On Fibonacci numbers with few prime divisors

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Abstract: If n is a positive integer, write F_n for the nth Fibonacci number, and $\omega(n)$ for the number of distinct prime divisors of n. We give a description of Fibonacci numbers satisfying $\omega(F_n) \le 2$. Moreover, we prove that the inequality $\omega(F_n) \ge (\log n)^{\log 2+o(1)}$ holds for almost all n. We conjecture that $\omega(F_n) \gg \log n$ for composite n, and give a heuristic argument in support of this conjecture.

Key words: Fibonacci numbers; arithmetic functions; prime divisors.

1. Introduction. Let F_n be the nth Fibonacci number and L_n be the nth Lucas number. In a previous paper [3], the following result was proved.

Theorem 1. *The only solutions to the equation*

$$F_n = y^m, \qquad m \ge 2$$

are given by $n = 0,\ 1,\ 2,\ 6$ *and* 12 *which correspond respectively to* $F_n = 0,\ 1,\ 1,\ 8$ *and* 144. *Moreover, the only solutions to the equation*

$$L_n = y^p, \qquad m \ge 2$$

are given by $n = 1$ *and* 3 *which correspond respectively to* $L_n = 1$ *and* 4.

The proof involves an intricate combination of Baker's method and the modular method; it also needs about one week of computer verification using the systems `pari` [1] and `magma` [2]. In this paper we use the above theorem, together with well-known results of Carmichael and Cohn to give a description of Fibonacci numbers with at most two distinct prime divisors.

If n is a positive integer, write $\omega(n)$ for the number of distinct prime divisors of n. We prove that the inequality $\omega(F_n) \ge (\log n)^{\log 2+o(1)}$ holds for almost all n. We conjecture that $\omega(F_n) \gg \log n$ for composite n, and give a heuristic argument in support of this conjecture.

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2. The theorems of Carmichael and Cohn. We need the following two celebrated results on Fibonacci numbers. The first is due to Carmichael [4] and the second to Cohn [5].

Theorem 2. *Let* $n > 2$ *and* $n \ne 6,\ 12$ *then* F_n *has a prime divisor which does not divide any* F_m *for* $0 < m < n$; *such a prime is called a* primitive divisor *of* F_n.

Theorem 3. *Let* $0 < m < n$ *and suppose that the product* F_mF_n *is a square, then* $(m,n) = (1,2)$, $(1,12)$, $(2,12)$ *or* $(3,6)$.

3. Fibonacci numbers with $\omega(F_n) \le 2$. Write $\omega(m)$ for the number of distinct prime factors of m. Notice that $\omega(F_n) > 0$ for $n > 2$ and that for integers m and n, with $1 < m < n$ and $mn \ne 6$, 12, we have

$$\omega(F_{mn}) > \omega(F_m) + \omega(F_n), \quad \text{if } (m,n) = 1, \tag{1}$$

because first F_m and F_n both divide F_{mn} and are coprime when the indices are coprime, and secondly Theorem 2 implies that there is a prime number dividing F_{mn} which does not divide the product F_mF_n.

Lemma 3.1. *Suppose* n *is a positive integer and* $\omega(F_n) \le 2$. *Then either* $n = 1,\ 2,\ 4,\ 8,\ 12$ *or* $n = \ell,\ 2\ell,\ \ell^2$ *for some odd prime number* ℓ.

Proof. The Lemma follows straightforwardly from the fact that $\omega(F_n) \ge 3$ in the following cases:
(a) if $16 \mid n$, or $24 \mid n$, or
(b) if $pq \mid n$ where p, q are distinct odd primes, or
(c) if $\ell^3 \mid n$ or $2\ell^2 \mid n$ or $4\ell \mid n$ for some odd prime ℓ, unless $n = 12$.

