221. *Axiomatic Treatment of Fullsuperharmonic Functions and Submarkov Resolvents*

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In 1963, P. A. Meyer [6] proved under very mild assumptions that, for any harmonic space satisfying Brelot's axioms there exists a semigroup such that the excessive functions with respect to this semigroup are exactly the nonnegative superharmonic functions. Our aim in the present paper is to show, under the same kind of assumptions adopted by Meyer, that there exists a submarkov resolvent $(V_\lambda)_\lambda\geqq 0$ such that: a) the excessive functions with respect to this resolvent are exactly the nonnegative fullsuperharmonic functions in the theory of axiomatic fullharmonic structures developed by F. Y. Maeda [4]; b) for any continuous function f with compact support the function $\lambda V_\lambda f$ converges uniformly to f as λ tends to infinity; c) $Vf=V_0 f$ is a bounded continuous fullsuperharmonic function of potential type if f is a nonnegative Borel function.

1. Preliminary results. First we shall give a brief summary of some results of F. Y. Maeda. Let S' be a (not compact) harmonic space with countable basis satisfying Brelot's axioms 1. 2. 3 [2]. The space of all harmonic functions on an open set U and the cone of superharmonic functions on U are denoted by $\mathcal{H}(U)$ and $\mathcal{S}(U)$ respectively. Let $\mathcal{D}$ be the family of domains D is S' such that D is not relatively compact and the boundary ∂D of D is compact. Let $\mathcal{G}$ be the family of open sets in S' with compact boundary. We will assume that for each $D\in\mathcal{D}$ we are given a linear subspace $\tilde{\mathcal{H}}(D)$ of $\mathcal{H}(D)$ satisfying:

(I) If $D, D'\in\mathcal{D}$, $D'\subset D$ and $u\in\tilde{\mathcal{H}}(D)$, then $u|_{D'}$, the restriction of u on D', $\in\tilde{\mathcal{H}}(D')$.

(II) If $u\in\mathcal{H}(D)$ and if there exists a compact set K in S' such that $\mathring{K}$ (the interior of K) $\supset\partial D$ and $u|_{D-K}\in\tilde{\mathcal{H}}(D-K)$, then $u\in\tilde{\mathcal{H}}(D)$.

A domain $D\in\mathcal{D}$ is said to be *regular* if any continuous function f on ∂D has a unique continuous extension $\tilde{H}^D_f$ on $\bar{D}$ such that $\tilde{H}^D_f|_D\in\tilde{\mathcal{H}}(D)$, and $f\geqq 0$ implies $\tilde{H}^D_f\geqq 0$. A set $G\in\mathcal{G}$ is said to be *regular* if every component of G is either relatively compact and regular in the sense of [2] or not relatively compact and regular in the sense described above. We will assume the next axiom:

(III) For any compact set K_0 in S', there exists another compact set K such that $\mathring{K}\supset K_0$ and $S'-K$ is regular.

A superharmonic function u on $G\in\mathcal{G}$ is *fullsuperharmonic* if, for any regular set $D\in\mathcal{D}$, $\bar{D}\subset G$ and any continuous function f on ∂D, we have

$$f\leqq u \text{ on } \partial D \Rightarrow \tilde{H}_f^D\leqq u \text{ on } D.$$

We will denote by $\tilde{\mathcal{S}}(G)$ the cone of fullsuperharmonic functions on $G\in\mathcal{G}$. If $G\in\mathcal{G}$ is relatively compact we have $\tilde{\mathcal{H}}(G)=\mathcal{H}(G)$, $\tilde{\mathcal{S}}(G)=\mathcal{S}(G)$.

Hereafter we take a regular domain $S\in\mathcal{D}$ and fix it. $C(S)$, $C_b(S)$, $B_b(S)$ respectively are the spaces of continuous functions, bounded continuous functions, bounded Borel measurable functions on S. Similar notations $C(A)\cdots$ are used for any locally compact subspace $A\subset S$.

Let $\mathcal{P}$ be the set of all nonnegative fullsuperharmonic functions on S such that; for any fullsuperharmonic function u on S, $p+u\geqq 0$ implies $u\geqq 0$. $p\in\mathcal{P}$ is said to be a *fullsuperharmonic function of potential type*. We shall define the *specific order* $p\succ q$ in $\mathcal{P}$ by $p-q\in\mathcal{P}(p,\ q\in\mathcal{P})$.

Besides (I)$\sim$(III) we adopt the next assumptions:

(IV) $1\in\tilde{\mathcal{S}}(S)$.

