52. *Index, Localization and Classification of Characteristic Surfaces for Linear Partial Differential Operators*

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Let $P(z,\partial)$ be a linear partial differential operator of order m with holomorphic coefficients in $\Omega\subset C^{n+1}$, and K be a connected nonsingular hypersurface in Ω. Characteristic indices and the localization on K of $P(z,\partial)$ are defined by means of a special coordinate in [6]. In the present paper we give another definition of them, new notions and a classification of characteristic surfaces of $P(z,\partial)$.

§1. **Definitions.** $z=(z_0,z_1,\cdots,z_n)=(z_0,z')$ denotes a point in C^{n+1}, $\partial=(\partial_0,\partial_1,\cdots,\partial_n)=(\partial_0,\partial')$, $\partial_i=\partial/\partial_{z_i}$. For a domain $U\subset C^{n+1}$, $\mathcal{O}(U)$ is the set of all holomorphic functions in U and $\mathcal{L}(U)$ is the set of all holomorphic vector fields in U. For a multi-index $\alpha=(\alpha_0,\alpha_1,\cdots,\alpha_n)=(\alpha_0,\alpha')$, $\alpha_i\in Z_+=N\cup\{0\}$, $|\alpha|=\alpha_0+|\alpha'|=\alpha_0+\alpha_1+\cdots+\alpha_n$ and for $X=(X_0,X_1,\cdots,X_n)=(X_0,X')\in\mathcal{L}(U)^{n+1}$, $X^\alpha=(X_0)^{\alpha_0}(X_1)^{\alpha_1}\cdots(X_n)^{\alpha_n}=(X_0)^{\alpha_0}(X')^{\alpha'}$.

For a point $p\in K$, there is a neighbourhood U of p such that $K\cap U=\{z\in U;\varphi(z)=0\}$ with $\varphi(z)\in\mathcal{O}(U)$ and $d\varphi(z)\neq 0$ on K. For $f(z)\in\mathcal{O}(U)$, $|f|=j\in Z_+\cup\{+\infty\}$ means that $f(z)=\varphi(z)^j g(z)$ with $g(z)\not\equiv 0$ on K. If $f(z)\equiv 0$, $|f|=+\infty$.

We can find $X=(X_0,X_1,\cdots,X_n)\in\mathcal{L}(U)^{n+1}$, by shrinking U if necessary, such that

(1.1) $\langle d\varphi, X_0\rangle\neq 0$ on K and $\langle d\varphi, X_i\rangle=0$ on K for $1\leq i\leq n$,

(1.2) $\{X_i\}$ $(0\leq i\leq n)$ are linearly independent at each point in U,

where $\langle\ ,\ \rangle$ means the product of cotangent and tangent vectors.

Hence we can write $P(z,\partial)$ in U, by using $\{X_i\}$, as follows:

$$P(z,\partial)=\sum_{k=0}^{m}\Big(\sum_{|\alpha|=k}A_\alpha(z)X^\alpha\Big)=\sum_{k=0}^{m}\Big(\sum_{|\alpha|=k}a_\alpha(z)\varphi(z)^{j_\alpha}(X_0)^{\alpha_0}(X')^{\alpha'}\Big),\tag{1.3}$$

where $A_\alpha(z)=a_\alpha(z)\varphi(z)^{j_\alpha}$ and $j_\alpha=|A_\alpha|$. Put

$$\begin{cases}d_k=\min\{j_\alpha+|\alpha'|;|\alpha|=k\}, & J_k=\min\{j_\alpha;|\alpha|=k,\ j_\alpha+|\alpha'|=d_k\},\\ L_k=d_k-J_k.\end{cases}\tag{1.4}$$