These may be verified using a combination of Theorem 2 and inequality (1). For example, if $4\ell \mid n$ where $\ell \ne 3$ is an odd prime then

$$\omega(F_n) \ge \omega(F_{4\ell}) > \omega(F_{2\ell}) > \omega(F_\ell) \ge 1,$$

where the strict inequalities follow from Theorem 2. Thus $\omega(F_n) \geq 3$. □

If $n = 2\ell$ with $\ell > 3$ then $F_n = F_\ell L_\ell$ and $\omega(F_n) = 2$ implies that F_ℓ and L_ℓ are prime numbers (because of Theorem 1).

Consider now the case $n = \ell^2$ with $\omega(F_n) = 2$. Then $\omega(F_\ell) = 1$ and, by Theorem 1, $F_\ell = p = $ prime. If $\ell = 5$ then $F_\ell = 5$ and $F_{25} = 5^2 \times 3001$. If $\ell \neq 5$ then p does not divide F_n/F_ℓ so that $F_n = pq^t$, say. By Theorem 3, the product $F_\ell F_n = p^2q^t$ is not a square, thus t is odd.

We have proved the following result.

Theorem 4. *The only solutions to the equation*

$$\omega(F_n) = 1$$

are given by $n = 4$, $n = 6$ or n is an odd prime number for which F_n is prime.

The only solutions to the equation

$$\omega(F_n) = 2$$

are given for even n by $n = 8$ or $n = 12$ or $n = 2\ell$ where ℓ is some odd prime number for which $F_n = F_\ell L_\ell$ where F_ℓ and L_ℓ are both prime numbers. For odd n, the only possible cases are $n = \ell$ or $n = \ell^2$. Moreover if $\omega(F_n) = 2$ and $n = \ell^2$ then F_ℓ must be prime, say $F_\ell = p$; and if $\ell \neq 5$, then

$$F_n = pq^t$$

where q is a prime number distinct from p, and the exponent t is odd.

A short computer search reveals examples of all of the above possibilities. Thus

- examples of $\omega(F_n) = 1$ with n prime are: $F_5 = 5$, $F_7 = 13$, $F_{11} = 89$, $F_{13} = 233, \ldots$
- examples of $\omega(F_n) = 2$ with $n = 2\ell$ are: $F_{10} = 55$, $F_{14} = 13 \times 29$, $F_{22} = 89 \times 199$, $F_{26} = 233 \times 521, \ldots$
- examples of $\omega(F_n) = 2$ with $n = \ell$, ℓ^2 are: $F_9 = 34$, $F_{19} = 37 \times 113$, F_{25}, $F_{31} = 557 \times 2417, \ldots, F_{121} = 89 \times 97415813466381445596089, \ldots$

Remark. Lemma 3.1 can also be proved using some results from [7]. For example, one of the results from [7] is that, if $\tau(m)$ denotes the number of divisors of the positive integer m, then we have $\tau(F_n) \geq F_{\tau(n)}$ for any positive integer n, with equality only for $n = 1, 2, 4$. Assuming that $\omega(F_n) \leq 2$, one gets immediately that (up to a few exceptions) $\tau(n) \leq 4$, giving that n is the product of two distinct prime numbers or the square of a prime number, which together with Theorem 2 yields the conclusion of Lemma 3.1.

4. Fibonacci numbers rarely have few prime factors. Theorem 4 in effect tells us that $\omega(F_n) \geq 3$ for almost all n. We mean by this that the set of integers for which this inequality holds has density 1. Indeed, much more is true. For example, by the following theorem, $\omega(F_n) \geq C$ holds for almost all n, whatever the value of C.

