25. Contiguity Relations of Generalized Confluent Hypergeometric Functions

By Yoshishige HARAOKA*) and Hironobu KIMURA**)

(Communicated by Kiyosi ITÔ, M. J. A., May 12, 1993)

In [3] we have introduced the generalized confluent hypergeometric function (CHG function, for short). It is a generalization of many special functions - in fact, by various specialization, it is reduced to the Gauss hypergeometric function, Appell-Lauricella hypergeometric functions F_D, Aomoto-Gelfand hypergeometric functions $F(k, n)$; the Kummer confluent hypergeometric function, the Bessel function, the Hermite function and the Airy function. Moreover by other specializations we can define new special functions in several variables of confluent type - e. g. the Kummer, Bessel, Hermite and Airy functions in several variables.

CHG function is a function of several complex variables with several complex parameters. Differential operators which send a CHG function to another CHG function are called *contiguity operators* if the parameters of the CHG functions differ by integers, and such linear differential relations between two CHG functions are called *contiguity relations.* Let us give two examples. For the Gauss hypergeometric function

$$F(\alpha, \beta, \gamma ; x) = \sum_{n=0}^{\infty} \frac{(\alpha, n)(\beta, n)}{(\gamma, n)(1, n)} x^n,$$

it is known that

$$\alpha F(\alpha+1, \beta, \gamma ; x) = \alpha F(\alpha, \beta, \gamma ; x) + x \frac{d}{dx} F(\alpha, \beta, \gamma ; x) \tag{0.1}$$

([2]). Then the differential operator

$$x \frac{d}{dx} + \alpha \tag{0.2}$$

sends $F(\alpha, \beta, \gamma ; x)$ to another hypergeometric function with parameters $(\alpha+1, \beta, \gamma)$. For the Bessel function

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{\nu+2n} n! \Gamma(\nu+n+1)} x^{\nu+2n},$$

it is known that

$$\frac{d}{dx} J_\nu(x) = \frac{\nu}{x} J_\nu(x) - J_{\nu+1}(x) \tag{0.3}$$

([9]). In this case the differential operator

$$-\frac{d}{dx} + \frac{\nu}{x} \tag{0.4}$$

sends a Bessel function with a parameter ν to one with a parameter $\nu+1$.

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*) Department of Mathematics, Kumamoto University.

**) Department of Mathematical Science, University of Tokyo.

Thus the relations (0.1) and (0.3) are contiguity relations of the Gauss hypergeometric function and the Bessel function, respectively. These relations are substantial for the study of special functions. In fact the study of the Lie algebras of contiguity operators yields many formulas for special functions ([5], [6]), and moreover, in a sense, the contiguity relations govern special functions ([5], [2]).

For Aomoto-Gelfand hypergeometric functions (which are CHG function of *nonconfluent* type) the contiguity relations have been obtained by T. Sasaki [8]. In this paper we shall obtain contiguity relations of general CHG function. The set of contiguity operators makes a Lie algebra. By specializations these contiguity relations yield contiguity relations of the Kummer functions, Bessel fuctions etc. of several variables.

§1. Generalized confluent hypergeometric functions. Let r and n be positive integers with $r < n$, and let $Z_{r,n}$ be the set of $r \times n$ matrices with complex entries of maximal rank. $GL(r, \boldsymbol{C})$ and $GL(n, \boldsymbol{C})$ act on $Z_{r,n}$ by the left and right matrix multiplication, respectively:

$$\begin{array}{ccc} GL(r, \boldsymbol{C}) \times Z_{r,n} \times GL(n, \boldsymbol{C}) & \longrightarrow & Z_{r,n} \\ (g, z, c) & \mapsto & gzc. \end{array}$$

Let H be a maximal commutative subgroup of $GL(n, \boldsymbol{C})$, and take a character χ of the universal covering group $\tilde{H}$ of H. We consider a function $F(z)$ on $Z_{r,n}$ which has the following covariance properties with respect to the above actions:

(1.1) $$F(gz) = (\det g)^{-1} F(z), \quad \text{for } g \in GL(r, \boldsymbol{C}),\ z \in Z_{r,n},$$

(1.2) $$F(zc) = F(z)\chi(c), \quad \text{for } z \in Z_{r,n},\ c \in H.$$

We denote by z_{ij} the (i, j)-entry of $z \in Z_{r,n}$ for $i = 1, \dots, r$ and for $j = 1, \dots, n$.