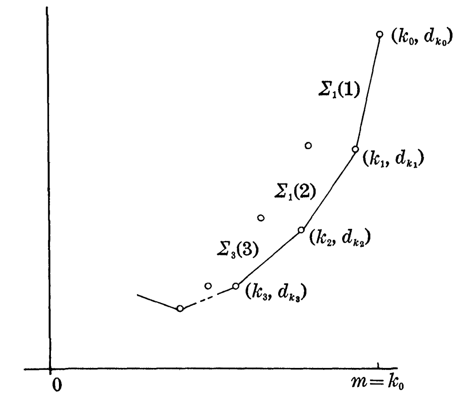
We can define quantities $\{\sigma_i\}$ and $\{\sigma_{p,j}\}$ by the same way as in [6]. Let $A=\{(k,d_k)\in R^2;d_k\neq+\infty,\ 0\leq k\leq m\}$ and $\hat{A}$ be the convex hull of A. If $A=\{(m,d_m)\}$, we put $\sigma_1=1$. Otherwise the lower convex part of the boundary $\partial\hat{A}$ of $\hat{A}$ consists of segments $\Sigma_1(i)$ $(1\leq i\leq l)$ (see Fig. 1). Let $A_1=\{(k_i,d_{k_i});0\leq i\leq l\}$, $m=k_0>k_1>\cdots>k_l\geq 0$, be the set of vertices of $\bigcup_{i=1}^{l}\Sigma_1(i)$. Put

$$\sigma_i=\max\{(d_{k_{i-1}}-d_{k_i})/(k_{i-1}-k_i),1\}.\tag{1.5}$$

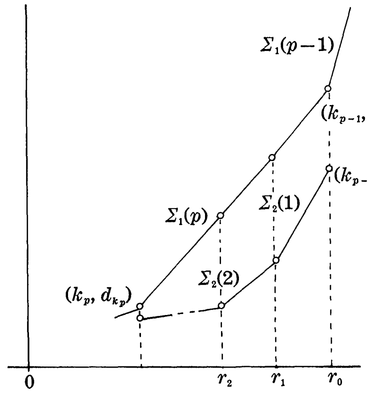
Then there is a $p \in N$ such that $\sigma_1 > \sigma_2 > \cdots > \sigma_p = 1$. Put

(1.6) $$C(i) = \{\alpha \in Z_+^{n+1}\,;\, |\alpha| = k_i,\ |\alpha'| = L_{k_i},\ j_\alpha = J_{k_i}\},$$

(1.7) $$\tilde{P}_i(z, \partial) = \textstyle\sum_{\alpha \in C(i)} a_\alpha(z)(X')^{\alpha'} \qquad (0 \le i \le p-1).$$



(Fig. 1)



(Fig. 2)

Let us also consider $B = \{(k, J_k) \in R^2\,;\, d_{k_{p-1}} - d_k = k_{p-1} - k,\ 0 \le k \le k_{p-1}\}$ to define $\{\sigma_{p,i}\}$. If $B = \{(k_{p-1}, J_{k_{p-1}})\}$, we put $\sigma_{p,1} = 1$. Otherwise the lower convex part of the boundary $\partial\hat{B}$ of the convex hull $\hat{B}$ of B consists of segments $\Sigma_2(i)$ $(1 \le i \le l')$ (see Fig. 2). Let $B_1 = \{(r_i, J_{r_i})\,;\, 0 \le i \le l'\}$, $k_{p-1} = r_0 > r_1 > \cdots > r_{l'} > 0$. Put

(1.8) $$\sigma_{p,i} = \max\{(J_{r_{i-1}} - J_{r_i})/(r_{i-1} - r_i),\ 1\}.$$

Hence there is a $q \in N$ such that $\sigma_{p,1} > \sigma_{p,2} > \cdots > \sigma_{p,q} = 1$. Put

(1.9) $$C(p, i) = \{\alpha \in \mathrm{Z}_+^{n+1}\,;\, |\alpha| = r_i,\ |\alpha'| = L_{r_i},\ j_\alpha = J_{r_i}\},$$

(1.10) $$\tilde{P}_{p,i}(z, \partial) = \textstyle\sum_{\alpha \in C(p,i)} a_\alpha(z)(X')^{\alpha'} \qquad (1 \le i \le q-1).$$

Definition 1.1. We call σ_i $(1 \le i \le p)$ the i-th characteristic index of K and $\sigma_{p,i}$ $(1 \le i \le q)$ the (p, i)-characteristic index of K for $P(\mathrm{z}, \partial)$.

We may restrict $\tilde{P}_i(\mathrm{z}, \partial)$ $(0 \le i \le p-1)$ and $\tilde{P}_{p,i}(z, \partial)$ $(1 \le i \le q-1)$ on K, have operators on K and denote them by $P_{\mathrm{loc},K,i}$ and $P_{\mathrm{loc},K,(p,i)}$ respectively.

