}\delta_n(\boldsymbol{c}_j^\nu,f)$.

Proof. By our assumption $\delta_n(\boldsymbol{e}_j,f)=1$ $(1\le j\le n+1, j\ne\nu)$ and Corollary 2.1 we have the inequality

$$\sum_{\boldsymbol{a}\in X-X_\nu^0(0)}\delta_n(\boldsymbol{a},f)\le N+1,$$

from which we obtain our inequality. □

Lemma 2.4. *Let $\boldsymbol{a}_1,\dots,\boldsymbol{a}_{n+1}$ be $n+1$ linearly independent vectors in X and let A be the $(n+1)\times(n+1)$ matrix whose j-th row is $\boldsymbol{a}_j$ $(1\le j\le n+1)$, $(\boldsymbol{a}_j,f)=F_j$ $(1\le j\le n+1)$ and $Y=\{\boldsymbol{a}A^{-1}\mid \boldsymbol{a}\in X\}$. Then, we have the followings*:

(a) *A is regular and $\boldsymbol{a}_jA^{-1}=\boldsymbol{e}_j$ $(j=1,\dots,n+1)$.*

(b) *Y is in N-subgeneral position.*

(c) *$F_1,\dots,F_{n+1}$ are entire functions without common zeros and linearly independent over $\boldsymbol{C}$.*

(d) *$T(r,F)=T(r,f)+O(1)$ and so F is transcendental, where $F=[F_1,\dots,F_{n+1}]$.*

(e) *$\delta_n(\boldsymbol{a},f)=\delta_n(\boldsymbol{b},F)$, where $\boldsymbol{b}=\boldsymbol{a}A^{-1}$ $(\boldsymbol{a}\in X)$.*

Proof. (a) and (b) are trivial. (c) As $f_1,\dots,f_{n+1}$ are entire functions without common zeros and linearly independent over $\boldsymbol{C}$, so are $F_1,\dots,F_{n+1}$.

(d) As $c||f(z)||\le||F(z)||\le C||f(z)||$ for positive constants c and C, we have our relation by the definition of the characteristic function.

(e) As $(\boldsymbol{a},f)=(\boldsymbol{b},F)$, we obtain our relation by (d). □

3. Theorem. Let f, X, $X_\nu(0)$ etc. be as in Section 1 or 2. We put $D_n^+(X,f)=\{\boldsymbol{a}\in X\mid \delta_n(\boldsymbol{a},f)>0\}$ and $D_n^1(X,f)=\{\boldsymbol{a}\in X\mid\delta_n(\boldsymbol{a},f)=1\}$.

Theorem 3.1. *Suppose that there exist $n+1$ linearly independent vectors $\boldsymbol{a}_1,\dots,\boldsymbol{a}_{n+1}$ in $D_n^1(X,f)$. Then, $\#D_n^+(X,f)\le(n+1)(N+1-n)$.*

Proof. Let $\boldsymbol{a}$ be any vector in $D_n^+(X,f)$. The vector $\boldsymbol{a}$ can be represented as a linear combination of $\boldsymbol{a}_1,\dots,\boldsymbol{a}_{n+1}:\boldsymbol{a}=c_1\boldsymbol{a}_1+\cdots+c_{n+1}\boldsymbol{a}_{n+1}$.

Then, at least one of $c_1,\dots,c_{n+1}$ is equal to 0. In fact, suppose to the contrary that none of $c_1,\dots,c_{n+1}$ is equal to zero. As $\boldsymbol{a}_1,\dots,\boldsymbol{a}_{n+1},\boldsymbol{a}$ are in general position, from Theorem A for $N=n$ and $q=n+2$, we obtain the inequality

$$\sum_{j=1}^{n+1}\delta_n(\boldsymbol{a}_j,f)+\delta_n(\boldsymbol{a},f)\le n+1,$$

which implies that $\delta_n(\boldsymbol{a},f)=0$. This is a contradiction. We have that at least one of $c_1,\dots,c_{n+1}$ is equal to 0. Let

$$X'_\nu(0)=\{\boldsymbol{a}=c_1\boldsymbol{a}_1+\cdots+c_{n+1}\boldsymbol{a}_{n+1}\in X\mid c_\nu=0\}.$$

