44. On Closed Subvarieties of Parabolic Type in Certain Quasi-Projective Spaces of Hyperbolic Type

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Introduction. Recently S. Iitaka has developed a theory of logarithmic forms for algebraic varieties from proper birational geometric viewpoint and as an application he classified varieties of the form $V=(P^n-$a union of hyperplanes) by means of logarithmic Kodaira dimension $\bar{\kappa}$ [1]. The present note is based on these results. We study closed subvarieties Γ's of V with $\bar{\kappa}(\Gamma)=0$ for V with $\bar{\kappa}(V)=n$. Recall that $\Gamma\simeq G_m^r$, where G_m^r denotes the r-dimensional algebraic torus. For our purpose, the maximal ones among V's are useful.

1. Maximality. Let $V^n=P^n(C)-L_0\cup\cdots\cup L_q$ where L_j's are distinct hyperplanes in $P^n(C)$. The conditions in terms of coordinates for V^n with $\bar{\kappa}(V^n)=n$ can be described as follows. We may assume L_j is defined by $X_j=0$, $0\leqq j\leqq n$. For the other equations, putting $s=q-n$, define $I_1,\cdots,I_s\subset\{0,1,\cdots,n\}$ by $I_j=\{i\,|\,\text{coef. of } X_i \text{ of } L_{n+j} \text{ is not zero.}\}$ Then renumbering j if necessary, the following conditions 0) and 1) are satisfied.

0) $I_1\cup\cdots\cup I_s=\{0,1,\cdots,n\}$

1) $I_1\cup\cdots\cup I_{j-1}$ is not disjoint to I_j for $2\leqq j\leqq n$.

Proposition 1. *Let Ca_j be the one dimensional subspace of A^{n+1} corresponding dually to L_j, $0\leqq j\leqq q$. Let (A^r, A^s) denote a pair of proper subspaces of A^{n+1} with $A^r\cap A^s=\{0\}$. Then V^n satisfies the above conditions* 0) *and* 1), *if and only if the following* (C) *holds.*

(C) *$A^r\cup A^s$ dose not contain all of Ca_j's for any (A^r, A^s).*

Proposition 2. *If V^n with $\bar{\kappa}(V^n)=n$ is maximal, we can impose on V^n the following additional conditions* 2) *and* 3):

2) *There are s numbers, $2\leqq i(1)<\cdots<i(s)=n$, such that*

$$I_1=\{i\,|\,0\leqq i\leqq i(1)\}$$
$$I_j-I_1\cup\cdots\cup I_{j-1}=\{i\,|\,i(j-1)<i\leqq i(j)\},\ 2\leqq j\leqq s.$$

3) *Any two of I_j's never have only one common element.*

Proof of Proposition 2. 2) is obvious. Assume that $I_{j1}\cap I_{j2}=\{k\}$. Let $Ce_0,\cdots,Ce_n,Ca_1,\cdots,Ca_s$ be corresponding dually to $L_0,\cdots,L_n,L_{n+1},\cdots,L_{n+s}$. Let A_0 be the subspace of A^{n+1} spanned by $\{e_i\,|\,i\in I_{j1}\cup I_{j2}\}$. Since we are assuming that V^n is maximal, there is, by Proposition 1, (A^r, A^s) such that $\{e_0,\cdots,\check{e}_k,\cdots,e_n,a_1,\cdots,a_s\}\subset A^r\cup A^s$. This also induces a splitting $(A_0\cap A^r, A_0\cap A^s)$ for $\{e_i\,|\,i\in I_{j1}\cup I_{j2}, i\neq k\}\cup\{a_{j1},a_{j2}\}$ in

A_0. This is impossible.

Lemma. *If V^n is of maximally hyperbolic type under* 0), $\cdots$, 3) *and moreover if $i(s-1)=n-1$, then*

$$V^{n-1}=\boldsymbol{P}^{n-1}(\boldsymbol{C})-L_0\cup\cdots\cup L_{n-1}\cup L_{n+1}\cup\cdots\cup L_{n+s-1}$$

is also of maximally hyperbolic type.

Proposition 3. *If V^n is of maximally hyperbolic type, then $1+q \leqq 2n$ holds. When the equality holds, V^n is uniquely determined by the equations, $L_{n+j}: X_0+X_1+X_{1+j}=0$, $1\leqq j\leqq n-1$.*

Proof. By the condition 2), we obtain the inequality. When $s=n-1$, we may assume L_{n+j} is as in the above for $1\leqq j\leqq n-2$, by the lemma. By the condition 3), we deduce $I_{n-1}=\{0,1,n\}$, that is L_{n+s}: $X_0+cX_1+X_n=0$, $c\neq 0$. But unless $c=1$, V^n is not maximal.

