157. Some Trigonometrical Series. XVII

By Shin-ichi IZUMI and Masako SATÔ
Mathematical Institute, Tokyo Metropolitan University, Tokyo
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1. G. H. Hardy and J. E. Littlewood [1] have proved the following

Theorem 1. *Let $0<\alpha<1, p>1$ and $\alpha>1/p$. If $f(x)$ belongs to the $Lip(\alpha, p)$ class, then $f(x)$ is equivalent to a function in the $Lip(\alpha-1/p)$ class.*

This was generalized by one of the authors in the following form [2]:

Theorem 2. *Under the assumption of Theorem 1,*

(1) $$|s_n(x,f)-f(x)| \leqq A/n^{\alpha-1/p},$$

where $s_n(x,f)$ denotes the nth partial sum of Fourier series of $f(x)$ and A is an absolute constant.

It is well known that (1) implies that $f(x)$ belongs to the Lip $(\alpha-1/p)$ class.

On the other hand, G. H. Hardy and J. E. Littlewood [3] (cf. [4], p. 225) have proved the following

Theorem 3. (i) *Let $0<\alpha<1$, $\beta>0$ and $\alpha+\beta<1$. If $f(x)$ belongs to the $Lip\,\alpha$ class, then the βth integral of $f(x)$ belongs to the $Lip\,(\alpha+\beta)$ class.*

(ii) *Let $0<\beta<\alpha\leqq 1$. If $f(x)$ belongs to the $Lip\,\alpha$ class, then the βth derivative of $f(x)$ belongs to the $Lip\,(\alpha-\beta)$ class.*

In this theorem, the conclusion can not be replaced by (1) with $\alpha\pm\beta$ instead of $\alpha-1/p$.

We can in fact prove

Theorem 4. (i) *Let $0<\alpha<1$, $p>1$, $\alpha-1/p=\beta>0$ and $\gamma>0$, $\alpha+\gamma<1$. Then if $f(x)$ belongs to the $Lip\,(\alpha, p)$ class, then*

(2) $$|s_n(x, f_\gamma)-f_\gamma(x)| \leqq A/n^{\beta+\gamma}, \text{ a.e. unif.}$$

where $f_\gamma(x)$ is the γth integral of $f(x)$.

(ii) *Let $0<\alpha<1$, $p>1$, $\alpha-1/p=\beta>0$ and $0<\gamma<\alpha$. Then if $f(x)$ belongs to the $Lip\,(\alpha, p)$ class, then*

$$|s_n(x, f^\gamma)-f^\gamma(x)| \leqq A/n^{\beta-\gamma}$$

where $f(x)$ is the γth derivative of $f(x)$.

By Theorem 1, the Lip (α, p) class is contained in the Lip $(\alpha-1/p)$ class; hence both the assumption and the conclusion of Theorem 4 are stronger than those of Theorem 3, respectively.

Further G. H. Hardy and J. E. Littlewood [3] (cf. [4], p. 227) have proved the following

Theorem 5. *Let $p>1$, $1/p<\alpha<1/p+1$. If $f(x)$ is L^p-integrable, then αth integral of $f(x)$ belongs to the Lip $(\alpha-1/p)$ class.*

We can generalize this in the following form, in the case of $\alpha<1$,

Theorem 6. *Let $p>1$, $1/p<\alpha<1$. If $f(x)$ is L^p-integrable, then we have*

$$|s_n(x, f_\alpha)-f_\alpha(x)| \leqq A/n^{\alpha-1/p}.$$

For the proof of Theorems 4 and 6 we use the method in [5]. In § 2 we prove Theorem 4, (i). Since Theorem 4, (ii) is proved quite similarly, we omit its proof (cf. [4]). In § 4, we prove Theorem 6.

