Existence of bounded solutions for semilinear degenerate elliptic equations with absorption term

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1. Introduction. Let $N \geq 1$ and $p > 1$. Let F be a compact set and Ω be a bounded open set of $\boldsymbol{R}^N$ satisfying $F \subset \Omega \subset \boldsymbol{R}^N$. We also set $\Omega' = \Omega \setminus \partial F$, where $\partial F = F \setminus \mathring{F}$ and $\mathring{F}$ denotes the interior of F. Define

$$P = -\operatorname{div}(A(x)\nabla\cdot), \tag{1.1}$$

where $A(x) \in C^1(\Omega')$ is positive in $\Omega \setminus F$ and vanishes in $\mathring{F}$. First we shall consider removable singularities of solutions for degenerate semilinear elliptic equations. Assume that $u \in C^0(\Omega') \cap C^2(\Omega \setminus F)$ satisfies

$$Pu + B(x)Q(u) = f(x), \qquad \text{in } \Omega', \tag{1.2}$$

for $f/B \in L^\infty(\Omega)$. Here $Q(u)$ is a nonlinear term defined in the section 2. Then we shall show the existence of a bounded solution in Ω which coincides with u in $\Omega' = \Omega \setminus \partial F$. This result was established by H. Brezis and L. Veron in [2], under the assumptions that F consists of finite points, $Q(t) = |t|^{p-1}t$ and $A(x)$, $B(x)$, $C(x)$ are positive constants. (see also [5]). In this paper we generalize their results for an arbitrary compact set F in place of finite set and for a wider class of (degenerate) elliptic operators P.

Secondly as an application, we shall consider the Dirichlet boundary problem for genuinely degenerate semilinear elliptic operators:

$$\begin{cases} Pu + B(x)Q(u) = f(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Then we shall establish the existence and uniqueness of bounded solutions u for this problem. When P is uniformly elliptic on $\bar{\Omega}$, this problem has been treated by many authors. But the development of the theory seems to be rather limitted in the study of genuinely degenerate operators.

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2. Main results and applications. Let $N \geq 1$. Let F and Ω be a compact set and bounded open subset of $\boldsymbol{R}^N$ respectively, satisfying $F \subset \Omega$, and set $\Omega' = \Omega \setminus \partial F$. Here ∂F is defined as $\partial F = F \setminus \mathring{F}$. In the next we define a modified distance to ∂F.

Definition 1. Let $d(x) \in C^\infty(\Omega')$ be a nonnegative function satisfying

$$C^{-1} \leq \frac{d(x)}{dist(x, \partial F)} \leq C, \tag{2.1}$$
$$|\partial^\gamma d(x)| \leq C(\gamma) dist(x, \partial F)^{1-|\gamma|}, \; x \in \Omega',$$

where C and $C(\gamma)$ are positive numbers independent of each point x.

We suppose the following four assumptions:

[H-1] (*Coefficients*).

$$\begin{cases} A(x) \in C^1(\Omega') \cap L^1_{\mathrm{loc}}(\Omega), \\ A(x) = 0 \text{ in } \mathring{F} = F \setminus \partial F, \\ A(x) > 0 \text{ in } \Omega \setminus F, \end{cases} \tag{2.2}$$
$$\begin{cases} B(x) \in L^\infty_{\mathrm{loc}}(\Omega') \cap L^1_{\mathrm{loc}}(\Omega), \\ B(x) > 0 \text{ in } \Omega' = \Omega \setminus \partial F, \\ C(x) \in L^\infty_{\mathrm{loc}}(\Omega') \cap L'_{\mathrm{loc}}(\Omega), \\ C(x) \geq 0 \text{ in } \Omega. \end{cases}$$

[H-2] (*Nonlinear term*).

$$\begin{cases} Q(t) \text{ is monotone increasing and continuous on } \mathbf{R} \\ \text{such that } Q(0) = 0 \text{ and } Q(t)t > 0 \text{ for any } t \in \mathbf{R} \setminus \{0\}. \end{cases} \tag{2.3}$$