Then, $\#X'_\nu(0)\le N$ $(\nu=1,\dots,n+1)$ since X is in N-subgeneral position. From the fact that $D_n^+(X,f)$ is a subset of $\cup_{\nu=1}^{n+1}X'_\nu(0)$, we obtain the inequality

$$\begin{aligned}\#D_n^+(X,f)&\le\#\left\{\bigcup_{\nu=1}^{n+1}X'_\nu(0)\right\}\\&\le n+1+(N-n)(n+1)\\&=(N+1-n)(n+1)\end{aligned}$$

since the vector $\boldsymbol{a}_j$ belongs to the set $\bigcup_{\nu=1;\nu\ne j}^{n+1}X'_\nu(0)$ $(1\le j\le n+1)$. □

Theorem 3.2. *Suppose that*

(i) *there exist $n+1$ linearly independent vectors $\boldsymbol{a}_1,\dots,\boldsymbol{a}_{n+1}$ in $D_n^1(X,f)$;*

(ii) $\sum_{\boldsymbol{a}\in D_n^+(X,f)}\delta_n(\boldsymbol{a},f)=2N-n+1$.

Then, we have that

$$D_n^+(X,f)=D_n^1(X,f)\ \textit{and}\ \#D_n^1(X,f)=2N-n+1.$$

Proof. Let

$$D_n^+(X,f)=\{\boldsymbol{a}_1,\dots,\boldsymbol{a}_{n+1},\boldsymbol{a}_{n+2},\dots,\boldsymbol{a}_q\}.$$

Then, we have that $q\le(N+1-n)(n+1)$ by Theorem 3.1. Let A,F and Y be as in Lemma 2.4 and put $\boldsymbol{b}_j=\boldsymbol{a}_jA^{-1}$ $(j=1,\dots,q)$. Then, by Lemma 2.4, we have that

(α) $\boldsymbol{b}_j=\boldsymbol{e}_j$ $(j=1,\dots,n+1)$;

(β) $\delta_n(\boldsymbol{b}_j,F)=\delta_n(\boldsymbol{a}_j,f)$ $(j=1,\dots,q)$

and by the assumption (i) and (β) we have that

(γ) $\delta_n(\boldsymbol{e}_j, F) = \delta_n(\boldsymbol{a}_j, f) = 1$ $(j = 1, \ldots, n+1)$.

We put for $\nu = 1, \ldots, n+1$

$$Y_\nu(0) = \{\boldsymbol{b} = (b_1, b_2, \ldots, b_{n+1}) \in Y \mid b_\nu = 0\}.$$

Then, $0 \leq \#Y_\nu(0) \leq N$ as Y is in N-subgeneral position.

By Lemma 2.1, we have the inequality

$$\#Y_\nu(0) - d(Y_\nu(0)) \leq N - n. \tag{4}$$

Let $Y_\nu^1(0) = \{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{n+1}\} - \{\boldsymbol{e}_\nu\}$ $(1 \leq \nu \leq n+1)$. We have that $\#Y_\nu^1(0) = d(Y_\nu(0)) = n$.

Next, we put $Y_\nu^0(0) = Y_\nu(0) - Y_\nu^1(0)$ $(1 \leq \nu \leq n+1)$. From (4) we have that $\#Y_\nu^0(0) \leq N - n$. Let $\boldsymbol{a}$ be any vector in $\{\boldsymbol{a}_j \mid n+2 \leq j \leq q\}$ and put $\boldsymbol{b} = \boldsymbol{a}A^{-1}$. Then, $\boldsymbol{b} \in \{\boldsymbol{b}_j \mid n+2 \leq j \leq q\}$. The vector $\boldsymbol{b}$ can be represented as a linear combination of $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{n+1} : \boldsymbol{b} = b_1\boldsymbol{e}_1 + \cdots + b_{n+1}\boldsymbol{e}_{n+1}$.

Then, at least one of $b_1, \ldots, b_{n+1}$ is equal to 0 from Theorem A for $N = n$ and $q = n+2$ as in the proof of Theorem 3.1. For simplicity we suppose that $b_{n+1} = 0$. Let $Y_{n+1}^0(0) = \{\boldsymbol{b}_{j_1}, \ldots, \boldsymbol{b}_{j_p}\}$. $\boldsymbol{b}$ is in $Y_{n+1}^0(0)$. As $\#Y_{n+1}^0(0) \leq N - n$, we have that $p \leq N - n$. By applying Lemma 2.3 to this case and by the assumption (ii) with (β), we obtain the inequality

$$\begin{aligned} 2N - n + 1 &= \sum_{j=1}^{q} \delta_n(\boldsymbol{b}_j, F) \\ &\leq N + 1 + \sum_{k=1}^{p} \delta_n(\boldsymbol{b}_{j_k}, F) \\ &\leq 2N - n + 1. \end{aligned}$$