Definition 1. Let $F(z)$ be a multi-valued function on $Z_{r,n}$ satisfying (1.1) and (1.2). When $F(z)$ satisfies the differential equations

(1.3) $$\square_{i,j}^{p,q} F(z) := \left(\frac{\partial^2}{\partial z_{ip}\partial z_{jq}} - \frac{\partial^2}{\partial z_{iq}\partial z_{jp}}\right) F(z) = 0$$

for $i, j = 1, \dots, r$ and for $p, q = 1, \dots, n$, we call $F(z)$ *a generalized confluent hypergeometric function* (*CHG function,* for short) of type $(r, n\,; H)$.

We describe H and χ explicitly.

Definition 2. For a positive integer m, we define the Jordan group $J(m)$ of size m by

$$J(m) = \left\{c = \sum_{i=0}^{m-1} c_i \Lambda^i\,;\ c_i \in \boldsymbol{C},\ c_0 \neq 0\right\},$$

where

$$\Lambda = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix} \in M(m, \boldsymbol{C}).$$

$J(m)$ is a maximal commutative subgroup of $GL(m, \boldsymbol{C})$.

Let $b = 1 + b_1 T + b_2 T^2 + \cdots \in \boldsymbol{C}[[T]]$ be a formal power series in the indeterminate T. The polynomials $\theta_i(b_1,\ldots,b_i)$ $(i=1,2,\ldots)$ are defined by

$$\log b = \sum_{i=1}^{\infty} \theta_i(b_1,\ldots,b_i)T^i.$$

Proposition 1. *For any character χ of $\tilde{J}(m)$, there are $\alpha_0,\ldots,\alpha_{m-1} \in \boldsymbol{C}$ such that*

$$\chi\left(\sum_{i=0}^{m-1} c_i\tau^i\right) = c_0^{\alpha_0} \exp\left(\sum_{i=1}^{m-1} \alpha_i\theta_i(c_1/c_0,\ldots,c_i/c_0)\right).$$

We denote the above χ by $\chi_{m,\alpha}$ with $\alpha = (\alpha_0, \alpha_1,\ldots,\alpha_{m-1})$.

Proposition 2. *For any maximal commutative subgroup H of $GL(n, \boldsymbol{C})$, there is a partition $\lambda = (\lambda_1, \lambda_2,\ldots,\lambda_l)$ of n (i.e. $\lambda_1,\ldots,\lambda_l \in \boldsymbol{N}$, $0<\lambda_1 \le \lambda_2 \le \cdots \le \lambda_l$, $\lambda_1+\lambda_2+\cdots+\lambda_l = n$) such that H is conjugate to the direct product*

$$H_\lambda = J(\lambda_1) \times J(\lambda_2) \times \cdots \times J(\lambda_l).$$

We remark that any character of $\tilde{H}_\lambda$ is a product of each character of $\tilde{J}(\lambda_i)$, $i=1,\ldots,l$.

Noting that $J(1) = \boldsymbol{C}^\times$, we can rewrite the above H_λ as

$$(\boldsymbol{C}^\times)^{\mu_0} \times J(\mu_1) \times \cdots \times J(\mu_k), \tag{1.4}$$

where $\mu_0, \mu_1,\ldots,\mu_k \in \boldsymbol{Z}$, $\mu_0 \ge 0$, $1<\mu_1 \le \cdots \le \mu_k$, and $\mu_0+\mu_1+\cdots+\mu_k = n$. We denote (1.4) also by H_μ with $\mu = (\mu_0, \mu_1,\ldots,\mu_k)$.