(V) For $\forall x\in S$ there exists a $p\in\mathcal{P}$ such that $0<p(x)<\infty$.

By virtue of these assumptions we can prove that there is a fullsuperharmonic function of potential type which is bounded continuous and is strictly fullsuperharmonic on S.

Definitions. (1) For any $p\in\mathcal{P}$, $G\in\mathcal{G}$, $\bar{G}\subset S$, put

$\mathcal{B}_G(p)=\{u\in\mathcal{P},\ \exists\in\tilde{\mathcal{S}}(G)$ such that $u=p+s$ on $G\}$.

$p_G(x)=\inf\{u(x),\ u\in\mathcal{B}_G(p)\}$.

(2) For any compact set K in S' contained in S, or containing ∂S, put

$$p_K(x)=p(x)-p_{S-K}(x).$$

(3) $\mathcal{P}_b=\mathcal{P}\cap\mathcal{H}(S)$.

(4) $Bp(x)=\sup\{u(x),\ u\in\mathcal{P}_b,\ u\prec p\}$.

The next properties (1.1)$\sim$(1.3) were obtained by F. Y. Maeda.

(1.1) $p_G\in\mathcal{P}\cap\mathcal{H}(S-\bar{G})$, $p_G\prec p$, the function $p-p_G$ is fullharmonic on G.

(1.2) $Bp\in\mathcal{P}_b$, $Bp\prec p$, and Bp is the smallest upper bound of $\{u\in\mathcal{P}_b,\ u\prec p\}$ with respect to the specific order.
$B(p+q)=Bp+Bq$.

(1.3) $Bp(x)=\inf\{p_{S-K}(x)$, K runs through all compact subset such that $S-K$ is regular.$\}$

We shall list some further properties of the set functions p_G and p_K.

(1.4) p_G is the greatest lower bound of $\mathcal{B}_G(p)$ with respect to the specific order.

(1.5) p_K is the greatest specific minorant of p which is fullharmonic on $S-K$.

(1.6) $$G_1\subset G_2\Rightarrow p_{G_1}\prec p_{G_2}\,;\ K_1\subset K_2\Rightarrow p_{K_1}\prec p_{K_2}$$ $$G\subset K\Rightarrow p_G\prec p_K\,;\ K\subset G\Rightarrow p_K\prec p_G.$$

(1.7) $$K\subset G\Rightarrow p_K=(p_G)_K.$$

(1.8) $$p_G=\sup(p_K\,;\ \subset G)$$ $$p_K=\inf(p_G\,;\ G\supset K).$$

(1.9) $$(p_{K_1})_{K_2}=p_{K_1\cap K_2},\ (p_{G_1})_{G_2}=p_{G_1\cap G_2}.$$

(1.10) $$p_{G_1\cup G_2}+p_{G_1\cap G_2}=p_{G_1}+p_{G_2},$$ $$p_{K_1\cup K_2}+p_{K_1\cap K_2}=p_{K_1}+p_{K_2}.$$

(1.11) If $p\in\mathcal{P}\cap\tilde{\mathcal{H}}(S-K)$ for some compact set K, then $Bp=0$ and $p=p_K$.

(1.12) Let p, $p_i(i=1,2)\in\mathcal{P}$ and $\alpha\geqq 0$, then $(p_1+p_2)_K=(p_1)_K+(p_2)_K$, $(\alpha p)_K=\alpha p_K$.

(1.13) If $p\in\mathcal{P}$ is strictly fullsuperharmonic on $G\in\mathcal{G}$, $\bar{G}\subset S$, then p_G is also strictly fullsuperharmonic on G.

2. Complete maximum principle and submarkov resolvents. Let $p\in\mathcal{P}$ and x be any point of S. From properties (1.6), (1.8), and (1.10) the set function $K\rightarrow p_K(x)$ may be extended to all subsets of S as an outer capacity. We shall denote by $V^p(x,\cdot)$ the restriction of this outer capacity to the Borel sets, which is a measure on S. The next lemma is a consequence of the minimum principle described in [4].

Lemma 2.1. *Let u be a fullsuperharmonic function on $G\in\mathcal{G}$, $\bar{G}\subset S$, with the following two properties:*

1) $\liminf\limits_{G\ni x\to z} u(x)\geqq 0$ on ∂G.

2) *There exists a $p\in\mathcal{P}$ such that $u+p\geqq 0$ on G. Then $u\geqq 0$.*

The proofs of the next theorems are carried out in the same manner as in sections 2 and 3 of [6] with the help of Lemma 2.1.

