56. *On the Extremal Ray of Higher Dimensional Varieties*

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The purpose of this note is to outline our recent results concerning the structure of the contraction of an extremal ray. Details will be published elsewhere.

Let X be a non-singular projective variety with $\dim X=n$ over an algebraically closed field k of characteristic zero. We fix the following notation.

$N^1(X):=(\{\text{Cartier divisors on } X\}/\approx)\otimes_{\boldsymbol{Z}}\boldsymbol{R}$

$N_1(X):=(\{\text{1-cycles on } X\}/\approx)\otimes_{\boldsymbol{Z}}\boldsymbol{R}$

$\overline{NE}(X):=$ the closure of the closed convex cone generated by effective 1-cycles in $N_1(X)$.

Here the symbol $\approx$ denotes the numerical equivalence.

A Cartier divisor D is called *numerically effective* or simply *nef* if $(D\cdot C)_X\geqq 0$ for all curves C on X.

A 1-cycle Z is also said to be *numerically effective* or *nef* if $(D\cdot Z)_X\geqq 0$ for any effective Cartier divisors D.

The *numerical Kodaira dimension* of a nef Cartier divisor D is defined by $\kappa_{\text{num}}(D):=\max\{d\,|\,D^d\not\approx 0\}$. Then $\kappa(D)\leqq\kappa_{\text{num}}(D)\leqq n=\dim X$.

A Cartier divisor is called *big* if $\kappa(D,X)=n$.

A linear system is called *free* if it has neither fixed components nor base points.

Definition. A curve C on X is called *extremal* if

(i) $(K_X\cdot C)<0$, and

(ii) given $A, B\in\overline{NE}(X)$, $A, B\in\boldsymbol{R}_+[C]$ if $A+B\in\boldsymbol{R}_+[C]$.

Definition. Let C be an extremal curve. A Cartier divisor H is called a *good supporting divisor* with respect to C, if

(i) H is nef,

(ii) for $Z\in\overline{NE}(X)$, $(K_X\cdot Z)_X=0$ if and only if $Z\in\boldsymbol{R}_+[C]$.

Definition (see Fujita [1]). A Gorenstein projective variety with $\dim X\geqq 3$ is called a *Del Pezzo variety* if

(i) there exists an ample Cartier divisor L such that $-K_X\sim(n-2)L$,

(ii) $H^i(X,tL)=0$ for all $t\in\boldsymbol{Z}$, $0<i<n$.

Definition (see Mukai [6]). A Gorenstein projective variety X with $\dim X\geqq 4$ is called a *Mukai variety* if

(i) there exists an ample Cartier divisor L such that $-K_X \sim (n-3)L$,

(ii) $H^i(X, tL)=0$ for all $t \in \boldsymbol{Z}$, $0<i<n$.

Assume that the canonical divisor K_X of X is not nef. By the following theorem, we have an extremal curve C and a good supporing divisor H with respect to C.

Cone theorem (Kawamata [4], Mori [5], János Kollár [3]). *Assume that X has only canonical singularities. Fix an ample divisor L. Then for any $\varepsilon>0$, there exist entremal curves $C_1, \cdots, C_r$ such that*

$$\overline{NE}(X)=\sum \boldsymbol{R}_+[C_i]+\overline{NE}_\varepsilon(X).$$

Here $\overline{NE}_\varepsilon(X):=\{Z \in \overline{NE}(X) \mid (K_X \cdot Z)>-\varepsilon(L \cdot Z)\}$.

By the following theorem, $|mH|$ is free for $m \gg 0$. The morphism $f: X \to Y$ associated with $|mH|$ for $m \gg 0$ is called the *contraction* of $\boldsymbol{R}_+[C]$. We study the structure of f.

Base point free theorem (Shokurov [7], Kawamata [4]). *Let X be a projective variety with only canonical singularities. Assume that a Cartier divisor H is nef and that $aH-K_X$ is nef and big for some $a \in \boldsymbol{N}$. Then $|mH|$ is free, for $m \gg 0$.*

The following Corollary is a consequence of above two theorems.

