37. On the Zeros of an Entire Function which is Periodic mod a Non-Constant Entire Function of Order Less than One

By Hironobu URABE*) and Chung-Chun YANG**)

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In this paper, we will consider the entire function which is periodic mod a non-constant entire function of order less than one, and state two theorems, one of which shows as a special case that periodic entire functions are uniquely determined by the sets of their fixed points (Corollary 2). The detailed proofs of these results will appear elsewhere [6].

1. For a non-zero constant b, we denote by $\boldsymbol{G}(b)$ the class of entire functions of the form:

$$f(z)=h(z)+H(z),$$

where $H(z)$ ($\not\equiv$const.) is entire, periodic with period b, i.e., $H(z+b)\equiv H(z)$, and $h(z)$ is a non-constant entire function of order less than one. Especially when $h(z)$ is a polynomial of degree one, the above class is denoted by $\boldsymbol{J}(b)$. A function $f(z)$ in $\boldsymbol{G}(b)$ (or $\boldsymbol{J}(b)$) is said to be periodic (with period b) mod $h(z)$.

These classes possess a significant position in factorization theory of transcendental entire functions (cf. [1], [2], and [5]).

2. Statement of our results. Theorem 1. *Let $f(z)=c_1z+H(z)$ and $g(z)=c_2z+K(z)$ belong to the class $\boldsymbol{J}(b)$ ($H(z)$ and $K(z)$ have period b, and c_1, c_2 are non-zero constants). Assume that the sets of the zeros of $f(z)$ and $g(z)$ are identical except (at most) a set whose exponent of convergence is less than one, then we have that $f(z)\equiv c\cdot g(z)$ and hence $H(z)\equiv c\cdot K(z)$ with $c=c_1/c_2$.*

Corollary 1. *Let $H(z)$ and $K(z)$ be two periodic entire functions with the same non-zero period. Assume that the sets of the fixed points of $H(z)$ and $K(z)$ are identical except (at most) a set whose exponent of convergence is less than one, then $H(z)$ and $K(z)$ are identically equal.*

Theorem 2. *Let $f_j(z)\in \boldsymbol{G}(b_j)$ ($j=1,2$). Assume that the sets of the zeros of $f_j(z)$ are identical for $j=1,2$, including multiplicities. Then we have that $f_1(z)\equiv c\cdot f_2(z)$ for some non-zero constant c and hence b_1/b_2 is a rational number.*

*) Kyoto Univ. of Education, Fushimi-ku, Kyoto 612.
**) Naval Research Laboratory, Washington, D.C. 20375.

Corollary 2. *Let $H_j(z)$ $(j=1,2)$ be non-constant periodic entire functions with period $b_j \neq 0$. Assume that the sets of the fixed points of $H_j(z)$ are identical, including multiplicities. Then we must have that $H_1(z)$ and $H_2(z)$ are identically equal and b_1/b_2 is a rational number.*

3. For the proofs of the above results, we shall need the following lemma (unicity theorem of Borel type) due to K. Niino and M. Ozawa.

Lemma (cf. [4]). *Let $G_j(z)$ $(1 \leqq j \leqq n)$ be transcendental entire functions and c_j be non-zero constants, and let $g(z)$ $(\not\equiv 0)$ be an entire function such that $T(r, g)=o(T(r, G_j))$ as r tends to infinity for any j. Assume that there exists an identical relation*

$$\sum_{j=1}^{n} c_j G_j(z)=g(z),$$

then we must have

$$\sum_{j=1}^{n} \delta(0, G_j) \leqq n-1.$$

Here $T(r, *)$ and $\delta(a, *)$ denote the Nevanlinna's characteristic function and deficiency, respectively.

4. Outline of the proof of Theorem 2. Let $f_1(z)=h(z)+H_1(z)$ and $f_2(z)=k(z)+H_2(z)$, where $H_j(z)$ is entire, non-constant, with $H_j(z+b_j)=H_j(z)$ $(j=1,2)$, $h(z)$ and $k(z)$ are non-constant entire functions of order less than one. By the assumption, there exists an identical relation such as

$$(*) \qquad h(z)+H_1(z)=(k(z)+H_2(z))e^{p(z)},$$

where $p(z)$ is an entire function.

Now it is possible to show that $p(z)$ would be constant if and only if b_1/b_2 is a rational number.

Owing to this fact, it is sufficient to prove that the identity $(*)$ does not hold under the additional assumption that b_1/b_2 is not a rational number and $p(z)$ is non-constant. From this assumption, one can show that $p(z+mb_j)-p(z)$ is non-constant for any non-zero integer m $(j=1,2)$.

Then by cancelling $H_1(z)$ and $H_2(z)$ from the identity $(*)$, we shall obtain the following new identity.

$$(**) \qquad \begin{aligned} &(r_2-r_1)+(r_1-r)e^{p-p_2-q+q_2}+(h_2-h_1)e^{p-p_1-p_2-q+q_1+q_2} \\ &\quad -(r_2-r)e^{p-p_1-q+q_1}-(h_2-h)e^{-p_2+q_2}+(h_1-h)e^{-p_1+q_1} \\ &\quad +(s-k)e^{p-p_2+q_2}-(s-k)e^{p-p_1+q_1}+(s_2-k_2)e^{p-p_1-q+q_1+q_2} \\ &\quad -(s-k)e^{p-p_2-q+q_1+q_2}-(s_2-k_2)e^{q_2}+(s_1-k_1)e^{q_1}=0, \end{aligned}$$

where $p_j(z)=p(z+jb_1)$, $q(z)=p(z+b_2)$, $q_j(z)=q(z+jb_1)=p(z+jb_1+b_2)$, $h_j(z)=h(z+jb_1)$, $r(z)=h(z+b_2)$, $r_j(z)=r(z+jb_1)$, $k_j(z)=k(z+jb_1)$, $s(z)=k(z+b_2)$ and $s_j(z)=s(z+jb_1)$ for $j=1,2$.

Now the coefficients of exponential functions in $(**)$ such as r_2-r_1, r_1-r, h_2-h_1, etc. are all of order less than one, and the function r_2-r_1

is not identically zero, since $r(z)$ is a non-constant entire function of order less than one and $r_2-r_1=r(z+2b_1)-r(z+b_1)$. And further the functions $-p_2+q_2$, $-p_1+q_1$, q_2 and q_1 are non-constant. Now we can prove, by using the lemma repeatedly, that the following seven functions

$$p-p_2-q+q_2,\ p-p_1-p_2-q+q_1+q_2,\ p-p_1-q+q_1,$$
$$p-p_2+q_2,\ p-p_1+q_1,\ p-p_1-q+q_1+q_2 \text{ and}$$
$$p-p_2-q+q_1+q_2$$

are all non-constant (cf. [6]). Then Lemma implies that the identity (**) cannot hold, a contradiction. Hence the assertion of Theorem 2 follows.

References

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