PAPERS COMMUNICATED

116. *On Locally Convex Topological Spaces.*

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Let L be a vector space and D a directed system. If there exists a real valued function $|x|_d$ on the domain $L\times D$ such that

(1) $|x|_d \geqq 0$; if $|x|_d=0$ for all $d \varepsilon D$ then $x=\theta$,

(2) $|ax|_d=|a|\cdot|x|_d$ for any real a,

(3) for any given $e\varepsilon D$ there exists $d\varepsilon D$ such that $|x|_d\to 0$ and $|y|_d\to 0$ imply $|x+y|_e\to 0$,

(4) $d<e$ implies $|x|_d\leqq|x|_e$,

then L is said to be a pseudo-normed linear space. It is proved by D. H. Hyers [1] that the pseudo-normed linear space is a linear topological space, which was defined by A. Kolmogoroff [2] and J. v. Neumann [3]. The triangular inequality

(3′) $|x+y|_d\leqq|x|_d+|y|_d$

is stronger than (3). If we take therefore the condition (3′) instead of (3) in addition of (1), (2), (4), then the space L is said to be a locally convex linear topological space. In this paper we concern the locally convex linear topological space L and its conjugate spaces $\bar{L}$ and $\bar{\bar{L}}$.

§ **1.** *Space* $\bar{L}$. The family of the sets $u(d, \delta)\equiv(x\,;|x|_d<\delta)$ $(\delta>0)$ is said to be a fundamental system of the origin θ; we denote it by $\{u(d, \delta)\,;\ \delta>0\}$.

Theorem 1. Referring the fundamental system $\{u(d,\delta)\,;\ \delta>0\}$, L is a locally convex linear topological space.

For a linear functional $f(x)$ on the domain L, if there exist some $d\varepsilon D$ and $M(d)>0$ such that

(1) $|f(x)|\leqq M(d)\cdot|x|_d$ for all $x\varepsilon L$,

then $f(x)$ is said to be bounded.

Theorem 2. For linear functionals continuity is equivalent to boundedness.

For the linear continuous functional $f(x)$ the set of all d with condition (1) is denoted by D_f, and for a given $d\varepsilon D$ the set of all $f(x)$ with condition (1) is denoted by $\bar{L}_d$.

Theorem 3. D_f is a cofinal subsystem of D.

Proof. If d' and d'' are two elements of D_f, then $|f(x)|\leqq M(d')\cdot|x|_{d'}$, and $|f(x)|\leqq M(d'')\cdot|x|_{d''}$ for all $x\varepsilon L$. Since D is a directed system, there exists a d such that $d'<d$ and $d''<d$. Consequently $|f(x)|\leqq M(d')\cdot|x|_d$ and $|f(x)|\leqq M(d'')\,|x|_d$ for all $x\varepsilon L$. That is $d\varepsilon D_f$. For any d in D there exists d'' such as $d''>d$ and $d''>d'$, so that $|f(x)|\leqq M(d')\,|x|_{d'}\leqq M(d')\,|x|_{d''}$, which shows that D_f is a cofinal subsystem of D.

Theorem 4. (i) $|f|_d\geqq 0$; and if $|f|_d=0$ then $f(x)\equiv 0$,

(ii) $|\alpha f|_d = |\alpha|\,|f|_d$,
(iii) for every elements f, g, there exists a $d \varepsilon D$ such that $|f+g|_d \leqq |f|_d + |g|_d$,
(iv) for any $f(x) \varepsilon \bar{L}_d$, $d<e$ implies $|f|_d \geqq |f|_e$.

Proof. (i) and (ii) are evident by the condition (1), and the definition of $|f|_d$. Concenring (iii) and (iv), we have $|f(x)| \leqq |f|_{d'}\,|x|_{d'}$ and $|g(x)| \leqq |g|_{d''}\,|x|_{d''}$. If $d' < d$ and $d'' < d$, then $|f(x)| \leqq |f|_{d'} \cdot |x|_d$ and $|g(x)| \leqq |g|_{d''} \cdot |x|_d$. Consequently $|f|_{d'} \geqq |f|_d$, $|g|_{d''} \geqq |g|_d$, and then $|f(x)| \leqq |f|_d \cdot |x|_d$; $|g(x)| \leqq |g|_d \cdot |x|_d$. Above inequalities give $|f(x)+g(x)| \leqq |f(x)| + |g(x)| \leqq [\,|f|_d + |g|_d]\,|x|_d$. That is $|f+g|_d \leqq |f|_d + |g|_d$, and $d<e$ implies $|f|_d \geqq |f|_e$.

Cor. 1. For any two linear continuous functionals $f(x)$ and $g(x)$, $0 \neq D_f \frown D_g \subset D_{f+g}$.

Cor. 2. For $f(x)$ and $g(x)$ in $\bar{L}_d$, then $|f+g|_d \leqq |f|_d + |g|_d$.

From this last corrolary it is evident that $\bar{L}_d$ is a normed space and we can easily show that $\bar{L}_d$ is complete. Hence $\bar{L}_d$ is a space of type (B). Now we shall denote by $\bar{L}$ the family of all linear continuous functions on L. We have easily $\bar{L} = \bar{L}_d + \bar{L}_{d'} + \cdots$. Now let $\|f\|$ be the $\underset{d \varepsilon D}{\text{g. l. b.}}\ |f|_d$, then we have (i′) $\|f\| \geqq 0$, (ii′) $\|\alpha f\| = |\alpha|\,\|f\|$, and (iii′) $\|f+g\| \leqq \|f\| + \|g\|$. But it is not true in general that $\|f\| = 0$ implies $f = \theta$. In this space L we can prove analogue of theorem in (B)-space. Among them we will state important ones without proof.