Corollary. *Given any extremal curve C, there exists a good supporting divisor H, which satisfies*

(i) *$|mH|$ is free for any $m \gg 0$,*

(ii) *if E is a Cartier divisor such that $(E \cdot C)_X>0$, then for $m \gg 0$, $mH+E$ is ample. Especially, $mH-K_X$ is ample. Moreover if X is non-singular, then $H^i(X, mH+E)=0$ and $H^i(X, mH)=0$ for $i>0$ and $m \gg 0$.*

First assume f to be birational. Let $E \subset X$ be the exceptional set of f. It is easy to see that $\dim E \geqq 2$. Note that if $\dim E=n-1$, then E is a prime divisor and further that $(E \cdot C)_X<0$.

Theorem 1. *Assume that* $\dim E=n-1$. *Let F be a general fiber of $f_E: E \to f(E)$ (note that if* $\dim f(E)=0$, *then $F=E$). Then there exists a Cartier divisor L on X such that*

(i) $\operatorname{Im}(\operatorname{Pic} X \to \operatorname{Pic} F)=\boldsymbol{Z}[L|_F]$ *and $L|_F$ is ample on F.*

(ii) *$\mathcal{O}_F(-K_X) \cong \mathcal{O}_F(pL)$ and $\mathcal{O}_F(-E) \cong \mathcal{O}_F(qL)$ for some $p, q \in \boldsymbol{N}$.*

(iii) *$H^i(F, tL)=0$ for $0<i<$* $\dim F$ *unless $-q<t<-p$. Especially $H^i(F, tL)=0$ for all $t \in \boldsymbol{Z}$ if* $\dim F \leqq 4$.

By these properties, we can classify F in the lower dimensional cases as follows.

(a) If $\dim F=1$, $F \cong \boldsymbol{P}^1$.

(b) If $\dim F=2$, F is $\boldsymbol{P}^2$ of Q^2.

(c) If $\dim F=3$, F is $\boldsymbol{P}^3$, Q^3, or a Del Pezzo 3-fold.

(d) If $\dim F=4$, F is $\boldsymbol{P}^4$, Q^4, a Del Pezzo 4-fold or a Mukai 4-fold.

Theorem 2. *If $\dim f(E)=\dim E-1=n-2$ and f_D is equi-dimensional then both Y and $f(E)$ are non-singular and moreover $f: X\to Y$ is the blowing up along a smooth center $f(E)$.*

Next we consider the case $\dim X>\dim Y$.

Theorem 3. (i) *A general fiber of f is a Fano r-fold, where $r:=\dim X-\dim Y$.*

(ii) *If $\dim Y=n-1$ and if f is equi-dimensional, then Y is non-singular and f induces a conic bundle structure on X.*

Sketchy proof of Theorem 1. Since $|mH|$ is free and $H|_F\approx 0$, it follows that $\mathcal{O}_F(H)\cong\mathcal{O}_E$. Thus any curve Z in F belongs to $\boldsymbol{R}_+[C]$. This means that $\operatorname{rank}(\operatorname{Im}(N_1(F)\to N_1(X)))=1$. Take $M\in\operatorname{Pic}X$ such that $M|_F\approx 0$. Since both $mH+M-K_X$ and $mH+M-E-K_X$ are ample for $m\gg 0$ in X, we have $H^i(X, mH+M-K_X)=0$ and $H^i(X, mH+M-E-K_X)=0$ for $i>0$. By the standard exact sequence $0\to\mathcal{O}_X(mH+M-E)\to\mathcal{O}_X(mH+M)\to\mathcal{O}_E(mH+M)\to 0$, we have $H^i(\mathcal{O}_E(mH+M))=0$ for any $i>0$. Let $A_1,\cdots,A_b$ $(b:=\dim f(E))$ be ample divisors in Y such that $f(E)\cap A_1\cap\cdots\cap A_b\ni f(F)$. Repeating this process we have $H_i(\mathcal{O}_{E\cap H_1\cap\cdots\cap H_b}(mH+M))=0$ for $i>0$. On the other hand, since F is a general fiber, $F=H_1\cap\cdots\cap H_b$. Noting that $\mathcal{O}_F(H)\cong\mathcal{O}_F$, we have $H^i(\mathcal{O}_F(M))=0$ for $i>0$. Therefore $h^0(\mathcal{O}_F(M))=\chi(\mathcal{O}_F(M))=\chi(\mathcal{O}_F)=h^0(\mathcal{O}_E)=1$. So $\mathcal{O}_F(M)\cong\mathcal{O}_F$. This implies that $\operatorname{Im}(\operatorname{Pic}X\to\operatorname{Pic}F)\cong\boldsymbol{Z}$. Let $L\in\operatorname{Pic}X$ be a generator of $\operatorname{Im}(\operatorname{Pic}X\to\operatorname{Pic}F)$ such that $L|_F$ is ample. Then $\mathcal{O}_F(-K_X)\cong\mathcal{O}_F(pL)$ and $\mathcal{O}_F(-E)\cong\mathcal{O}_F(qL)$ for some $p, q\in\boldsymbol{N}$. By the adjunction formula $\omega_F\cong\mathcal{O}_F((-p-q)L)$. By making use of the above exact sequences and Kawamata vanishing theorem, we conclude