Definition 2. Let us set for $x \in \Omega' = \Omega \setminus \partial F$

$$\tilde{A}(x) = A(x) + d(x)|\nabla A(x)|, \tag{2.4}$$
$$\Phi(x) = \text{ess-sup}_{|y-x|<\frac{d(x)}{2}} \frac{\tilde{A}(y)}{B(y)}.$$

[H-3]. There is a positive number $\delta_0 > 0$ such that

$$\sup_{t \in \mathrm{R}\setminus\{0\}} \frac{|t|^{1+\delta_0}}{|Q(t)|} < +\infty, \text{ Super-linearlity.} \tag{2.5}$$

and

$$\liminf_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_{0<d(x)<\varepsilon} \tilde{A}(x) \left[\left(\frac{\Phi(x)}{d(x)^2} \right)^{\frac{1}{\delta_0}} + 1 \right] \frac{dx}{d(x)} < +\infty. \tag{2.6}$$

[H-4].

(2.7) $$\sup_{x\in\Omega}\frac{C(x)}{B(x)} < +\infty$$

Now we are able to state our main results.

Theorem 1. *Assume that* [H-1], [H-2], [H-3] *and* [H-4]. *Assume that* $u \in L^{\infty}_{\mathrm{loc}}(\Omega')$ *satisfies* $Pu \in L^{1}_{\mathrm{loc}}(\Omega')$ *in the distribution sense. Moreover we assume that for almost all* $x \in \{x \in \Omega'; u(x) \geq 0\}$,

(2.8) $$Pu + B(x)Q(u) \leq C(x).$$

Then we have $u_+ \in L^{\infty}_{\mathrm{loc}}(\Omega)$, *where* $u_+ = \max(u, 0)$.

The following is a direct consequence of this theorem. (The proof is omitted).

Corollary 1. *Assume that* [H-1], [H-2] *and* [H-3]. *Instead of* [H-4], *assume that* $f(x) \in L^{\infty}_{\mathrm{loc}}(\Omega') \cap L^{1}_{\mathrm{loc}}(\Omega)$ *satisfies for some positive number* C

(2.9) $|f(x)| \leq C\cdot B(x)$, *for almost all* $x \in \Omega$.

Assume that $u \in L^{\infty}_{\mathrm{loc}}(\Omega')$ *satisfies*

(2.10) $Pu + B(x)Q(u) = f$, *in* $D'(\Omega')$.

Then there exists a function $v \in L^{\infty}_{\mathrm{loc}}(\Omega)$ *such that*

(2.11) $$\begin{cases} Pv + B(x)Q(v) = f, & \text{in } D'(\Omega), \\ v|_{\Omega'} = u. \end{cases}$$

As an application we consider the Dirichlet boundary value problem for degenerate semilinear elliptic equation:

(2.12) $$\begin{cases} Pu + B(x)Q(u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Then we have.

Theorem 2. *Assume that* [H-1], [H-2] *and* [H-3]. *Instead of* [H-4] *assume that* $f(x) \in L^{\infty}(\bar{\Omega})$ *satisfies for some positive number* C

(2.13) $|f(x)| \leq C\cdot B(x)$, *for almost all* $x \in \Omega$.

Moreover we assume that $A(x)$, $B(x) \in C^0(\bar{\Omega})$. *Then there exists a unique function*

(2.14) $$u \in L^{\infty}(\Omega) \cap H^1_{\mathrm{loc}}(\bar{\Omega}\setminus F)$$

which satisfies (2.12) *in the distribution sense and satisfies*

(2.15) $$\int_{\Omega}[A(x)|\nabla u|^2 + B(x)Q(u)u]\,dx \leq C\|f/B\|^{\lambda}_{\infty}.$$

Here $\lambda = \dfrac{2+\delta_0}{1+\delta_0}$ *and* C *is a positive number independent of each function* f.