Remark. Maximal V^n's do not have a parameter for $n\leqq 4$. But when $n=5$, $s=2$, V^n is determined by $L_6: X_0+X_1+X_2+X_3+X_4=0$, L_7: $X_0+X_1+cX_2+cX_3+X_5=0$, where c is a complex parameter.

2. $\boldsymbol{\Gamma}$ of codimension 1 with $\bar{\kappa}(\boldsymbol{\Gamma})=0$.

Proposition 4. *V^n $(n\geqq 3)$ with $\bar{\kappa}(V^n)=n$ has at most one closed subvariety Γ of codimension 1 with $\bar{\kappa}(\Gamma)=0$. When V^n has Γ as in the above, V^n is uniquely determined as the V^n in Proposition 3, if it is maximal.*

Lemma. *If V^n $(n\geqq 3)$ is of maximally hyperbolic type described under* 0), $\cdots$, 3) *and if $\sharp I_j\geqq 4$ for some j, $1\leqq j\leqq s$, then V^n has no Γ as in Proposition* 4.

Proof of Lemma. Recall that Γ is a closed subvariety of $\boldsymbol{G}_m^n=\boldsymbol{P}^n(\boldsymbol{C})-L_0\cup\cdots\cup L_n=\operatorname{Spec}\boldsymbol{C}[X_1/X_0,\cdots,X_n/X_0,X_0/X_1,\cdots,X_0/X_n]$ defined by $u_1=1$ for some new variables $u_1,\cdots,u_n$ of the $\boldsymbol{G}_m^n$ such that $X_i/X_0=a_iu_1^{\varepsilon(i1)}u_2^{\varepsilon(i2)}\cdots u_n^{\varepsilon(in)}$, $a_i\neq 0$, $1\leqq i\leqq n$, with the matrix E of exponents in $\mathrm{GL}(n,\boldsymbol{Z})$. We may assume $L_{n+1}: X_0+X_1+\cdots+X_k=0$, $k\geqq 3$. Since Γ lies on $\boldsymbol{G}_m^n-L_{n+1}$, the following indeterminate equation must hold with a unit of $\boldsymbol{C}[u_2,\cdots,u_n,1/u_2,\cdots,1/u_n]$ in the right hand side:

$$1+a_1u_2^{\varepsilon(12)}\cdots u_n^{\varepsilon(1n)}+\cdots+a_ku_2^{\varepsilon(k2)}\cdots u_n^{\varepsilon(kn)}=cu_2^{\alpha(2)}\cdots u_n^{\alpha(n)}.$$

But, since $E\in\mathrm{GL}(n,\boldsymbol{Z})$, $\sharp\{(\varepsilon(i2),\cdots,\varepsilon(in))\,|\,1\leqq i\leqq k\}=k$ or $k-1$. Thus the equation has no solution E and a_i's, if $k\geqq 3$.

Proof of Proposition 4. We may assume that V^n is maximal and satisfies 0), $\cdots$, 3). Then $\sharp I_j>2$ for all j. On the other hand, if V^n has Γ as in the statement, then by the lemma, $\sharp I_j<4$ for all j. Thus $\sharp I_j=3$ for all j. By this we deduce $s=n-1$, because V^n is maximal. Thus V^n is uniquely determined by Proposition 3. The V^n in Proposition 3 has actually only one Γ defined by $X_0+X_1=0$.

3. Example. We obtain a list of Γ's for $n=4$, solving the indeterminate equations as in the lemma for Proposition 4. There are only 3 maximal figures in this case.

$s=1$, $L_5: X_0+X_1+X_2+X_3+X_4=0$.

Γ	Aspect in V
G_m^3	none
G_m^2	15 pieces
$G_m^1(\not\subset G_m^2)$	A fibre of 25 fibred spaces

$s=2$, $L_5: X_0+X_1+X_2+X_3=0$, $L_6: X_0+X_1+X_4=0$.

G_m^3	none
G_m^2	A fibre of one fibred space
$G_m^1(\not\subset G_m^2)$	i) A fibre of 3 fibred spaces
	ii) A fibre of 12 fibred planes

$s=3$, $L_5: X_0+X_1+X_2=0$, $L_6: X_0+X_1+X_3=0$, $L_7: X_0+X_1+X_4=0$.

G_m^3	only one
$G_m^2(\not\subset G_m^3)$	none
$G_m^1(\not\subset G_m^3)$	8 pieces

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References

[1] Iitaka, S.: Logarithmic forms of algebraic varieties. J. Fac. Sci. Univ. Tokyo, **23**, 525–544 (1976).

[2] ——: Algebraic Geometry III. Iwanami (1977) (in Japanese).