2. Proof of Theorem 4, (i). It is sufficient to prove (2), replaced s_n by s_n^*. The γth integral of $f(x)$ is defined by

$$f_\gamma(x)=\frac{1}{\Gamma(\gamma)}\int_0^\infty f(x-t)t^{\gamma-1}dt. \tag{3}$$

Let $F(x)=\int_0^x f(t)dt$, then we get by integration by parts

$$f_\gamma(x)=\frac{1-\gamma}{\Gamma(\gamma)}\int_0^\infty [F(x)-F(x-t)]t^{\gamma-2}dt. \tag{4}$$

Let us write

$$s_n^*(x, f_\gamma)-f_\gamma(x)=\frac{1}{\pi}\int_0^\pi D_n^*(u)[f_\gamma(x+u)+f_\gamma(x-u)-2f_\gamma(x)]du$$
$$=\frac{1}{\pi}\Big[\int_0^{\pi/n}+\int_{\pi/n}^\pi\Big]=I+J.$$

We shall begin to estimate I. By (4),

$$I=\frac{1-\gamma}{\pi\Gamma(\gamma)}\int_0^{\pi/n}D_n^*(u)du\int_0^\infty[\{F(x+u)-F(x+u-t)\}$$
$$+\{F(x-u)-F(x-u-t)\}-2\{F(x)-F(x-t)\}]t^{\gamma-2}dt$$
$$=\frac{1-\gamma}{\pi\Gamma(\gamma)}\int_0^{\pi/n}du\Big(\int_0^{1/n}+\int_{1/n}^\infty\Big)dt=I_1+I_2.$$

Now

$$\int_0^{\pi/n}D_n^*(u)du\int_0^{1/n}[\{F(x+u)-F(x+u-t)\}-\{F(x)-F(x-t)\}]t^{\gamma-2}dt$$
$$=\int_0^{\pi/n}D_n^*(u)du\int_0^{1/n}t^{\gamma-2}dt\int_0^t[f(x+u-v)-f(x-v)]dv.$$

By Theorem 1, $f(x+u-v)-f(x-v)=O(u^\beta)$, and then the last integral is of order $1/n^{\beta+\gamma}$. Hence $|I_1|\leqq A/n^{\beta+\gamma}$. Further

$$\Big|\int_0^{\pi/n}D_n^*(u)du\int_{1/n}^\infty[\{F(x+u)-F(x+u-t)\}-\{F(x)-F(x-t)\}]t^{\gamma-2}dt\Big|$$
$$\leqq An\int_0^{\pi/n}du\int_{1/n}^\infty \max\Big|\{F(x+u)-F(x)\}-\{F(x+u-t)-F(x-t)\}\Big|t^{\gamma-2}dt$$

$$\leqq An\int_0^{\pi/n} du\, u\int_{1/n}^{\infty} t^{\beta+\gamma-2}dt \leqq A/n^{\beta+\gamma}.$$

Thus we have $|I| \leqq A/n^{\beta+\gamma}$.

On the other hand, we write

$$J=\int_{\pi/n}^{\pi} du\int_0^{\infty} dt=\int_{\pi/n}^{\pi} du\Big(\int_0^{1/n}+\int_{1/n}^{\infty}\Big)dt=J_1+J_2.$$

Since we can suppose that n is odd,

$$(5)\quad J_1=\int_0^{1/n} t^{\gamma-2}dt\int_{\pi/n}^{\pi}[\{F(x+u)-F(x+u-t)\}+\{F(x-u)-F(x-u-t)\}$$
$$-2\{F(x)-F(x-t)\}]D_n^*(u)du$$
$$=\frac{1}{2}\int_0^{1/n} t^{\gamma-2}dt\sum_{k=1}^{(n-1)/2}\int_0^{\pi/n}\Big[\Big\{\Big(\frac{F(x+u+2k\pi/n)}{\sin(u+2k\pi/n)/2}-\frac{F(x+u+(2k-1)\pi/n)}{\sin(u+(2k-1)\pi/n)/2}\Big)$$
$$-\Big(\frac{F(x+u-t+2k\pi/n)}{\sin(u+2k\pi/n)/2}-\frac{F(x+u-t+(2k-1)\pi/n)}{\sin(u+(2k-1)\pi/n)/2}\Big)\Big\}$$
$$+\text{similar terms}\Big]\sin nudu,$$