Theorem 5. *The inequality $\omega(F_n) \geq (\log n)^{\log 2 + o(1)}$ holds for almost all n.*

Proof. By Theorem 2, we know that if a divisor d of n is not 1, 2, 6 or 12 then F_d has a primitive prime factor. This translates immediately in saying that

$$\omega(F_n) \geq \tau(n) - 4,$$

where $\tau(n)$ is the total number of divisors of n. Certainly, $\tau(n) \geq 2^{\omega(n)}$. Since $\omega(n) = (1+o(1)) \log\log n$ for almost all n, the desired inequality follows. □

5. Heuristic results. Given the present state of knowledge in analytic number theory, it seems unrealistic to obtain precise results as to whether each of the possibilities from Theorem 4 occurs finitely many times or infinitely often. There is, however, a standard heuristic argument which suggests that there are infinitely many primes ℓ with F_ℓ prime, but only finitely many primes ℓ with $\omega(F_{2\ell}) = 2$ or $\omega(F_{\ell^2}) = 2$.

The heuristic argument goes as follows (compare with [6, page 15]): from the Prime Number Theorem, the 'probability' that a positive integer m is prime is $1/\log m$. Thus the 'expected number' of primes ℓ such that F_ℓ is also prime is

$$\sum_{\ell \text{ is prime}} \frac{1}{\log F_\ell} \geq A \sum_{\ell \text{ is prime}} \frac{1}{\ell}$$

for some positive constant A. Since this last series diverges (albeit very slowly), it is reasonable to guess that there are infinitely prime F_ℓ.

Applying the same heuristic argument suggests that there are only finitely many primes ℓ with $\omega(F_{2\ell}) = 2$ or $\omega(F_{\ell^2}) = 2$. For example, the 'expected number' of primes ℓ with $\omega(F_{2\ell}) = 2$ is

$$\sum_{\ell \text{ is prime}} \frac{1}{\log F_\ell \times \log L_\ell} \leq B \sum_{\ell \text{ is prime}} \frac{1}{\ell^2} < \infty$$

where B is some positive constant.

In fact we can go even further. We conjecture the following.

Conjecture 5.1. *$\omega(F_n) \gg \log n$ holds for all composite positive integers n.*

In order to 'justify' this conjecture, let us make the following heuristic principle.

Heuristic 5.2. *Let $k : \mathbf{N} \to \mathbf{N}$ be any function. Let $\mathcal{A}$ be a subset of positive integers such that there is no algebraic reason for $a \in \mathcal{A}$ to have more than $k(a)$ prime factors. Assume that for every $a \in \mathcal{A}$ there exists a proper divisor of a, let us call it $\tilde{a} > 1$, such that the greatest common divisor of $\tilde{a}$ and $a/\tilde{a}$ has at most one prime factor and the series*

$$\sum_{a\in\mathcal{A}} \sum_{k_1+k_2\le k(a)+1} \frac{1}{(k_1-1)!}\frac{1}{(k_2-1)!} \frac{(\log\log\tilde{a})^{k_1-1}(\log\log(a/\tilde{a}))^{k_2-1}}{\log\tilde{a}\times\log(a/\tilde{a})}$$

is convergent. Then $\omega(a) \le k(a)$ should hold only for finitely many $a \in \mathcal{A}$.

Heuristic 5.2, is based on the fact that the 'probability' for a positive integer n to have k distinct prime factors is

$$\frac{1}{(k-1)!}\frac{(\log\log n)^{k-1}}{\log n}.$$

So, if $a \in \mathcal{A}$ and $\omega(a) = k \le k(a)$, then there exist nonnegative integers k_1 and k_2 such that $\omega(\tilde{a}) = k_1$ and $\omega(a/\tilde{a}) = k_2$. Furthermore, since the greatest common divisor of $\tilde{a}$ and $a/\tilde{a}$ has at most one prime factor, it follows that either $k_1 + k_2 = k \le k(a)$ or $k_1+k_2 = k+1 \le k(a)+1$. Thus, the above sum represents just the sum of the 'probabilities' that $\omega(\tilde{a}) = k_1$ and $\omega(a/\tilde{a}) = k_2$ assuming that such events are independent.