This implies that $p = N - n$ and $\delta_n(\boldsymbol{b}_{j_k}, F) = 1$ $(1 \leq k \leq N - n)$. We have that $\delta_n(\boldsymbol{b}, F) = 1$. By ($\beta$), $\delta_n(\boldsymbol{a}, f) = 1$. This means that $D_n^+(X, f) = D_n^1(X, f)$ and we have that $\#D_n^1(X, f) = 2N - n + 1$ from the assumption (ii). □

Corollary 3.1. *Suppose that*

(i) $\#D_n^1(X, f) \geq N + 1$;

(ii) $\sum_{\boldsymbol{a} \in D_n^+(X,f)} \delta_n(\boldsymbol{a}, f) = 2N - n + 1$.

Then, we have that

$$D_n^+(X, f) = D_n^1(X, f) \ \textit{and}\ \#D_n^1(X, f) = 2N - n + 1.$$

Proof. As X is in N-subgeneral position, there are $n+1$ linearly independent vectors in $D_n^1(X, f)$ by the assumption (i). We have this corollary from Theorem 3.2 immediately. □

Theorem 3.3. *Suppose that*

(i) *there exist n linearly independent vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ in $D_n^1(X, f)$;*

(ii) $\sum_{\boldsymbol{a} \in D_n^+(X,f)} \delta_n(\boldsymbol{a}, f) = 2N - n + 1$.

(iii) $\#D_n^1(X, f) < 2N - n + 1$.

Then, we have that $\#D_n^1(X, f) = N$.

Proof. Let

$$D_n^+(X, f) = \{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n, \boldsymbol{a}_{n+1}, \ldots, \boldsymbol{a}_q\}.$$

Then, by the assumptions (ii) and (iii) we have that $q \geq 2N - n + 2 > N + 1$. As X is in N-subgeneral position, we can choose $n+1$ linearly independent vectors containing $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ from $D_n^+(X, f)$. We may suppose without loss of generality that $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{n+1}$ are linearly independent. Let A, F and Y be as in Lemma 2.4 and put $\boldsymbol{b}_j = \boldsymbol{a}_j A^{-1}$ $(j = 1, \ldots, q)$. Then, by Lemma 2.4, we have that

(α) $\boldsymbol{b}_j = \boldsymbol{e}_j$ $(j = 1, \ldots, n+1)$;

(β) $\delta_n(\boldsymbol{b}_j, F) = \delta_n(\boldsymbol{a}_j, f)$ $(j = 1, \ldots, q)$

and by the assumption (i) and (β) we have that

(γ) $\delta_n(\boldsymbol{e}_j, F) = \delta_n(\boldsymbol{a}_j, f) = 1$ $(j = 1, \ldots, n)$.

We put

$$Y(0) = \{\boldsymbol{b} = (b_1, b_2, \ldots, b_{n+1}) \in Y \mid b_{n+1} = 0\}.$$

Then, $0 \leq \#Y(0) \leq N$ as Y is in N-subgeneral position. By Lemma 2.1, we have the inequality

$$\#Y(0) - d(Y(0)) \leq N - n. \tag{5}$$

Let $Y^1(0) = \{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$. We have that $\#Y^1(0) = d(Y(0)) = n$.

Next, we put $Y^0(0) = Y(0) - Y^1(0)$. From (5) we have the inequality $\#Y^0(0) \leq N - n$. Let

$$Y^0(0) = \{\boldsymbol{b}_{j_1}, \ldots, \boldsymbol{b}_{j_p}\}\ (j_k \geq n+2; k = 1, \ldots, p).$$

As $\#Y^0(0) \leq N - n$, we have that $p \leq N - n$. By applying Lemma 2.3 to this case $(\nu = n+1)$ and by the assumption (ii) with (β), we obtain the inequality

$$\begin{aligned} 2N - n + 1 &= \sum_{j=1}^{q} \delta_n(\boldsymbol{b}_j, F) \\ &\leq N + 1 + \sum_{k=1}^{p} \delta_n(\boldsymbol{b}_{j_k}, F) \leq 2N - n + 1. \end{aligned}$$

This implies that $p = N - n$ and $\delta_n(\boldsymbol{b}_{j_k}, F) = 1$ $(k = 1, \ldots, N - n)$. This means that

$$D_n^1(Y, F) = \{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\} \cup \{\boldsymbol{b}_{j_1}, \ldots, \boldsymbol{b}_{j_{N-n}}\}.$$

We have that $\#D_n^1(X, f) = \#D_n^1(Y, F) = N$. □

Remark 3.1. By using the inequality (1) and Theorem A we are able to obtain results for $\delta(\boldsymbol{a}, f)$

corresponding to the results obtained for $\delta_n(\boldsymbol{a}, f)$ in this section.