where the term in the brackets [ ] is

$$\{(F(x+u+2k\pi/n)-F(x+u+(2k-1)\pi/n))/\sin(u+2k\pi/n)/2$$
$$+F(x+u+(2k-1)\pi/n)(1/\sin(u+2k\pi/n)/2-1/\sin(u+(2k-1)\pi/n)/2)$$
$$+\text{similar terms}\}+\text{similar terms}$$
$$=(f(x+u+2k\pi/n+\theta t)-f(x+u+(2k-1)\pi/n+\theta t))t/\sin(u+2k\pi/n)/2$$
$$+(f(x+u+(2k-1)\pi/n+\theta t)-f(x+\theta t))\cdot O\Big(\frac{t}{n}\Big/\Big(\frac{k}{n}\Big)^2\Big)+\text{similar terms}$$

by the mean value theorem, where $0<\theta<1$. By the assumption and Theorem 1, we get[1)]

$$|J_1|\leqq A\int_0^{1/n} t^{\gamma-1}dt\sum_{k=1}^{(n-1)/2}\int_{-\pi/n}^{\pi/n} du$$
$$\{|f(x+u+2k\pi/n+\theta t)-f(x+u+(2k-1)\pi/n+\theta t)|/(u+2k\pi/n)$$
$$+|f(x+u+(2k-1)\pi/n+\theta t)-f(x+\theta t)|\,n/k^2\}$$
$$\leqq A\int_0^{1/n} t^{\gamma-1}dt\Big[\Big(\int_0^{2\pi}|f(v+\pi/n)-f(v)|^p dv\Big)^{1/p}\Big(\int_{\pi/n}^{\pi} u^{-p'}du\Big)^{1/p'}$$
$$+\sum_{k=1}^{(n-1)/2}\Big(\frac{k}{n}\Big)^{\beta}\frac{n}{k^2}\frac{1}{n}\Big]$$
$$\leqq \frac{A}{n^{\gamma}}\Big[\Big(\frac{\pi}{n}\Big)^{\alpha}n^{-1/p}+\frac{1}{n^{\beta}}\Big]\leqq A/n^{\beta+\gamma}.$$

Finally, in order to estimate J_2, we write it in the form (5) where the range of integration with respect to t is replaced by $(1/n, \infty)$. By the mean value theorem, the term in the brackets is

$$\pi\{f(x+u+(2k-\theta)\pi/n)-f(x+u+(2k-\theta)\pi/n-t)\}/n\sin(u+2k\pi/n)/2$$
$$+(u+(2k-1)\pi/n)\{f(x+\lambda(u+(2k-1)\pi/n))-f(x+\lambda(2k-1)\pi/n)-t\}\cdot$$
$$\cdot O(1/n(u+2k\pi/n)^2),$$

1) $1/p+1/p'=1$.

where $0<\theta<1$, $0<\lambda<1$, and then

$$|J_2|\leqq A\int_{1/n}^{\infty} t^{\gamma-2}dt\, n^{-1}t^{\alpha}n^{1/p}\leqq A/n^{\beta+\gamma}.$$

Thus we have proved that $|J|\leqq A/n^{\beta+\gamma}$. Combining this with the estimation of I, we get the required result.