Definition 1.2. We call $P_{\mathrm{loc},K,i}$ the i-th localization on K of $P(z, \partial)$ and $P_{\mathrm{loc},K,(p,i)}$ the (p, i)-localization on K of $P(z, \partial)$.

Remark 1.3. In [6] we call $\sigma_{p,i}$ the i-th subcharacteristic index and only $P_{\mathrm{loc},K,0}$ is defined and called the localization.

The characteristic indices and the localizations are defined by $\varphi(z)$ and $\{X_i\}$ $(0 \le i \le n)$. Let $\psi(z) \in \mathcal{O}(V)$ be another function defining K and $Y = (Y_0, Y_1, \cdots, Y_n) \in \mathcal{L}(V)^{n+1}$ be vector fields with properties (1.1)–(1.2) for $\psi(z)$ and $\{Y_i\}$ $(0 \le i \le n)$ in a neighbourhood V of p. Then we have in $U \cap V$,

(1.11) $$\varphi(z) = \chi(z)\psi(z), \quad X_i = \textstyle\sum_{j=0}^n a_{i,j}(z) Y_j \quad (0 \le i \le n),$$

where $\chi(z)$, $a_{i,j}(z) \in \mathcal{O}(U \cap V)$, $\chi(z) \neq 0$ and $\det(a_{i,j}(z)) \neq 0$ in $U \cap V$. From (1.1)–(1.2), we have

Lemma 1.1. *The following holds in* (1.11):

(1.12) $a_{0,0}(z)|_{K\cap U\cap V}\neq 0$ and $a_{i,0}(z)_{K\cap U\cap V}=0$ for $1\leq i\leq n$.

We denote by $\sigma_i(\psi, Y)$ $(1\leq i\leq p')$ and $\sigma_{p',j}(\psi, Y)$ $(1\leq j\leq q')$ characteristic indices, and by $P_{\mathrm{loc},K,i}(\psi, Y)$ and $P_{\mathrm{loc},K,(p',j)}(\psi, Y)$ localizations of $P(z,\partial)$ defined by $\psi(z)$ and Y. Let $Q(z,\partial)$ be an operator homogeneous in X with degree k,

(1.13) $Q(z,\partial)=\sum_{|\alpha|=k} B_\alpha(z)X^\alpha=\sum_{|\alpha|=k} B_\alpha(z)\{\prod_{i=0}^n (\sum_{j=0}^n a_{i,j}(z)Y_j)^{\alpha_i}\}$.

In the sequel P.S.L means the principal symbol of an operator L. We have from Lemma 1.1,

Lemma 1.2. *The followings hold for* $Q(z,\partial)$:

(1.14) $\sigma_1(\varphi, X)=\sigma_{1,1}(\varphi, X)=\sigma_1(\psi, Y)=\sigma_{1,1}(\psi, Y)=1$,

(1,15) $\mathrm{P.S.Q}_{\mathrm{loc},K,0}(\varphi, X)=h(s)\,\mathrm{P.S.Q}_{\mathrm{loc},K,0}(\psi, Y)$,

where $h(s)$ $(s\in K\cap U\cap V)$ is a holomorphic function on $K\cap U\cap V$ and $h(s)$ does not vanish on $K\cap U\cap V$.

Consequently we have from Lemma 1.2,

Theorem 1.3. *The characteristic indices $\{\sigma_i\}$ $(1\leq i\leq p)$ and $\{\sigma_{p,j}\}$ $(1\leq j\leq q)$ don't depend on $\varphi(z)$ defining K and vector fields $\{X_i\}$ $(0\leq i\leq n)$ satisfying* (1.1)–(1.2).

For the localization we have

Theorem 1.4. *There are holomorphic functions $h_i(s)$ $(0\leq i\leq p-1)$ and $h_{p,j}(s)$ $(1\leq j\leq q-1)$ on $K\cap U\cap V$ such that*

(i) *$h_i(s)\neq 0$ and $h_{p,j}(s)\neq 0$ on $K\cap U\cap V$,*

(ii)-(a) $\mathrm{P.S.P}_{\mathrm{loc},K,i}(\varphi, X)=h_i(s)\,\mathrm{P.S.P}_{\mathrm{loc},K,i}(\psi, Y)$,

-(b) $\mathrm{P.S.P}_{\mathrm{loc},K,(p,j)}(\varphi, X)=h_{p,j}(s)\,\mathrm{P.S.P}_{\mathrm{loc},K,(p,j)}(\psi, Y)$.