We fix an H_μ, and consider a CHG function of type $(r, n\,; H_\mu)$. As we have remarked above, any character χ is given by

$$\chi = \chi_\alpha = \prod_{j=1}^{\mu_0} \chi_{1,\alpha_j^{(0)}} \prod_{p=1}^{k} \chi_{\mu_p,\alpha^{(p)}},$$

where

$$\alpha_j^{(0)} \in \boldsymbol{C}, \quad j=1,\ldots,\mu_0,$$
$$\alpha^{(p)} = (\alpha_0^{(p)},\ldots,\alpha_{\mu_0-1}^{(p)}) \in \boldsymbol{C}^{\mu_p},\ p=1,\ldots,k,$$
$$\alpha = (\alpha_1^{(0)},\ldots,\alpha_{\mu_0}^{(0)}, \alpha^{(1)},\ldots,\alpha^{(k)}) \in \boldsymbol{C}^n.$$

For the compatibility of (1.1) and (1.2), we assume

$$\sum_{j=1}^{\mu_0} \alpha_j^{(0)} + \sum_{p=1}^{k} \alpha_0^{(p)} = -r. \tag{1.5}$$

We set

$$\mu_0+\mu_1+\cdots+\mu_i =: \nu_i$$

for $i=0,1,\ldots,k$; in particular we have

$$\nu_0 = \mu_0,\ \nu_k = \mu_0+\mu_1+\cdots+\mu_k = n.$$

Now we interpret the covariance properties (1.1) and (1.2) into infinitesimal expressions. The $GL(r, C)$ covariance (1.1) is interpreted as

$$\sum_{p=1}^{n} z_{ip}\frac{\partial}{\partial z_{jp}}F(z) = -\delta_{ij}F(z),\ i,j=1,\ldots,r. \tag{1.6}$$

The H_μ covariance (1.2) is interpreted as

$$\sum_{q=1}^{r} z_{qj}\frac{\partial}{\partial z_{qj}}F(z) = \alpha_j^{(0)}F(z),\ j=1,\ldots,\nu_0,$$

$$\sum_{q=1}^{r}\sum_{t=j}^{\nu_1} z_{q,t-j+(\nu_0+1)}\frac{\partial}{\partial z_{qt}}F(z) = \alpha_{j-(\nu_0+1)}^{(1)}F(z),\ \nu_0+1 \le j \le \nu_1, \tag{1.7}$$

$$\sum_{q=1}^{r}\sum_{t=j}^{\nu_2} z_{q,t-j+(\nu_1+1)}\frac{\partial}{\partial z_{qt}}F(z) = \alpha^{(2)}_{j-(\nu_1+1)}F(z),\ \nu_1+1\le j\le \nu_2,$$

$$\cdots\cdots\cdots$$

$$\sum_{q=1}^{r}\sum_{t=j}^{n} z_{q,t-j+(\nu_{k-1}+1)}\frac{\partial}{\partial z_{qt}}F(z) = \alpha^{(k)}_{j-(\nu_{k-1}+1)}F(z),\ \nu_{k-1}+1\le j\le n.$$

Theorem 1. *The system of differential equations* (1.6), (1.7) *and* (1.3) *is completely integrable.*

We call the system (1.6), (1.7), (1.3) the *generalized confluent hypergeometric system* (*CHG system,* for short) of type $(r, n ; H_\mu)$ with parameters α, and denote it by $E(r, n ; H_\mu ; \alpha)$. The left ideal of the Weyl algebra $\boldsymbol{C}[z_{ij}, \partial/\partial z_{ij} ; i=1,\dots,r ; j=1,\dots,n]$ generated by the differential operators

$$M_{ij} := \sum_{p=1}^{n} z_{ip}\frac{\partial}{\partial z_{jp}} + \delta_{ij},\quad i,j=1,\dots,r,$$

$$L_{ij}(\alpha) := \sum_{q=1}^{r}\sum_{t=j}^{\nu_i} z_{q,t-j+(\nu_{i-1}+1)}\frac{\partial}{\partial z_{qt}} - \alpha^{(i)}_{j-(\nu_{i-1}+1)},$$

$$i=1,\dots,k,\ \nu_{i-1}+1\le j\le \nu_i,$$

$$\square_{ij}^{pq} := \frac{\partial^2}{\partial z_{ip}\partial z_{jq}} - \frac{\partial^2}{\partial z_{iq}\partial z_{jp}},\ i, j=1,\dots,r,\ p, q=1,\dots,n,$$

of $E(r, n ; H_\mu ; \alpha)$ is denoted by $\mathscr{L}(r, n ; H_\mu ; \alpha)$, or simply $\mathscr{L}(\alpha)$. The solution space of $E(r, n ; H_\mu ; \alpha)$ is denoted by $S(r, n ; H_\mu ; \alpha)$, or simply $S(\alpha)$.

