55. Accretivity and Duality Map in Banach Space

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(Communicated by Kôsaku YOSIDA, M. J. A., June 12, 1984)

Introduction. Let X be a real Banach space with the dual space X^*. Let F be the *duality map* on X to X^*: for every $x \in X$

$$F(x)=\{f \in X^* : (x, f)=\|x\|^2=\|f\|^2\}.$$

A multi-valued operator A in X is said to be *accretive* if

(M) for any $[x, x'], [y, y'] \in A$, there exists $f \in F(x-y)$ such that $\quad (x'-y', f)\geq 0,$

or equivalently if

(A) for any $[x, x'], [y, y'] \in A$ and $t>0$

$$\|(x-y)+t(x'-y')\|\geq\|x-y\|,$$

where $[x, x'] \in A$ means that $x \in D(A)$ and $x' \in Ax$.

The following lemma, which was first proved by T. Kato [3], implies the equivalence of two definitions.

Lemma (Kato [3]). *Let x and y be elements of X. Then*

(M)′ *there exists $f \in F(x)$ such that $(y, f)\geq 0$, if and only if*

(A)′ $\quad \|x+ty\|\geq\|x\| \quad$ *for any $t>0$.*

The purpose of this note is to give a new proof of the "if" part of this lemma. We use only the Hahn-Banach separation theorem, while the proof in [3] is based on the Banach-Alaoglu theorem.

§ 1. Proof of Kato's Lemma.

Lemma 1.1. *Let x be an element of X and let K be a convex subset in X which contains 0. Then the following two assertions are equivalent:*

(1.1) *there exists $f \in F(x)$ such that*

$$(y, f)\geq 0 \qquad \text{for all } y \in K,$$

(1.2) $\quad \|x+y\|\geq\|x\| \quad$ *for all $y \in K$.*

If in addition K is a linear subspace then (1.2) *holds if and only if*

(1.3) *there exists $f \in F(x)$ such that*

$$(y, f)=0 \qquad \text{for all } y \in K.$$

Proof. If $x=0$ then there is nothing to prove, so we may assume that $x\neq 0$. Let $f \in F(x)$ be as in (1.1). Then for every $y \in K$, we have

$$\|x+y\|^2-\|x\|^2\geq 2(x+y-x, f)\geq 0,$$

which implies (1.2).

Conversely, let us assume that (1.2) holds. Let B be the open ball about 0 with the radius $\|x\|$. Then $K+x$ and B are mutually disjoint convex subsets of X. By Hahn-Banach separation theorem (see e.g.

[5], p. 29, Proposition 5) there exist a non-zero $f \in X^*$ and a real number r such that

a) $\|f\| = \|x\|$;

b) $(z, f) \leq r$ for all $z \in B$;

c) $(x+y, f) \geq r$ for all $y \in K$.

Since $0 \in K$, we have

$$\|x\| \cdot \|f\| \leq r \leq (x, f) \leq \|x\| \cdot \|f\|$$

from b) and c). This together with a) implies $f \in F(x)$ and

$$(y, f) \geq r - (x, f) = 0 \qquad \text{for all } y \in K,$$

which completes the proof of Lemma 1.1.

Kato's lemma is nothing but Lemma 1.1 in which K is a half-line with the end-point 0 and the direction y.

Lemma 1.1 was first noticed by M. G. Crandall and A. Pazy [1], however they assumed the uniform convexity of X^*. When K is a linear subspace, it was proved by R. C. James [2]. His proof was based on the Hahn-Banach extension theorem.

Definition 1.2. The *directional derivative* $G_+(y, x)$ *of norm* at x in the direction of y is defined by

$$G_+(y, x) = \lim_{t \downarrow 0} [\|x+ty\| - \|x\|]/t = \inf_{t>0} [\|x+ty\| - \|x\|]/t.$$

The latter equality is a consequence of the convexity of $\|x+ty\|$ with respect to t.

Corollary 1.3. *Let x and y be elements in X. Then*

$$\|x\| \cdot G_+(y, x) = \max\{(y, f) : f \in F(x)\}. \tag{1.4}$$

Proof. If $x=0$ then the equality is trivial. We consider the case of $x \neq 0$. Let f be an arbitrary element in $F(x)$. Then for any $t>0$,

$$\|x\| \cdot \|x+ty\| - \|x\|^2 \geq (x+ty-x, f) = t(y, f).$$

This implies the left hand side of (1.4) is greater than (y, f) for all $f \in F(x)$. On the other hand, from the definition of G_+,

$$\|x + t(y - \beta x)\| \geq \|x+ty\| - t\beta\|x\| \geq \|x\|$$

for any $t>0$, where $\beta = G_+(y, x)/\|x\|$. By Lemma 1.1 there exists $g \in F(x)$ such that $(y - \beta x, g) \geq 0$. Then we have

$$\|x\| \cdot G_+(y, x) \leq (y, g).$$

This completes the proof.