Theorem 5. For $x_0 \varepsilon L$ and $d \varepsilon D$ there is an $f_0 \varepsilon \bar{L}$ such that $|f_0|_d = 1$ and $f_0(x_0) = |x_0|_d$.

Theorem 6. If E_0 is a linear subset of L and $f(x)$ is a linear continuous functional on E_0, then there is an $f(x) \varepsilon \bar{L}$ such that $f(x) = f_0(x)$ for all $x \varepsilon E_0$ and $|f|_d = |f_0|_d$.

Theorem 7. Suppose E_0 is a linear subset of L, $y_0 \varepsilon L\text{-}E_0$ and for all $d \varepsilon D$ $\underset{x \varepsilon E_0}{\text{g. l. b.}}\ |x-y_0|_d = k > 0$, then there is an $f(x) \varepsilon \bar{L}$ such that $f(E_0) = 0$ $f(y_0) = 1$ and $|f|_d = 1/k$.

Theorem 8. For an $E_0 \subset L$ and functional $f_0(x)$ defined in E_0, necessary and sufficient conditions that for any $M > 0$ and any $d \varepsilon D$ there exists an $f(x) \varepsilon \bar{L}$ such that $f(x) = f_0(x)$ for $x \varepsilon E_0$ and $|f|_d \leqq M$, is that for any finite sequence $\{x_1, x_2, \ldots, x_r\}$ of E_0 and any real finite sequence $\{H_1, H_2, \cdots, H_r\}$, we have

$$\left|\sum_{i=1}^{r} H_i f_0(x_i)\right| \leqq M \left|\sum_{i=1}^{r} h_i x_i\right|.$$

2. On the space L. We will now consider a new topology in the space $\bar{L}$. A subset Γ of $\bar{L}$ is called to be closed in $\bar{L}$ if for any $d \varepsilon D$ $\Gamma \frown \bar{L}_d$ is closed in the space $\bar{L}_d$. A linear functional $X(f)$ on $\bar{L}$ is called to be continuous if it is continuous on the subspaces $\bar{L}_d$; by the same way we can define the boundedness of $X(f)$.

Theorem 9. For a linear functional continuity is equivalent to boundedness.

This theorem is evident. Since $\bar{L}_d$ is a (B)-space, $|X|_d$ is a norm of $X(f)$ in $\bar{L}_d$. Consequentlly $|X|_d \equiv \underset{f \varepsilon \bar{L}d}{\text{l. u. b.}}\ |X(f)|\,/\,|f|_d$. By $\bar{\bar{L}}$ we denote the family of all the linear continuous functionals on $\bar{L}$. The bar operation for $\bar{L}$ is different from that for L.

Theorem 10. For any $x_0 \varepsilon L$ and any $d \varepsilon D$ there is an $X_0 \varepsilon \bar{\bar{L}}$ such that $|x_0|_d = |X_0|_d$.

Proof. Put $X_0(f) = f(x_0)$, then the functional $X_0(f)$ is evidently linear on $\bar{L}$ and moreover for every $\bar{L}_d$, $|X_0(f)| = |f(x_0)| \leqq |f|_d |x_0|_d$. So that X_0 is linear continuous on $\bar{L}$, on the other hand by theorem 5, for any $d \varepsilon D$ there is an $f_0(x)$ such that $f_0(x_0) = |x_0|_d$ and $|f_0|_d = 1$. Consequently $|X_0(f_0)| = |f_0(x_0)| = |x_0|_d = |f_0|_d |x_0|_d$, and then $|X_0|_d = |x_0|_d$.

By this theorem we see that $L \subset \bar{\bar{L}}$. Since for any $x_0 \varepsilon L$ and $d \varepsilon D$ we have $|x_0|_d = |X_0|_d$, therefore there occurs problem of regulality as in (B)-space. Here we will not enter it, but we will show by an example difficulties of this problem.

Example. Let L be a (B)-space and $\bar{L}$ be its conjugate. If we define $f > g$ by $|f(x)| > |g(x)|$ for all $x \varepsilon L$, then $\bar{L}$ is a vector lattice (this was proved in [7].). Consequently $\bar{L}$ is adirected system by this ording. If we put $|x|_f \equiv |f(x)|$ for $x \varepsilon L$, then this pseudo-norm $|x|_f$ satisfies all conditions (1), (2), (3′) and (4) in the introduction. Consequently L is a locally convex linear topological space with respect to such the norm $|x|_f$.

Further if we define $u(f_1, f_2, \dots, f_k; \delta)$ by $u(f_1, f_2, \dots, f_k; \delta) \equiv (x; |x|_{f_i} < \delta; i = 1, 2, \dots, k)$, then we can easily prove that this topology implies also the original topology of L. In this example we denote the former topology by $T^{(1)}$ and latter by $T^{(2)}$, then in general the topology $T^{(1)}$ is not stronger than $T^{(2)}$, and $T^{(2)}$ is not stronger than the norm topology of L. Thus the problem of regularity becomes the problem of regularity with respect to weak convergence topology of L.

Literature.

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