§2. Contiguity relations of CHG function. We fix a H_μ given by (1.4), and consider a CHG system of type $(r, n ; H_\mu)$.

Definition 3. A linear differential operator P is called *a contiguity operator,* if there is a $\zeta \in \boldsymbol{Z}^n$ such that

(2.1) $$PS(\alpha)\subset S(\alpha+\zeta)$$

holds.

Proposition 3. *For a linear differential operator P,* (2.1) *holds if and only if*

(2.2) $$LP\in\mathscr{L}(\alpha)$$

holds for every $L\in\mathscr{L}(\alpha+\zeta)$.

We denote by e_i the element in $\boldsymbol{Z}^n$ with the only non-zero entry 1 in the i-th position, for $i=1,\dots,n$. We define a mapping

$$\pi : \{1,2,\dots,n\}\to \boldsymbol{Z}$$

by

$$\pi(j) = j \quad\text{if}\quad 1\le j\le \nu_0,$$
$$\pi(j) = \nu_{p-1}+1 \quad\text{if}\quad \nu_{p-1}+1\le j\le \nu_p\ (p=1,\dots,k).$$

Our main result is the following.

Theorem 2. *Let $a, b\in\{1,2,\dots,n\}$ satisfy*

(2.3) $$a\in\{1,\dots,\nu_0,\ \nu_0+1,\ \nu_1+1,\dots,\nu_{k-1}+1\},\quad b\in\{1,\dots,\nu_0,\ \nu_1,\ \nu_2,\dots,\nu_k\}.$$

Set

$$P_{ab} = \sum_{q=1}^{r} z_{qa}\frac{\partial}{\partial z_{qb}}.$$

Then we have

$$LP_{ab}\in\mathscr{L}(\alpha)$$

for any $L\in\mathscr{L}(\alpha+e_{\pi(a)}-e_{\pi(b)})$. Thus P_{ab} is a contiguity of CHG function of *type* $(r, n ; H_\mu)$.

We set

$$N_1 := \{1,\dots,\nu_0,\ \nu_0+1,\ \nu_1+1,\dots,\nu_{k-1}+1\},$$
$$N_2 := \{1,\dots,\nu_0,\ \nu_1,\ \nu_2,\dots,\nu_k\}.$$

Let $\mathscr{G}$ be a $\boldsymbol{C}$-linear space spanned by P_{ab} with a,b satisfying (2.3):

$$\mathscr{G} = \langle P_{ab} \,;\, a \in N_1,\ b \in N_2\rangle.$$

Lemma 1.

$$[P_{ab}, P_{cd}] = \delta_{bc}P_{ad} - \delta_{ad}P_{cb}$$

for $a, c \in N_1$, $b, d \in N_2$.

Thus we obtain

Theorem 3. *$\mathscr{G}$ makes a Lie algebra over $\boldsymbol{C}$.*

We define a subspace $\mathscr{G}_0$ of $\mathscr{G}$ by

$$\mathscr{G}_0 = \langle P_{ab} \,;\, a,\ b \in \{1,2,\dots,\mu_0\}\rangle.$$

It is shown by Sasaki [8] that $\mathscr{G}_0$ is isomorphic to the Lie algebra gl_{μ_0} of general linear matrics. By the definition it is easy to see that P_{ab} belongs to the center of $\mathscr{G}$ if $\pi(a) > \mu_0$ and $\pi(b) > \mu_0$. Summing up the above, we have