Theorem 2.2. *Let $p\in\mathcal{P}$ be continuous on S. There is one and only one positive kernel V^p which satisfies the following properties:*

a) *For every positive bounded Borel measurable function g on S, the function V^pg belongs to $\mathcal{P}$, is continuous and is fullharmonic on $S-\overline{\{g>0\}}$ if the closed support of g is compact.*

b) $V^p1=p-Bp$.

Theorem 2.3. *Let $p\in\mathcal{P}$ be bounded continuous. The kernel V^p associated with p defines a bounded positive operator in the Banach space $C_b(S)$ and satisfies the complete maximum principle.*

Proposition 2.4. *Let $p\in\mathcal{P}\cap C_b(S)$.*

a) *If p is fullharmonic on $S-K$ for some compact set K in S' such that $K\subset S$ or $\mathring{K}\supset\partial S$, then V^pf $(f\in B_b(S))$ is fullharmonic on $S-K$, and, for every $x\in S$, the support of $V^p(x,\cdot)$ is contained in K.*

b) *For any $\varepsilon>0$, there is a compact subset K of S such that*

$$\sup_{x\in H} V^p(x,S-K)<\varepsilon$$

for any compact subset H of S.

c) *If p is strictly fullsuperharmonic on $\mathring{K}$, then V^p is a strictly positive kernel on $\mathring{K}$, that is, $V^p(x, U)>0$ for every open subset U of $\mathring{K}$.*

d) *Let $q \in \mathcal{P}$ and $p \succ q$. Then $V^p = V^q + V^{p-q}$.*

Let f be measurable, positive, bounded by 1; then $-V^p f = -p + Bp + V^p(1-f) \in \tilde{\mathcal{S}}(S-K)$, so $V^p f \in \tilde{\mathcal{H}}(S-K)$. From $p \in \tilde{\mathcal{H}}(S-K)$ we have $p = p_K$ and $Bp=0$. This yields $V^p(x, S) = p(x) = p_K(x) = V_P(x, K)$. b) follows from axiom (III), (1.3), and Dini's theorem. c) and d) are immediate consequences of (1.13) and (1.12) respectively. ($q \in C_b(S)$ follows from $q \prec p$).

From Theorem 2.3, there exists a unique submarkov resolvent $(V_\lambda^p)_{\lambda>0}$ such that

$$V_\lambda^p - V^p = \lambda V_\lambda^p V^p = \lambda V^p V_\lambda^p.$$

Proposition 2.5. *Every positive fullsuperharmonic function s is supermedian with respect to the resolvent (V_λ^p).*

It follows from Lemma 2.1 and Theorem 2.2 a) that any positive fullsuperharmonic function s is a V^p-dominant function. But this says that s is supermedian with respect to (V_λ^p) [5, Ch. IX, T. 70].

Proposition 2.6.

$$V_\lambda^p s \geqq V_\lambda^{p_K} s \quad \text{for } \forall_s \in \tilde{\mathcal{S}}_+(S).$$

First we note that $V^p(1_{S-K} g)(x) = V^{p_{S-K}} \cdot g(x)$ ($g \in B_b(S)$), which follows from the uniqueness assertion of Theorem 2.2 and Proposition 2.3 d). Let $t = V_\lambda^p s - V_\lambda^{p_K} s$, we have $(I + \lambda V^p)t = V^{p-p_K}(s - \lambda V_\lambda^{p_K} s)$ by virtue of the resolvent equations for (V_λ^p) and $(V_\lambda^{p_K})$, so $(I+\lambda V^p)t = V^p(1_{S-K} \cdot h)$ where $h = s - \lambda V_\lambda^{p_K} s \geqq 0$. Therefore $0 \geqq V^p(\lambda t - 1_{S-K} \cdot h)$ on the set $\{\lambda t - (1_{S-K} \cdot h) > 0\}$, and we have $0 \geqq V^p(\lambda t - 1_{S-K} \cdot h)$ everywhere from the complete maximum principle, that is, $t \geqq 0$.

3. The regularity of the resolvent (V_λ^p) and the excessive functions. The next theorem is an analogous formula of the corresponding theorem in the axiomatic theory of superharmonic functions. The proof of it is based on the assumptions (IV) and (V). See, for example, [1].