Definition 1.4. Let K be a subset of X. Then the *metric projection* P_K of K is a multi-valued map from X to K defined by

$$P_K(x) = \{y \in K : \|x-y\| = d(x, K)\}$$

for every $x \in X$, where $d(x, K) = \inf\{\|x-z\| : z \in K\}$.

Corollary 1.5. *Let K be a closed convex subset of X and let x be an element in X. Then the following assertions are equivalent:*

(1.5) $y \in P_K(x)$,

(1.6) $y \in K$ *and there is* $f \in F(y-x)$ *such that*

$$(z-y, f) \geq 0 \qquad \text{for all } z \in K,$$

(1.7) *$y \in K$ and there is $f \in F(y-x)$ such that*

$$(z-y', f) \geq 0 \qquad \text{for all } z \in K \quad \text{and} \quad y' \in P_K(x).$$

Proof. It is an immediate consequence of Lemma 1.1 that (1.5) implies (1.6) and (1.7) implies (1.5). It suffices to show that (1.6) implies (1.7). Let y' be an element in $P_K(x)$ and let $f \in F(y-x)$ be as in (1.6). Then we have

$$0 \geq \|z-x\|^2 - \|y-x\|^2 \geq (y'-x-y+x, f) \geq 0.$$

Therefore

$$(z-y', f) = (z-y, f) + (y-y', f) \geq 0$$

for any $z \in K$. This completes the proof.

If X is a Hilbert space, this corollary is already known (see [4], p. 9, Theorem 2.3).

§ 2. 1-complemented subspace in Banach space.

Definition 2.1. A closed linear subspace E of a Banach space X is said to be 1-*complemented* if there exists a norm-one projection P (i.e. continuous linear operator with the property $P^2=P$) on X to E. In this case, X is the direct sum of E and the null space $N(P)$ of P.

Let A be a multi-valued operator from X to Y and let E be a subset in $D(A)$, where X and Y are normed spaces. Then a map S from E to Y is called a *selection* of A on E if $S(x) \in A(x)$ for any $x \in E$.

Proposition 2.2. *Let E be a closed subspace of X and let F be the duality map on X to X^*. Then the following assertions are equivalent:*

(2.1) *E is a* 1-*complemented subspace of X,*

(2.2) *$E+S(E)^{\perp}=X$ for some selection S of F on E,*

(2.3) *there exists a dense subspace G of E such that $G+T(G)^{\perp}$ is dense in X for some selection T of F on G,*

where $M^{\perp}=\{f \in X^* : (x, f)=0$ for any $x \in M\}$ for arbitrary subset M of X.

Proof. We see from Lemma 1.1 that (2.1) implies (2.2). Since the implication of (2.2) to (2.3) is trivial, it remains to show that (2.3) implies (2.1). Suppose that (2.3) holds. Then we have

1) $\|y+z\| \geq \|y\|$ for all $y \in E$ and $z \in T(G)^{\perp}$.

In fact, let $y \in E$. Then there exists a sequence $\{y_n\}$ in G which converges to y. We have $\|y_n+z\| \geq \|y_n\|$ for any $n \in N$ from Lemma 1.1. Going to limit $n \to +\infty$, we obtain 1).

Let $x \in X$. Then there exists a sequence $\{x_n\}$ which converges to x. By assumption we can write $x_n = y_n + z_n$ for some $\{y_n\}$ in G and $\{z_n\}$ in $T(G)^{\perp}$. Noting that $y_n - y_m \in E$ and $z_n - z_m \in T(G)^{\perp}$, it follows from 1)

$$\|x_n - x_m\| = \|(y_n - y_m) + (z_n - z_m)\| \geq \|y_n - y_m\|$$

for all n and m in N. Since $\{x_n\}$ is a Cauchy sequence, so are $\{y_n\}$ and

$\{z_n\}$. Then $\{y_n\}$ and $\{z_n\}$ converge to some y in E and z in $T(G)^{\perp}$, respectively. Thus for every $x \in X$, there exist $y \in E$ and $z \in T(G)^{\perp}$ such that $x = y + z$. It is easily obtained from 1) that this representation is unique. Setting $Px = y$, we obtain a projection P on X to E. It follows from 1) that $\|P\| = 1$. This completes the proof.

Acknowledgement. The author wishes to express his sincere gratitude to Prof. T. Shibata and Prof. N. Okazawa for their advice and encouragement.

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