Proposition 4. (i) *When $\mu_0 = n$, $\mathscr{G}$ is isomorphic to gl_n.*

(ii) *When $\mu_0 = 0$, $\mathscr{G}$ is abelian.*

§3. Example. Contiguity relations of the Bessel function. The Bessel function is obtained from the CHG function of type $(2,4 \,;\, J(2) \times J(2))$ by the following specialization. For

$$z = \begin{pmatrix} z_{11} & z_{12} & z_{13} & z_{14} \\ z_{21} & z_{22} & z_{23} & z_{24} \end{pmatrix} \in Z_{2,4},$$

we set

$$g = \begin{pmatrix} z_{11} & z_{13} \\ z_{21} & z_{23} \end{pmatrix},$$

and we assume $g \in GL(2, C)$. Set

$$g^{-1}\begin{pmatrix} z_{12} & z_{14} \\ z_{22} & z_{24} \end{pmatrix} =: \begin{pmatrix} u_{12} & u_{14} \\ u_{22} & u_{24} \end{pmatrix},$$

and assume $u_{22} \neq 0$. Then, setting

$$h = \begin{pmatrix} 1 & \\ & u_{22} \end{pmatrix},\ c = \begin{pmatrix} 1 & -u_{12} & & \\ & 1 & & \\ & & u_{22} & -u_{22}u_{24} \\ & & & u_{22} \end{pmatrix},$$

we have $h \in GL(2, C)$ and $c \in J(2) \times J(2)$. We have

$$h^{-1}g^{-1}zc = \begin{pmatrix} 1 & 0 & 0 & x \\ 0 & 1 & 1 & 0 \end{pmatrix}, \tag{3.1}$$

where

$$x = u_{14}u_{22}.$$

We denote the right hand side of (3.1) also by x. Let $F(\alpha \,;\, z)$ be a CHG function of type $(2,4 \,;\, J(2) \times J(2))$ with parameter α. Then by the covariance (1.1) and (1.2) we have

$$F(\alpha \,;\, x) = \det h \cdot \det g \cdot F(\alpha \,;\, z) \cdot \chi_\alpha(c). \tag{3.2}$$

We denote $F(\alpha \,;\, x)$ by $f(\alpha \,;\, x)$ as a function of x. Then from the differential equations (1.3) we derive a differential equation for $f(\alpha \,;\, x)$:

$$xf''(\alpha\,;x)-\alpha_3 f'(\alpha\,;x)-\alpha_2\alpha_4 f(\alpha\,;x)=0. \tag{3.3}$$

By the change of the independent variable

$$x=\frac{\xi^2}{4}$$

and the gauge transformation

$$f(\alpha\,;x)=\xi^{\alpha_3+1}\varphi(\alpha\,;\xi), \tag{3.4}$$

we obtain the differential equation for $\varphi(\alpha\,;\xi)$:

$$\varphi''(\alpha\,;\xi)+\frac{1}{\xi}\varphi'(\alpha\,;\xi)+\left\{-\alpha_2\alpha_4-\frac{(\alpha_3+1)^2}{\xi^2}\right\}\varphi(\alpha\,;\xi)=0. \tag{3.5}$$

If we fix α_2 and α_4 so that $-\alpha_2\alpha_4=1$, (3.5) is just the Bessel differential equation with parameter α_3+1.

Tracing the above process of specialization, we obtain from the contiguity operators

$$P_{32}=z_{13}\frac{\partial}{\partial z_{12}}+z_{23}\frac{\partial}{\partial z_{22}},$$
$$P_{14}=z_{11}\frac{\partial}{\partial z_{14}}+z_{21}\frac{\partial}{\partial z_{24}}$$

of $F(\alpha\,;z)$ the contiguity relations of the Bessel function. Namely, noting (3.2) and (3.4), we obtain

$$\left(-\frac{d}{d\xi}+\frac{\alpha_3+1}{\xi}\right)\varphi(\alpha\,;\xi)=C\varphi(\alpha+e_3\,;\xi),$$
$$\left(\frac{d}{d\xi}+\frac{\alpha_3+1}{\xi}\right)\varphi(\alpha\,;\xi)=C\varphi(\alpha-e_3\,;\xi),$$

where C denotes an appropriate constant.

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