$$H^i(F, tL)=0 \quad\text{for}\quad i>0,\quad t\geqq -p \quad\text{and}\quad i<\dim F,\quad t\leqq -q.$$

Let $r:=\dim F$ and $\mathrm{d}:=(L^r)_F$. Now we classify F in the cases of $r\leqq 4$. Define a polynomial by

$$P(t):=\chi(\mathcal{O}_F(tL))=\frac{d}{r!}t^r+\frac{(p+q)d}{2(r-1)!}t^{r-1}+\text{lower term in } t.$$

Then by Serre duality, $P(-t)=(-1)^r P(t-p-q)$. $P(0)=\chi(\mathcal{O}_F)=1$. By (iii), $P(t)=0$ for any t such that $-p\leqq t<0$. By these we have

(a) if $r=1$, $P(t)=dt+1$,

(b) if $r=2$, $P(t)=(d/2)t(t+p+q)$,

(c) if $r=3$, $P(t)=(d/12)t(t+p+q)(2t+p+q)+(2t/(p+q))+1$,

(d) if $r=4$, $P(t)=(1/24)\{t^2(t+p+q)^2d+t(t+p+q)(pqd+(24/pq))\}+1$.

By computing Δ-genera of the pair (F, L) defined by $\Delta(F,\mathrm{L})=(n-1)+d-P(1)$ (see Fujita [1]), we can classify (F, L). □

The proofs of Theorems 2 and 3 use the following

Lemma 4. *Let X be a non-singular projective variety of dimen-*

sion n. Further let C be an irreducible curve on X such that

(i) $(K_X \cdot C) < 0$.

(ii) $\chi(\mathcal{O}_{C'}) \geqq 0$ *for any subscheme* C' *in* X *with* $(C')_{\mathrm{red}} = C$. *Then* $C \cong \boldsymbol{P}^1$ *and* $N_{C/X}$ *are classified into the following four cases*:

(1) $N_{C/X} \cong \mathcal{O}_C^{n-1}$,

(2) $N_{C/X} \cong \mathcal{O}_C(-1) \oplus \mathcal{O}_C^{n-2}$,

(3) $N_{C/X} \cong \mathcal{O}_C(1) \oplus \mathcal{O}_C(-2) \oplus \mathcal{O}_C^{n-3}$,

(4) $N_{C/X} \cong \mathcal{O}_C(1) \oplus \mathcal{O}_C(-1)^2 \oplus \mathcal{O}_C^{n-4}$.

Moreover, in the cases of (3) *and* (4), *letting* $J \subset \mathcal{O}_X$ *be an ideal such that* $I_C \supset J \supset I_C^2$ *and* $I_C/J \cong \mathcal{O}_C(-1)$, *we have* $J/J^2 \cong (\mathcal{O}_X/J)^{n-1}$, *where* I_C *is the ideal of* $\mathcal{O}_X$ *defining* C *in* X.

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