29. *A Note on the Fundamental Group of a Unirational Variety*

By Noriyuki SUWA
Department of Mathematics, University of Tokyo

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1. Introduction. Let k be an algebraically closed field and let X be a smooth projective variety over k. X is unirational (or separably unirational) if there is a dominant rational map $\boldsymbol{P} \dashrightarrow X$ where $\boldsymbol{P}$ is a projective space such that the extension of fields $k(\boldsymbol{P})/k(X)$ is finite (or finite separable). Serre showed in [5] the following results.

(1) An étale covering of a unirational (or separably unirational) variety is also unirational (or separably unirational).

(2) The fundamental group of a unirational variety is finite.

(3) If k is of characteristic 0, every unirational variety is simply connected.

Further the following facts are known about the fundamental variety in the case of characteristic $p>0$.

(4) If X is separably unirational and of dimension 3, then X is simply connected (Nygaard [4]).

(5) If X is unirational and of dimension ≤ 3, $\pi_1(X)$ is p-torsion-free (Katsura [3]*), Crew [1]).

(6) The order of the fundamental groups of unirational surfaces are not bounded (Shioda [6], remark 7).

In this note we will show the following:

Theorem. *Let k be an algebraically closed field of characteristic $p>0$ and X a separably unirational variety over k. Then the fundamental group $\pi_1(X)$ of X is p-torsion-free.*

2. Proof of the theorem. The proof is based on the theory of de Rham-Witt complex of Deligne-Illusie [2] and a recent result of Crew [1]. We follow the notation of [2].

Proof. Since X is separably unirational, we have

$$H^0(X, \Omega_X^i)=0 \qquad \text{for } i>0.$$

Now the isomorphism

$$W.\Omega_X^i/VW.\Omega_X^i \xrightarrow{\sim} Z.\Omega_X^i$$

induces the isomorphism

$$H^0(X, W\Omega_X^i/VW\Omega_X^i) \xrightarrow{\sim} \varprojlim_{C} H^0(X, Z_n\Omega_X^i)$$

*) Katsura has communicated to me orally that the method in [3] is also valid for unirational three-folds.

(Illusie [2], II.2.2). Then we have

$$H^0(X, W\Omega_X^i / VW\Omega_X^i) = 0 \qquad \text{for } i > 0.$$

Hence the Verschiebung V is surjective on $H^0(X, W\Omega_X^i)$ for $i>0$. Now V is topologically nilpotent on $H^0(X, W\Omega_X^i)$ (Illusie [2], II.2.5), hence we obtain

$$H^0(X, W\Omega_X^i) = 0 \qquad \text{for } i > 0,$$

and therefore

$$H^i(X/W)_K^{[i]} = 0 \qquad \text{for } i > 0$$

(Illusie [2], II.3.5). By hard Lefschetz theorem and Poincaré duality for crystalline cohomology, we obtain

$$H^i(X/W)_K^{[0]} = 0 \qquad \text{for } i > 0,$$

and therefore

$$H^i(X, \boldsymbol{Q}_p) = 0 \qquad \text{for } i > 0$$

(Illusie [2], II.5.2). It follows that

$$\chi_p(X) = \sum_{i \geq 0} (-1)^i \dim H^i(X, \boldsymbol{Q}_p) = 1$$

for any separably unirational variety X.

Now let $\tilde{X} \to X$ be the universal covering of X and $Y \to X$ the étale covering of X, corresponding to a p-Sylow subgroup of $\pi_1(X)$. Then $\tilde{X} \to Y$ is an étale Galois covering with degree a power of p. By Crew's formula ([1], 1.7), we have

$$\chi_p(\tilde{X}) = \deg(\tilde{X}/Y)\chi_p(Y).$$

Now $\tilde{X}$ and Y are separably unirational by (1), and so

$$\chi_p(\tilde{X}) = \chi_p(Y) = 1.$$

Hence $\tilde{X} = Y$, i.e. $\pi_1(X)$ is p-torsion-free. Q.E.D.

Remark. According to Illusie, this theorem has been independently proved by Ekedahl. Ekedahl has also given proofs of (4) and (5).

References

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