3. Prof. T. Tsuchikura informed us the following proof of Theorem 4. This follows from Theorem 2 and

Theorem 7. *Let $0\leqq\alpha<1$, $\beta>0$, $\alpha+\beta<1$ and $p>1$. If $f(x)$ belongs to the Lip(α, p) class, then the βth integral $f_\beta(x)$ belongs to the Lip$(\alpha+\beta, p)$ class.*

Proof. Let $F(x)=\int_0^x f(t)dt$. By (4),

$$\frac{\Gamma(\beta)}{1-\beta}\{f_\beta(x+h)-f_\beta(x)\}=\int_0^{\infty}[\{F(x+h)-F(x+h-t)\}$$
$$-\{F(x)-F(x-t)\}]t^{\beta-2}dt.$$

Hence, by the Minkowski inequality,

$$\Big(\int_0^{2\pi}|f_\beta(x+h)-f_\beta(x)|^p dx\Big)^{1/p}$$
$$\leqq A\left(\int_0^{2\pi}\left|\int_0^{\infty}[\{F(x+h)-F(x+h-t)\}-\{F(x)-F(x-t)\}]t^{\beta-2}dt\right|^p dx\right)^{1/p}$$
$$=\int_0^h+\int_h^{\pi}+\int_{\pi}^{\infty}=P+Q+R,$$

say. By the maximal theorem we get

$$P\leqq A\int_0^h t^{\beta-2}\Big[\int_0^{2\pi}\Big|\int_{x-t}^{x}\{f(u+h)-f(u)\}du\Big|^p dx\Big]^{1/p}dt$$
$$\leqq A\int_0^h t^{\beta}\Big[\int_0^{2\pi}|f(x+h)-f(x)|^p dx\Big]^{1/p}dt\leqq A\int_0^h t^{\beta-1}\cdot h^{\alpha}dt=Ah^{\alpha+\beta}.$$

Further

$$Q\leqq A\int_h^{\pi} t^{\beta-2}\Big[\int_0^{2\pi}\Big|\int_x^{x+h}\{f(u)-f(u-t)\}du\Big|^p dx\Big]^{1/p}dt$$
$$\leqq Ah\int_h^{\pi} t^{\beta-2}\Big[\int_0^{2\pi}\Big|\frac{1}{h}\int_x^{x+h}\{f(u)-f(u-t)\}du\Big|^p dx\Big]^{1/p}dt$$
$$\leqq Ah\int_h^{\pi} t^{\beta-2}\Big[\int_0^{2\pi}|f(x)-f(x-t)|^p dt\Big]^{1/p}dt$$
$$\leqq Ah\int_h^{\pi} t^{\beta-2}\cdot t^{\alpha}dt\leqq Ah(h^{\alpha+\beta-1}+O(1))=O(h^{\alpha+\beta})$$

and similarly $R\leqq Ah\int_{\pi}^{\infty}t^{\beta-2}O(1)dt=O(h)$. Thus we obtain

$$\Big(\int_0^{2\pi}|f_\beta(x+h)-f_\beta(x)|^p dx\Big)^{1/p}=O(h^{\alpha+\beta}),$$

which is the required.

4. Proof of Theorem 6. We have

$$s_n^*(x,f)-f(x)$$
$$=\frac{1}{\pi}\int_0^\pi D_n^*(u)du\cdot\frac{1}{\Gamma(\alpha)}\int_0^\pi[f(x+u-t)+f(x-u-t)-2f(x-t)]t^{\alpha-1}dt$$
$$=\frac{1}{\pi\Gamma(\alpha)}\int_0^\pi D_n^*(u)du\Big(\int_0^{1/n}+\int_{1/n}^\infty\Big)\varphi_{x-t}(u)t^{\alpha-1}dt=\frac{1}{\pi\Gamma(\alpha)}(I+J),$$

say, where $\varphi_y(u)=f(y+u)+f(y-u)-2f(y)$. By the M. Riesz theorem we have

$$I=\int_0^{1/n}t^{\alpha-1}dt\int_0^\pi\varphi_{x-t}(u)D_n^*(u)du=\pi\int_0^{1/n}t^{\alpha-1}s_n(x-t)dt,$$
$$|I|\leqq A\Big(\int_0^{1/n}t^{(\alpha-1)p'}dt\Big)^{1/p'}\Big(\int_0^{2\pi}|s_n(t)|^p\,dt\Big)^{1/p}$$
$$\leqq A\Big(\int_0^{1/n}t^{(\alpha-1)p'}dt\Big)^{1/p'}\Big(\int_0^{2\pi}|f(t)|^p\,dt\Big)^{1/p}\leqq A/n^{\alpha-1/p}.$$