In the rest of this subsection we shall show that in certain respects Theorem 1 gives best possible results. Let F be either the origin 0 or an m-dimensional C^{∞} compact submanifolds in $\boldsymbol{R}^N$ with $0 < m \leq N-1$. We put $d(x) = dist(x, F)$ which is smooth near F.

(2.16) $L_{\alpha}u = -\operatorname{div}(d(x)^{2\alpha}\nabla u)$ and $Q(u) = |u|^{p-1}u$.

Assume that real numbers α, β and γ satisfy the following four conditions which correspond to [H-1]—[H-4] in §2.

(h-1) $\beta > -\dfrac{N-m}{2}$ and $\gamma > -\dfrac{N-m}{2}$.

Define

(2.17) $$p^*_m = \begin{cases} 1 + 2\dfrac{1-\alpha+\beta}{N+2\alpha-2-m}, & \text{if } \alpha < \beta+1, \\ 1, & \text{if } \alpha \geq \beta+1. \end{cases}$$

(h-2) $$\begin{cases} p \geq p^*_m, & \text{if } \alpha < \beta+1, \\ p > p^*_m = 1, & \text{if } \alpha \geq \beta+1, \\ \alpha > -\dfrac{N-m-2}{2}. \end{cases}$$

(h-3) $$\beta \leq \gamma.$$

Then we immediately have

Theorem 3. *Let* F *and* L_{α} *be as above. Assume that* (h-1), (h-2) *and* (h-3). *Assume that* $u \in L^{\infty}_{\mathrm{loc}}(\Omega')$ *satisfies* $L_{\alpha}u \in L^1_{\mathrm{loc}}(\Omega')$ *in the distribution sense. Moreover we assume that for almost all* $x \in \{x \in \Omega; u(x) \geq 0\}$

(2.18) $L_{\alpha}u + b(x)d(x)^{2\beta}u^p \leq c(x)d(x)^{2\gamma}$,

for some positive smooth functions $b(x)$ *and* $c(x)$. *Then we have* $u^+ \in L^{\infty}_{\mathrm{loc}}(\Omega)$.

Proof of Theorem 3. Since $Q(u) = |u|^{p-1}u$, we can put $\delta_0 = p-1$ to obtain (2.5). To apply Theorem 1, it suffices to show that the condition (2.6) in [H-3] is satisfied. But a direct calculation leads us to

(2.19)

$$\frac{1}{\varepsilon}\int_{0<d(x)<\varepsilon}\left(d(x)^{-\frac{2(1-\alpha+\beta)}{\delta_0}}+1\right)d(x)^{2\alpha-1}\,dx =$$
$$C'\,diam(F)^m\left(\varepsilon^{\frac{N+2\alpha-m-2}{p-1}\left(p-\frac{N+2\beta-m}{N+2\alpha-m-2}\right)}+\varepsilon^{2\alpha+N-m-2}\right) < \infty.$$

Next we show the sharpness of the condition (h-2). We consider $U(x) = d(x)^{-M}$ for $M > 0$ in a small neighborhood of F. Then we shall solve

(2.20) $L_{\alpha}U(x) + b(x)d(x)^{2\beta}U(x)^p = 0$, near F.

Here we remark that

(2.21) $|\nabla d(x)| = 1$ (near F) and $$\lim_{x\to F} d(x)\,\Delta d(x) = N-m-1.$$

Hence if we choose α, β, p, M and $b(x)$ so that

(2.22) $$\begin{cases} p = 1 + \dfrac{2(1-\alpha+\beta)}{M}, \\ M(d(x)\,\Delta d(x) + 2\alpha - 1 - M) + b(x) = 0, \\ M > N-m-2+2\alpha, \end{cases}$$

then U clearly satisfies (2.20), but unbounded near F.