4. Application to another defect. Let f, X etc. be as in Section 1 or 2 and $\boldsymbol{a}$ be a vector in $\boldsymbol{C}^{n+1} - \{\boldsymbol{0}\}$. We say that

"$\boldsymbol{a}$ has multiplicity m if $(\boldsymbol{a}, f(z))$ has at least one zero and all the zeros of $(\boldsymbol{a}, f(z))$ have multiplicity at least m, while at least one zero has multiplicity m."

If $(\boldsymbol{a}, f(z))$ has no zero, we set $m = \infty$.

Then, as a corollary of Theorem A, Cartan ([1], $N = n$) and Nochka ([4], $N > n$) gave the following theorem (see [3, Theorem 3.3.15]):

Theorem B. *For any* $\boldsymbol{a}_1, \dots, \boldsymbol{a}_q \in X$ $(q < \infty)$, *let* $\boldsymbol{a}_j$ *have multiplicity* m_j. *Then,*

$$\sum_{j=1}^{q}(1 - n/m_j) \leq 2N - n + 1.$$

As the numbers "$1 - n/m_j$" are not always non-negative in this theorem, we define a new defect as follows:

Definition 4.1. For $\boldsymbol{a} \in \boldsymbol{C}^{n+1} - \{\boldsymbol{0}\}$ with multiplicity m we put

$$\mu_n(\boldsymbol{a}, f) = \left(1 - \frac{n}{m}\right)^{+} = 1 - \frac{n}{\max(m, n)},$$

where $a^{+} = \max(a, 0)$.

We call the quantity $\mu_n(\boldsymbol{a}, f)$ the μ_n-defect of $\boldsymbol{a}$ with respect to f. Note that $\mu_n(\boldsymbol{a}, f) < 1$ if (a, f) has zeros and $\mu_n(\boldsymbol{a}, f) = 1$ if (a, f) has no zero.

We put $M_n^{+}(X, f) = \{\boldsymbol{a} \in X \mid \mu_n(\boldsymbol{a}, f) > 0\}$ and $M_n^{1}(X, f) = \{\boldsymbol{a} \in X \mid \mu_n(\boldsymbol{a}, f) = 1\}$.

$\mu_n(\boldsymbol{a}, f)$ has the following properties.

Proposition 4.1. (a) $\mu_n(\boldsymbol{a}, f) = 1$ *if and only if* $(\boldsymbol{a}, f)$ *has no zero.*

(b) $0 \leq \mu_n(\boldsymbol{a}, f) \leq \delta_n(\boldsymbol{a}, f) \leq 1$.

(c) (μ_n*-defect relation*) *For any* $\boldsymbol{a}_1, \dots, \boldsymbol{a}_q \in X$, *we have the following inequality*:

$$\sum_{j=1}^{q} \mu_n(\boldsymbol{a}_j, f) \leq 2N - n + 1.$$

Proof. (a) This is trivial from the definition of $\mu_n(\boldsymbol{a}, f)$.

(b) When $(\boldsymbol{a}, f)$ has no zero, $\mu_n(\boldsymbol{a}, f) = \delta_n(\boldsymbol{a}, f) = 1$. When $(\boldsymbol{a}, f)$ has zeros, let m be the multiplicity of $\boldsymbol{a}$. Then, we obtain the inequality for $r \geq 1$

$$N_n(r, \boldsymbol{a}, f) \leq \frac{n}{\max(m, n)} N(r, \boldsymbol{a}, f)$$

$$\leq \frac{n}{\max(m, n)} T(r, f) + O(1),$$

from which we obtain the inequality

$$0 \leq \mu_n(\boldsymbol{a}, f) \leq \delta_n(\boldsymbol{a}, f) \leq 1.$$

(c) From (b) and Theorem A we obtain this relation. □

Theorem 4.1. $\#M_n^{+}(X, f) \leq (n + 1)(2N - n + 1)$.

