22. *Iteration Methods for Common Fixed Points of Nonexpansive Mappings*

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1. Introduction. In [1], DeMarr proved the existence theorem of common fixed points for commuting nonexpansive mappings: Let D be a nonempty compact convex subset of a Banach space. If T_i ($i \in J$, J is an index set) are commuting nonexpansive mappings of D into itself, then T_i, $i \in J$, have a common fixed point in D. Now we are interested in the constructing process of a sequence which converges to a common fixed point. From this point of view, Ishikawa [3] and Kuhfittig [4] have shown the following theorems respectively.

Theorem A (Ishikawa [3]). *Let D be a compact convex subset of a Banach space, and let $\{T_i : i=1,2,\cdots,k\}$ be a finite family of commuting nonexpansive self-mappings of D* (i.e., $\|T_i x - T_i y\| \leqq \|x-y\|$ *for all $x, y \in D$ and $T_i T_j = T_j T_i$ for all $i, j=1,2,\cdots,k$).*

Then $\bigcap_{i=1}^k F(T_i) \neq \phi$ and the sequence $\{x_{n_k}\}$ converges to a point in $\bigcap_{i=1}^k F(T_i)$ as $n_k \to \infty$, where x_{n_k} is defined by

$$[\prod_{n_{k-1}=1}^{n_k} [S_k \prod_{n_{k-2}=1}^{n_{k-1}} [S_{k-1} \cdots [S_3 \prod_{n_1=1}^{n_2} [S_2 \prod_{n_0=1}^{n_1} S_1]] \cdots]]] \, x$$

with $S_i = (1-\alpha_i)I + \alpha_i T_i$, $0 < \alpha_i < 1$ $(i=1,2,\cdots,k)$, and $F(T_i)$ stands for the set of fixed points of T_i.

Theorem B (Kuhfittig [4]). *Let D be a compact convex subset of a strictly convex Banach space, and let $T_i : i=1,2,\cdots,k$ be a finite family of nonexpansive self-mappings of D with a nonempty set of common fixed points. Define the mappings $U_i = (1-\alpha)I + \alpha T_i U_{i-1}$ for $0<\alpha<1$, $i=1,2,\cdots,k$ with $U_0 = I$, the identity mapping. Then for any point $x \in D$, the sequence $\{U_k^n x\}$ converges to a point in $\bigcap_{i=1}^k F(T_i)$ as $n \to \infty$.*

Comparing these two theorems, though in Theorem A the assumption of strict convexity of Banach space in B is removed, the condition of commuting mappings is stronger than that of existence of common fixed points ([1]), and further the iteration method in A is more complicated than that in B. The purpose of this paper is to present another simple iteration process for a common fixed point under slightly weaker assumptions than those in B.

We here note that in the case of a single mapping T_1 (i.e. $k=1$) Ishikawa showed the following iteration methods without any assumption on convexity. We shall later make use of this.

Theorem C (Ishikawa [2]). *Let D be a closed subset of a Banach space X and let T be a nonexpansive mapping from D into a compact subset of X. Suppose that a point $x_1 \in D$ and a sequence $\{t_n\}_{n=1}^{\infty}$ satisfy the conditions: $\sum_{n=1}^{\infty} t_n = \infty$, $0 \leq t_n \leq b < 1$ and $x_n \in D$ for all positive integer n, where $\{x_n\}_{n=1}^{\infty}$ is defined by*

(1) $$x_{n+1} = (1-t_n)x_n + t_n T x_n.$$

Then $F(T) \neq \phi$ and $\{x_n\}$ converges to a point in $F(T)$ as $n \to \infty$.

2. Iteration methods of nonexpansive mappings. Now we shall show an iteration process for common fixed points of nonexpansive mappings. We first prove the following lemmas.