Further, putting $F(t)=\int_0^t f(u)du$,

$$J=\int_0^\pi D_n^*(u)du\int_{1/n}^\infty[f(x+u-t)+f(x-u-t)-2f(x-t)]t^{\alpha-1}dt$$
$$=\int_0^\pi D_n^*(u)du\Big\{\Big[t^{\alpha-1}(F(x+u-t)+F(x-u-t)-2F(x-t))\Big]_{1/n}^\infty$$
$$+(1-\alpha)\int_{1/n}^\infty t^{\alpha-2}(F(x+u-t)+F(x-u-t)-2F(x-t))dt\Big\}$$
$$=(1-\alpha)\int_0^\pi D_n^*(u)du\int_{1/n}^\infty t^{\alpha-2}[\{F(x+u-t)-F(x-u-1/n)\}$$
$$+\{F(x-u-t)-F(x-u-1/n)\}-2\{F(x-t)-F(x-1/n)\}]dt$$
$$=(1-\alpha)\Big(\int_0^{\pi/n}du+\int_{\pi/n}^\pi du\Big)=J_1+J_2.$$

Now

$$J_1=(1-\alpha)\int_0^{\pi/n}D_n^*(u)du\int_{1/n}^\infty t^{\alpha-2}dt\int_0^u(f(x-t+v)+\text{similar terms})\,dv$$

and then

$$|J_1|\leqq An\int_0^{\pi/n}du\int_{1/n}^\infty t^{\alpha-2}dt\cdot$$
$$\cdot\Big[\Big(\int_0^u|f(x-t-v)|^p\,dv\Big)^{1/p}+\text{similar terms}\Big]\Big(\int_0^u dv\Big)^{1/p'}$$
$$\leqq An\int_0^{\pi/n}u^{1/p'}du\int_{1/n}^\infty t^{\alpha-2}dt\leqq A/n^{\alpha-1/p}.$$

It remains now to estimate J_2.

$$J_2=\int_{1/n}^{\infty} t^{\alpha-2}dt \int_0^{\pi/n} du \cdot \sum_{k=1}^{(n-1)/2}$$

$$[\{(F(x+u+2k\pi/n-t)-F(x+u+(2k-1)\pi/n-t))/\sin(u+(2k-1)\pi/n)/2 + \text{similar terms}\}$$
$$+\{(F(x+u+(2k-1)\pi/n-t)-F(x+u+(2k-2)\pi/n-t))\cdot O(n/k^2) + \text{similar terms}\}].$$

The terms in the brackets [ ] are less than, in absolute value, the sum of the terms of the type

$$n\,|\,F(y+k\pi/n)-F(y+(k-1)\pi/n)\,|\,/k$$

and easily estimatable terms. Now

$$\sum_{k=1}^{n} |\,F(y+k\pi/n)-F(y+(k-1)\pi/n)\,|\,/k$$
$$\leqq A\Big(\sum_{k=1}^{n}|\,F(y+k\pi/n)-F(y+(k-1)\pi/n)\,|^p\, n^{p-1}\Big)^{1/p}\Big(\sum_{k=1}^{n} k^{-p'}\Big)^{1/p'}\cdot n^{1/p-1}$$

where the first term is bounded by Young's theorem, and then

$$|\,J_2\,| \leqq \frac{A}{n^{-1/p}}\,\frac{1}{n}\int_{1/n}^{\infty} t^{\alpha-2}dt \leqq A/n^{\alpha-1/p}.$$

Thus we have proved the theorem.

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