Theorem 3.1. *There exists a countable collection of functions p_n ($n \geqq 1$) in $\mathcal{P} \cap C_b(S)$ such that:*

a) *For any compact subset K of S there is a subsequence (p_{n_j}) of (p_n) which is total in $C(K)$. Moreover each p_{n_j} can be chosen from $\tilde{\mathcal{H}}(S-K)$.*

b) *For every $N \geqq 1$, $r_N = \sum_{k=N}^{\infty} \frac{1}{2^k} \frac{p_k}{\|p_k\|} \in \mathcal{P} \cap C_b(S)$.*

In the sequel we will fix a $q \in \mathcal{P} \cap C_b(S)$ and let

$$p = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{p_n}{\|p_n\|} + q.$$

Lemma 3.2. *Let μ be a bounded Radon measure on S such that $\mu(s) \leqq s(x)$ for every $s \in \tilde{\mathcal{S}}_+(S)$ and some $x \in S$. Then $\mu(p) < p(x)$ provided $\mu \neq \varepsilon_x$.*

Corollary 3.3. *p is strictly fullsuperharmonic on S.*

Corollary 3.4. *Let K be a compact subset of S, $x \in K$ and $\mu \in C^*(K)$.* If *$\mu(s) \leqq s(x)$ for every $s \in \tilde{\mathcal{S}}_+(S)$, then $\mu(p_K) < p_K(x)$ or $\mu = \varepsilon_x$.*

Lemma 3.5. $\lim_{\lambda \to \infty} \lambda V_\lambda^{p_K} f(x) = f(x)$ *for every* $f \in C(K)$ *and* $x \in K$.

From Proposition 2.4 a), $\mu_\lambda(\cdot) = \lambda V_\lambda^{p_K}(x, \cdot)$ defines a Radon measure on K with its norm $\leqq 1$. Let μ be a weak limit point of $\mu_\lambda (\lambda \to \infty)$. Then we have $\mu(s) \leqq s(x)$ for every $s \in \tilde{\mathcal{S}}_+(S)$ from Proposition 2.5. Therefore $\mu(p_K) < p_K(x)$ or $\mu = \varepsilon_x$ by Corollary 3.4. But, since $\mu_\lambda(p_K) = \lambda V_\lambda^{p_K} V^{p_K} 1(x) \to V^{p_K} 1(x) = p_K(x)$ $(\lambda \to \infty)$, $\mu(p_K) = p_K(x)$ and $\mu = \varepsilon_x$.

From Lemma 3.5 and Proposition 2.6 we have the following:

Proposition 3.6. $\lim_{\lambda \to \infty} \lambda V_\lambda^p s(x) = s(x)$ *for* $\forall s \in \tilde{\mathcal{S}}_+(S) \cap C(S)$, $\forall x \in S$.

Lemma 3.7 (Maeda). *Let $(s_n)_{n \geqq 1}$ be an increasing sequence of fullsuperharmonic functions on S with* $\sup s_n < \infty$. *Then* $\sup s_n$ *is fullsuperharmonic on S.*

Lemma 3.8. *For any positive fullsuperharmonic function s on S there exists an increasing sequence of functions $p_n \in \mathcal{P} \cap C(S)$ such that $p_n \uparrow s$.*

Our main result is the following:

Proposition 3.9. *The positive fullsuperharmonic functions on S are exactly the finite excessive functions with respect to (V_λ^p).*

Proposition 2.5, Proposition 3.6, and Lemma 3.8 yield that every positive fullsuperharmonic function is excessive. The converse is easily proved from Lemma 3.7.

Proposition 3.10. *For every continuous function f with compact support the function $\lambda V_\lambda^p f$ converges uniformly to f as λ tends to infinity.*

Let K be a compact set such that $\overline{\{f > 0\}} \subset \mathring{K}$. By virtue of Theorem 3.1 a), for any $\varepsilon > 0$, we can find two functions u, $v \in \mathcal{P} \cap C_b(S) \cap \tilde{\mathcal{H}}(S-K)$ such that $|f-(u-v)| < \varepsilon$ on K. From Lemma 2.1 we have $-\varepsilon < -u+v = f-(u-v) < \varepsilon$ on $S-K$, that is, $|f-(u-v)| < \varepsilon$ on S. Since (V_λ^p) is submarkov, $\lambda V_\lambda^p(|f-(u-v)|) < \varepsilon$ on S. Now it follows from Proposition 3.9 and Dini's theorem that $0 \leqq u - \lambda V_\lambda^p u \leqq \varepsilon$ and $0 \leqq v - \lambda V_\lambda^p v \leqq \varepsilon$ on K for sufficiently large λ. Again applying Lemma 2.1 we have $0 \leqq u - \lambda V_\lambda^p u \leqq \varepsilon$ and $0 \leqq v - \lambda V_\lambda^p v \leqq \varepsilon$ on S. Therefore $|\lambda V_\lambda^p f - f| < 4\varepsilon$ uniformly on S for sufficiently large λ.

References

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