Lemma 1. *Let D be a closed subset of a Banach space X and let $\{T_i : i=1,2,\cdots,k\}$ be a finite family of nonexpansive mappings from D into a compact subset of X. Suppose that a point $x_1 \in D$ and a sequence $\{a_i\}_{i=0}^{k}$ satisfy the conditions: $0 < a_i < 1$ for $i=0,1,\cdots,k$, $\sum_{i=0}^{k} a_i = 1$ and $x_n \in D$ for all positive integer n, where $\{x_n\}_{n=1}^{\infty}$ is defined by*

(2) $$x_{n+1} = a_0 x_n + \sum_{i=1}^{k} a_i T_i x_n.$$

Then the sequence $\{x_n\}$ converges to a point y such that

(3) $$\sum_{i=1}^{k} a_i T_i y = \sum_{i=1}^{k} a_i y.$$

Proof. Putting $a_i' = a_i/(1-a_0)$, $i=1,2,\cdots,k$, (2) may be expressed as follows

$$x_{n+1} = a_0 x_n + (1-a_0) \sum_{i=1}^{k} a_i' T_i x_n$$

with $0 < a_i' < 1$, $i=1,2,\cdots,k$ and $\sum_{i=1}^{k} a_i' = 1$. Since the mappings T_i, $i=1,2,\cdots,k$, are nonexpansive mappings from D into a compact subset of X and $x_n \in D$ for all positive integer n, $T := \sum_{i=1}^{k} a_i' T_i$ maps D into a compact subset of X. Therefore Theorem C may be applicable, thus we have $F(\sum_{i=1}^{k} a_i' T_i) \neq \phi$ and the sequence $\{x_n\}$ of (2) converges to a $y \in F(\sum_{i=1}^{k} a_i' T_i) \cap D$ which implies $\sum_{i=1}^{k} a_i T_i y = \sum_{i=1}^{k} a_i y$.

Lemma 2. *Let X be a strictly convex Banach space and let y_i, $i=1,2,\cdots,k$, be any elements of X. Suppose that $y = \sum_{i=1}^{k} a_i y_i$ with $0 < a_i < 1$, $i=1,2,\cdots,k$, $\sum_{i=1}^{k} a_i = 1$ and there is at least an element y_i such that $y_i \neq y$. Then we have*

(4) $$\|y\| < \max\{\|y_i\| : \text{for all } y_i \text{ such that } y_i \neq y\}.$$

Proof. We shall prove the lemma by induction. When $k=2$ the assertion is true by the definition of strict convexity. Suppose that the assertion is true for any $k-1$ elements of X. Since

(5) $$y = \sum_{i=1}^{k} a_i y_i = a_1 y_1 + (1-a_1) \sum_{i=2}^{k} a_i' y_i$$

with $a_i' = a_i/(1-a_1)$, $i=2,3,\cdots,k$, $\sum_{i=2}^{k} a_i' = 1$, if $y_1 = y$, then we have $y = \sum_{i=2}^{k} a_i' y_i$ with some $y_i \neq y$, $i=2,3,\cdots,k$, $\sum_{i=2}^{k} a_i' = 1$. Hence by the assumption of induction, (4) for y_i, $i=2,3,\cdots,k$, is true. If $y_1 \neq y$, then $\sum_{i=2}^{k} a_i' y_i \neq y$. Otherwise, $y_1 = y$ by (5). Therefore again by the assumption of induction we have

$$\|y\| < \max(\|y_1\|, \|\textstyle\sum_{i=2}^k a_i' y_i\|)$$
$$\leqq \max(\|y_1\|, \max\{\|y_i\| : \text{for all } y_i \neq y,\ i=2,3,\cdots,k\}).$$

This completes the proof.

Making use of these lemmas we shall show the following main theorem.