5.1. Conjecture 5.1 follows from Heuristic 5.2. Let n be a composite integer. Observe first that, by Theorem 2, if a divisor d of n is different from 1, 2, 6 or 12, then F_d has a prime factor not dividing F_{d_1} for any positive integer $d_1 < d$. Consequently, if n has at least $0.1\log n$ divisors, then we have

$$\omega(F_n) \ge \max\{1, 0.1\log n - 6\} \gg \log n. \tag{2}$$

Let $\mathcal{A}$ be the set of Fibonacci numbers F_n, where n runs through the composite integers having less than $0.1\log n$ divisors. Let n be composite and write m for the largest proper divisor of n. Clearly, $m = n/p(n)$, where $p(n)$ is the smallest prime factor of n. Note that $m \ge n^{1/2}$. It is known that $\gcd(F_m, F_n/F_m) \mid p(n)$, therefore the two numbers F_m and F_n/F_m share at most one prime factor. Fix k and let k_1 and k_2 be such that $k_1 + k_2 = k$. One checks immediately that both inequalities $F_m < e^m$ and $F_n/F_m < e^{n-m}$ hold. Thus, we get

$$\begin{aligned}
&\sum_{k_1+k_2=k} \frac{1}{(k_1-1)!(k_2-1)!} (\log\log F_m)^{k_1-1}(\log\log(F_n/F_m))^{k_2-1}\\
&\le \frac{1}{(k-2)!}\sum_{k_1+k_2=k} \binom{k-2}{k_1-1}(\log m)^{k_1-1}(\log(n-m))^{k_2-1}\\
&= \frac{1}{(k-2)!}(\log m + \log(n-m))^{k-2}\\
&< \left(\frac{2e\log n}{k-2}\right)^{k-2},
\end{aligned}$$

where for the last inequality above we used Stirling's formula. Using the inequalities $\log F_m \gg m \ge n^{1/2}$ and $\log(F_n/F_m) \gg (n-m) \gg n$ (because m divides n), it follows that

$$\sum_{k_1+k_2=k} \frac{(\log\log F_m)^{k_1-1}(\log\log(F_n/F_m))^{k_2-1}}{(k_1-1)!\,(k_2-1)!\,\log F_m\times\log(F_n/F_m)} \ll \frac{1}{n^{3/2}}\left(\frac{2e\log n}{k-2}\right)^{k-2}.$$

For a fixed y, the function $x \longmapsto (2ey/x)^x$ is increasing for $x < 2y$. Thus, taking $k(F_n) = c_0\log n$, where $c_0 < 2$ is some constant, we get that

$$\sum_{k_1+k_2\le k(F_n)} \frac{(\log\log F_m)^{k_1-1}(\log\log(F_n/F_m))^{k_2-1}}{(k_1-1)!\,(k_2-1)!\,\log F_m\times\log(F_n/F_m)}$$
$$\ll \sum_{k\le k(F_n)} \frac{1}{n^{3/2}}\left(\frac{2e\log n}{k-2}\right)^{k-2} \ll \frac{\log n}{n^{3/2}}\left(\frac{2e}{c_0}\right)^{c_0\log n}.$$

Choosing c_0 such that $c_0\log(2e/c_0) = c_1 < 1/2$ (we can choose, say, $c_0 = 0.1$), we get that the right hand side of the above inequality is

$$\ll \frac{\log n}{n^{3/2-c_1}}$$

and summing up over n we get a convergent series. Hence, Heuristic 5.2 with $a = F_n$ and $\tilde{a} = F_m$ for F_n in $\mathcal{A}$ implies that $\omega(F_n) < 0.1\log n$ holds only for finitely many composite integers n, which, by (2) and the definition of $\mathcal{A}$, implies that $\omega(F_n) \gg \log n$ holds for all composite integers n. Actually, our choice of $\mathcal{A}$ takes into account *an algebraic reason* for which

F_n has more than $0.1 \log n$ divisors.

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