Proof. For any q vectors $\boldsymbol{a}_1, \dots, \boldsymbol{a}_q \in M_n^{+}(X, f)$, from Proposition 4.1 (c) we have the inequality

$$\sum_{j=1}^{q} \mu_n(\boldsymbol{a}_j, f) \leq 2N - n + 1. \tag{6}$$

As $\mu_n(\boldsymbol{a}_j, f) \geq 1 - n/(n + 1) = 1/(n + 1)$, we have the inequality $q/(n + 1) \leq (2N - n + 1)$ from (6), so that we have that $q \leq (n + 1)(2N - n + 1)$. This means that this theorem holds. □

Lemma 4.1 ([1, p. 10]). *For* $1 \leq i \neq j \leq n + 1$,

$$T(r, f_i/f_j) < T(r, f) + O(1).$$

Theorem 4.2. *Suppose that there exist* $n + 1$ *linearly independent vectors* $\boldsymbol{a}_1, \dots, \boldsymbol{a}_{n+1}$ *in* $M_n^{1}(X, f)$. *Then, we have the followings*:

(a) *If there exists*

$$\boldsymbol{a} \in M_n^{+}(X, f) - \{\boldsymbol{a}_1, \dots, \boldsymbol{a}_{n+1}\},$$

then $\boldsymbol{a} = c_j \boldsymbol{a}_j$ *for some* j $(1 \leq j \leq n + 1; c_j \neq 0)$.

(b) $M_n^{+}(X, f) = M_n^{1}(X, f)$.

Proof. (a) Let m be the multiplicity of $\boldsymbol{a} \in M_n^{+}(X, f) - \{\boldsymbol{a}_1, \dots, \boldsymbol{a}_{n+1}\}$. Note that $n < m \leq \infty$. The vector $\boldsymbol{a}$ can be represented as a linear combination of $\boldsymbol{a}_1, \dots, \boldsymbol{a}_{n+1} : \boldsymbol{a} = c_1 \boldsymbol{a}_1 + \dots + c_{n+1} \boldsymbol{a}_{n+1}$.

We put $(\boldsymbol{a}_j, f) = F_j$ $(1 \leq j \leq n+1)$ and $(\boldsymbol{a}, f) = F_0$. Then, $F_0 = c_1 F_1 + \dots + c_{n+1} F_{n+1}$. We prove that all coefficients $c_1, \dots, c_{n+1}$ except one are equal to zero.

First we prove that at least one of $c_1, \dots, c_{n+1}$ is equal to zero. In fact, suppose to the contrary that none of $c_1, \dots, c_{n+1}$ is equal to zero. Then, $\boldsymbol{a}_1, \dots, \boldsymbol{a}_{n+1}, \boldsymbol{a}$ are in general position, and so from Proposition 4.1 (c) for $N = n$, $q = n + 2$ we have that $\mu_n(\boldsymbol{a}, f) = 0$, which is a contradiction.

Next, let

$$\{j_1, \dots, j_k\} = \{j \mid c_j \neq 0,\ 1 \leq j \leq n + 1\}.$$

Then, k must be equal to 1. Suppose to the contrary that $k \geq 2$. Let $\varphi = [F_{j_1}, \dots, F_{j_k}]$.

As $F_{j_1}, \dots, F_{j_k}$ are entire functions without zeros and the function F_{j_1}/F_{j_k} is not constant, it is transcendental. By Lemma 4.1 we obtain that φ is transcendental. Note that for $\boldsymbol{a'} = (c_{j_1}, \dots, c_{j_k})$, $(\boldsymbol{a}, f) = (\boldsymbol{a'}, \varphi) = F_0$ and

$$0 < \mu_n(\boldsymbol{a}, f) = 1 - \frac{n}{m} \le 1 - \frac{k-1}{m} = \mu_{k-1}(\boldsymbol{a'}, \varphi).$$

We apply Proposition 4.1 (c) to $f = \varphi$, $N = n = k-1$, $q = k+1$ and $\boldsymbol{e}_1, \dots, \boldsymbol{e}_k, \boldsymbol{a}'(\in \boldsymbol{C}^k - \{\boldsymbol{0}\})$, which are in general position, to obtain that $\mu_{k-1}(\boldsymbol{a'}, \varphi) = 0$, which is a contradiction. This means that k must be equal to 1. That is to say, $\boldsymbol{a} = c_{j_1}\boldsymbol{a}_{j_1}$ $(c_{j_1} \ne 0)$.