3. Lemmas. We will state some preliminary facts that will be useful in the sequel.

Lemma 3.1 (Kato's inequality). *Assume that* $u \in L^1_{loc}(\Omega')$ *and* $Pu \in L^1_{loc}(\Omega')$. *Then we have*

(3.1) $Pu^+ \leq (Pu)sgn^+u$,

$$\text{in } D'(\Omega'), \text{ where } sgn^+u = \begin{cases} 1, & \text{for } u > 0, \\ 1/2, & \text{for } u = 0, \\ 0, & \text{for } u < 0. \end{cases}$$

Lemma 3.2 (Extension). *Assume that* [H-1], [H-2] *and* [H-3]. $f \in L^1_{loc}(\Omega)$ *and* $B(x) \cdot Q(u) \in L^1_{loc}(\Omega)$, *and assume that*

(3.2) $Pu \leq f, \quad \text{in } D'(\Omega')$.

Then we have

(3.3) $Pu \leq f, \quad \text{in } D'(\Omega)$.

Lemma 3.3 (Pointwise estimate). *Assume that* $u \in L^\infty_{loc}(\Omega')$ *satisfies* $Pu \in L^1_{loc}(\Omega')$ *in the distribution sense. Assume that* [H-1] — [H-4]. *Moreover we assume that for almost all* $x \in \{x \in \Omega\,;\, u(x) \geq 0\}$

(3.4) $Pu + B(x)Q(u) \leq C(x)$.

Then we have, for some positive numbers C *and* ε_0,

(3.5) $u(x) \leq C[\Phi(x)^{\frac{1}{\delta_0}} d(x)^{-\frac{2}{\delta_0}} + 1]$,

for x *with* $0 < d(x) \leq \varepsilon_0$.

Lemma 3.4 (Integrability). *Assume that* [H-1] — [H-3]. *Assume that* $u \in L^\infty_{loc}(\Omega')$ *satisfies* $Pu \in L^1_{loc}(\Omega')$ *in the distribution sense. Moreover we assume that for almost all* $x \in \{x \in \Omega\,;\, u(x) \geq 0\}$

(3.6) $Pu + B(x)Q(u) \leq C(x)$.

Then we have

(3.7) $B(x)Q(u^+) \in L^1_{loc}(\Omega)$.

4. A sketch of the proof of Theorem 1. From lemmas in the previous section we have for a sufficiently large $\mu > 0$,

(4.1) $P(u-\mu)_+ + B(x)sgn^+(u-\mu)(Q(u) - Q(\mu)) \leq 0, \quad \text{in } D'(\Omega)$.

Now we assume that without loss of generality $\{x : d(x) < 1\} \subset \Omega$ and $\mu \geq \sup_{1/2<d(x)<1} u(x)$. Moreover we assume that $(u-\mu)_+$ is smooth, because we can approximate it by a sequence of smooth functions in a standard way. Then we integrate (4.1) to obtain $u(x) \leq \mu$, for $d(x) < 1/2$. This proves the assertion.

5. A sketch of the proof of Theorem 2. We assume that $N > 1$ for simplicity. Since the uniqueness of solutions in $L^\infty(\Omega) \cap H^1_{loc}(\bar{\Omega} \setminus F)$ follows easily from the monotonicity of the operator, we will show the existence of solutions. Let $\{P_\varepsilon\}_{\varepsilon>0}$ be the ε-regulatization of P, namely $P_\varepsilon = -\operatorname{div}[(\varepsilon + A(x))\nabla \cdot]$. We consider the Dirichlet problem (2.12) with P replaced by P_ε. Then we can show that there is a unique solution $u_\varepsilon \in H^1_0(\Omega)$. For the detailed argument, see [1] for example. From Young's inequality we can also establish the estimate (2.15) for u_ε uniformly in $\varepsilon > 0$. Next by the method of a priori estimate and compactness, we derive a subsequence $\{u_{\varepsilon_j}\}_{j=1}^\infty$ from $\{u_\varepsilon\}_{\varepsilon>0}$ which converges weakly to some element $\bar{u} \in H^1_{loc}(\bar{\Omega} \setminus F)$. Since $\bar{u}$ satisfies (2.12) in $\Omega \setminus F$ in the sense of distribution, it follows from Theorem 1 and its corollary that there is a unique bounded function u satisfying (2.12) in Ω. This proves the asserti

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