Finally we give a classification of characteristic surfaces.

Definition 1.4. Suppose that K is characteristic for $P(z,\partial)$.

(a) If $\sigma_1>1$, then K is called irregular.

(b) If $\sigma_1=1$ and $\sigma_{1,1}>1$, then K is called weakly irregular.

(c) If $\sigma_1=\sigma_{1,1}=1$, then K is called regular.

§2. Remarks. (i) Let $P(z,\partial)$ be an operator with decomposable principal part, namely said to have constant multiple characteristics (see [2], [4]), and K be its characteristic surface. Let $\tilde{\sigma}$ be the irregularity of characteristic elements defined in [4]. Then $\sigma_1\leq\tilde{\sigma}$. For $P(z,\partial)=(\partial_1)^2+z_0\partial_0$ and $K=\{z_0=0\}$, we have $\sigma_1=1$, but $\tilde{\sigma}=2$. If $P(z,\partial)$ satisfies the Levi's condition, namely $\tilde{\sigma}=1$, we have $\sigma_1=\sigma_{1,1}=1$.

(ii) Let $P(z,\partial)$ be an operator treated in [1], [3], where characteristic initial value problems were considered. We have $\sigma_1=\sigma_{1,1}=1$ for their initial characteristic surface.

(iii) If K is generically noncharacteristic, that is, $\mathrm{P.S.P}(z,\xi)|_{z\in K,\xi=d\varphi(z)}\neq 0$, then $\sigma_1=\sigma_{1,1}=1$. If $P(z,\partial)$ is an ordinary differential operator, then $\sigma_{p,1}=1$.

We give some examples. Let $K=\{z_0=0\}$.

$$
(2.1)\quad \begin{cases} P(z,\partial)=(\partial_0)^{m-l}(\partial_1)^l+\sum_{|\beta|\le m,\beta\neq\alpha}a_\beta(z)\partial^\beta, & \alpha=(m-l,l,0,\cdots,0),\\ a_\beta(z)=0(|z_0|^\delta)\quad (\delta=\max(\beta_0-m+l,0)) & \text{for } |\beta|=m \text{ and } \beta\neq\alpha,\\ P_{\mathrm{loc},K,0}=(\partial_1)^l+\sum_{|\beta'|=l,\beta\neq\alpha}a_{(m-l,\beta')}(0,z')(\partial')^{\beta'}. \end{cases}
$$

$$
(2.2)\quad \begin{cases} P(z,\partial)=(z_0)^2(\partial_0)^2(\partial_1)^2+(\partial_1)^4+(z_0)^2(\partial_0)^3+(\partial_1)(\partial_0),\\ \sigma_1=2,\quad \sigma_2=1,\quad \sigma_{2,1}=2,\quad \sigma_{2,2}=1,\quad P_{\mathrm{loc},K,0}=(\partial_1)^4,\quad P_{\mathrm{loc},K,1}=I,\\ P_{\mathrm{loc},K,(2,1)}=\partial_1. \end{cases}
$$

$$
(2.3)\quad \begin{cases} P(z,\partial)=(z_0)^2(\partial_0)^2\partial_1+(\partial_1)^2+z_0b(z)\partial_0+c(z)\partial_1+d(z),\\ \sigma_1=1,\quad \sigma_{1,1}=2,\quad \sigma_{1,2}=1,\quad P_{\mathrm{loc},K,0}=\partial_1,\quad P_{\mathrm{loc},K,(1,1)}=(\partial_1)^2. \end{cases}
$$

Operators of the form (2.1) were treated in [7]. K is weakly irregular in (2.3) and irregular in (2.2).

In [6], some theorems concerning with existence of solutions with singularity on K for $P(z,\partial)u(z)=f(z)$ are stated, where σ_1, $\sigma_{1,1}$ and $P_{\mathrm{loc},K,0}$ are used. It follows from Theorem 1.3 and 1.4 that the conditions in [6] are invariant by coordinate transformations. Not only σ_1 and $P_{\mathrm{loc},K,0}$ but also other σ_i and $P_{\mathrm{loc},K,i}$ are used in [5], where the relation between genuine solutions and solutions of formal power series for characteristic Cauchy problems is studied.

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