Theorem 1. *Let D be a closed subset of a strictly convex Banach space X and $\{T_i : i=1,2,\cdots,k\}$ be a finite family of nonexpansive mappings from D into a compact subset of X such that $\bigcap_{i=1}^k F(T_i) \neq \phi$. Suppose that a point $x_1 \in D$ and a sequence $\{a_i\}_{i=0}^k$ satisfy the conditions in Lemma 1. Then the sequence $\{x_n\}$ defined by (2) converges to an element $y \in \bigcap_{i=1}^k F(T_i)$.*

Proof. We have proved in Lemma 1 that the sequence $\{x_n\}$ converges to an element $y \in \bigcap_{i=1}^k F(T_i)$ satisfying (3). Putting $a_i' = a_i/(1-a_0)$ for $i=1,2,\cdots,k$, this implies

(6) $$y = \textstyle\sum_{i=1}^k a_i' T_i y \quad \text{with} \quad 0 < a_i' < 1,\ \sum_{i=1}^k a_i' = 1.$$

On the other hand, from the assumptions of $\bigcap_{i=1}^k F(T_i) \neq \phi$ and nonexpansiveness of T_i, there exists an element $w : T_i w = w$ for $i=1,2,\cdots,k$, and we have

(7) $$\|T_i y - w\| \leqq \|y - w\| \qquad \text{for } i=1,2,\cdots,k.$$

Now we wish to show $y \in \bigcap_{i=1}^k F(T_i)$. Suppose not, then there exists at least a $T_i y$ such that $T_i y \neq y$. Since X is strictly convex, Lemma 2 is applicable to $T_i y - w$, $i=1,2,\cdots,k$ and $y-w$ instead of y_i, $i=1,2,\cdots,k$ and y respectively. Thus from (6) and (7) we have

$$\|y-w\| = \|\textstyle\sum_{i=1}^k a_i'(T_i y - w)\|$$
$$< \max\{\|T_i y - w\| : T_i y \neq y\} \leqq \|y-w\|.$$

This contradiction shows $T_i y = y$ for all $i=1,2,\cdots,k$, which completes the proof.

Generalizing Mann's iteration method $M(x_1, t_n, T)$ ([2], [5]) for a single mapping T to the case of a family of nonexpansive mappings, we may extend Theorem 1 to the following.

Theorem 2. *Let D be a closed subset of a strictly convex Banach space X and let $\{T_i : i=1,2,\cdots,k\}$ be a finite family of nonexpansive mappings from D into a compact subset of X such that $\bigcap_{i=1}^k F(T_i) \neq \phi$. Suppose that a point $x_1 \in D$ and a sequence $\{t_n\}$ satisfy the conditions: $0 \leqq t_n \leqq b < 1$ for any positive integer n, $\sum_{n=1}^\infty t_n = \infty$ and further $x_n \in D$ for all n. Here x_n is defined by*

(8) $$x_{n+1} = (1-t_n)x_n + \frac{t_n}{k} \textstyle\sum_{i=1}^k T_i x_n \qquad \text{for } n=1,2,\cdots.$$

Then the sequence $\{x_n\}$ converges to a common fixed point of T_i, $i=1,2,\cdots,k$.

The proof is based on the following lemma instead of Lemma 1.

Lemma 1′. *Let D and $\{T_i : i=1,2,\cdots,k\}$ be the same as in Lemma*

1. *For any sequence* $\{t_n\}_{n=1}^{\infty}$ *such that* $0 \leqq t_n \leqq b < 1$, $\sum_{n=1}^{\infty} t_n = \infty$, *we define the sequence* $\{x_n\}_{n=1}^{\infty}$ *by* (8). *If* $x_n \in D$ *for all positive integer* n, *then* $\{x_n\}$ *converges to a point* $y \in D$ *that satisfies*

$$y = \frac{1}{k} \sum_{i=1}^{k} T_i y. \tag{9}$$

As in the proof of Lemma 1, putting $T := (1/k) \sum_{i=1}^{k} T_i$, then T is nonexpansive. Therefore we can apply Theorem C to this T, which shows $F((1/k) \sum_{i=1}^{k} T_i) \neq \phi$ and $\{x_n\}$ converges to a $y \in F((1/k) \sum_{i=1}^{k} T_i)$. This implies (9).

Proof of Theorem 2. By making use of Lemmas 1′ and 2 in the place of Lemmas 1 and 2 in the proof of Theorem 1, we can prove the theorem in the same way of Theorem 1. Therefore we omit it.

References

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