(b) By definition, we have the relation $M_n^1(X, f) \subset M_n^+(X, f)$. On the other hand we obtain the relation $M_n^+(X, f) \subset M_n^1(X, f)$ from (a), so that we have $M_n^+(X, f) = M_n^1(X, f)$. □

Remark 4.1. Theorem 4.2 (a) is a generalization of Borel's theorem (see [1, p. 19, 1^o]).

Corollary 4.1. *If* $M_n^1(X, f) \ge N + 1$, *then* $M_n^+(X, f) = M_n^1(X, f)$.

Proposition 4.2. $\#M_n^1(X, f) \le N + N/n$.

Proof. Let $q = \#M_n^1(X, f)$. Then, by Theorem 4.1, $q \le (2N + 1 - n)(n + 1)$. We have only to prove this lemma when $q \ge N + 1$. Let

$$M_n^1(X, f) = \{\boldsymbol{a}_1, \dots, \boldsymbol{a}_{n+1}, \boldsymbol{a}_{n+2}, \dots, \boldsymbol{a}_q\},$$

where $\boldsymbol{a}_1, \dots, \boldsymbol{a}_{n+1}$ are linearly independent. Note that we can find $n + 1$ linearly independent vectors in $\#M_n^1(X, f)$ since X is in N-subgeneral position and $q \ge N + 1$.

By using Theorem 4.2 (a) or by Borel's theorem (see [1, p. 19, 1^o]), we obtain

$$\boldsymbol{a}_k = a_k \boldsymbol{a}_{j_k} \quad (k = 1, \dots, q;\ 1 \le j_k \le n + 1), \tag{7}$$

$(a_k \ne 0)$. Here, $a_k = 1$, $j_k = k$ for $1 \le k \le n + 1$.

When we represent $\boldsymbol{a}_k$ by $\boldsymbol{a}_1, \dots, \boldsymbol{a}_{n+1}$: $\boldsymbol{a}_k = a_{k1}\boldsymbol{a}_1 + \cdots + a_{kn+1}\boldsymbol{a}_{n+1}$ $(k = 1, \dots, q)$, we have by (7) that

$$\#\{a_{kj} = 0 \mid k = 1, \dots, q;\ j = 1, \dots, n + 1\} = qn.$$

As X is in N-subgeneral position, it must hold that $qn \le N(n + 1)$, from which we obtain that $q \le N + N/n$. □

Remark 4.2. This proposition is given in [2, Theorem 16, p. 41] in a different situation.

Theorem 4.3. *Suppose that there exist* $n + 1$ *linearly independent vectors* $\boldsymbol{a}_1, \dots, \boldsymbol{a}_{n+1}$ *in* $M_n^1(X, f)$. *Then,* $\sum_{\boldsymbol{a} \in M_n^+(X,f)} \mu_n(\boldsymbol{a}, f) \le N + N/n$.

Proof. As $M_n^+(X, f) = M_n^1(X, f)$ from Theorem 4.2 (b), we have the equality

$$\sum_{\boldsymbol{a} \in M_n^+(X,f)} \mu_n(\boldsymbol{a}, f) = \#M_n^1(X, f)$$

and by Proposition 4.2 we have our theorem. □

Corollary 4.2. *If* $\#M_n^1(X, f) \ge N + 1$, *then* $\sum_{\boldsymbol{a} \in M_n^+(X,f)} \mu_n(\boldsymbol{a}, f) \le N + N/n$.

Remark 4.3. $N + N/n \le 2N - n + 1$ and the equality holds if and only if $N = n$ or $n = 1$. This implies that the μ_n-defect relation of f is not extremal when $N > n \ge 2$ in Theorem 4.3 or Corollary 4.2.

Theorem 4.4. *Suppose that*

(i) *there exist* n *linearly independent vectors* $\boldsymbol{a}_1, \dots, \boldsymbol{a}_n$ *in* $M_n^1(X, f)$;

(ii) $\sum_{\boldsymbol{a} \in M_n^+(X,f)} \mu_n(\boldsymbol{a}, f) = 2N - n + 1$.

(iii) $\#M_n^1(X, f) < 2N - n + 1$.

Then, we have that $\#M_n^1(X, f) = N$.

Proof. As $0 \le \mu_n(\boldsymbol{a}, f) \le \delta_n(\boldsymbol{a}, f) \le 1$ for any $\boldsymbol{a} \in X$ (Proposition 4.1 (b)), from the assumption (ii) and Theorem A we obtain that $\mu_n(\boldsymbol{a}, f) = \delta_n(\boldsymbol{a}, f)$ for any $\boldsymbol{a}$ in X, so that we obtain this theorem from Theorem 3.3. □

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